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Research Article

Dynamical Analysis in a Delayed Predator-Prey Model with Two Delays

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A class of Beddington-DeAngelis functional response predator-prey model is considered. The conditions for the local stability and the existence of Hopf bifurcation at the positive equilibrium of the system are derived. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory. Some numerical simulations for justifying the theoretical analysis are also provided. Finally, main conclusions are given.

1. Introduction

In recent years, population dynamics (including stable, unstable, persistent, and oscillatory behavior) has become very popular since Vito Volterra and James Lotka proposed the seminal models of predator-prey models in the mid-1920s. Great attention has been paid to the dynamics properties of the predator-prey models which have significant biological background. Many excellent and interesting results have been obtained [1–21]. In 2009, Gakkhar et al. [4] investigated the local stability and Hopf bifurcation of the autonomous delayed predator-prey system with Beddington-DeAngelis functional response:

$$\begin{aligned}
\dot{u}_1(t) &= u_1(t)\left[1 - u_1(t-\tau_1)\right] - \frac{a_1 u_1(t) u_2(t)}{a + u_1(t) + b u_2(t)}, \\
\dot{u}_2(t) &= d_1 u_2(t)\left[-d + \frac{a_1 u_1(t-\tau_2)}{a + u_1(t-\tau_2) + b u_2(t-\tau_2)}\right],
\end{aligned} \tag{1.1}$$

where $u_1(t),\ u_2(t)$ represent the prey density and the predator density, respectively. The delay terms occur in growth as well as interaction terms. For this, it means that the prey takes time τ_1 to convert the food into its growth [11], whereas the predator takes time τ_2 for the same [22]. All the parameters in the model take positive values, that is, $a_1 > 0,\ d > 0,\ d_1 > 0,\ a > 0,\ b > 0$. The more detail biological meaning of the coefficients of system (1.1), one can see [11] or [22].

We would like to point out that Gakkhar et al. [4] studied the local stability and Hopf bifurcation of system (1.1) under the assumption: $\tau_1 = \tau_2 = \tau$ and obtained some excellent results. While in most cases, $\tau_1 \neq \tau_2$. Considering the factor, we further investigate the model (1.1) with $\tau_1 \neq \tau_2$ as a complementarity.

In this paper, we go on to study the stability, the local Hopf bifurcation for system (1.1). To the best of our knowledge, it is the first time to deal with the research of Hopf bifurcation for model (1.1) under the assumption $\tau_1 \neq \tau_2$.

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the positive equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2. Stability of the Positive Equilibrium and Local Hopf Bifurcations

In this section, we shall study the stability of the positive equilibrium and the existence of local Hopf bifurcations.

Since time delay does not change the equilibrium of system and according to [4], we know that the delayed prey predator model (1.1) has three equilibrium points: two boundary equilibrium $E_1(0,0)$ and $E_2(1,0)$, and a nontrivial equilibrium point $E_0(u_1^*, u_2^*)$, where u_1^*, u_2^* are the positive solutions of the following quadratic equations:

$$\begin{aligned} &u_1^2 + \alpha_1 u_1 + \beta_1 = 0, \\ &u_2^2 + \alpha_2 u_2 + \beta_2 = 0, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \alpha_1 &= \frac{a_1 - d a_1 - b}{b}, \qquad \beta_1 = -\frac{a d a_1}{d}, \\ \alpha_2 &= \frac{a_1 (d-1)^2 + b(2ad + d - 1)}{b^2 d}, \qquad \beta_2 = -\frac{a(ad + d - 1)}{b^2 d}. \end{aligned} \tag{2.2}$$

Since $\beta_1 < 0$, (2.1) in u_1 admits a unique positive solution. If one of the following conditions:

$$\text{(a)}\ \alpha_2 > 0, \quad \beta_2 < 0, \qquad \text{(b)}\ \alpha_2 < 0, \quad \beta_2 = \frac{\alpha_2^2}{4}, \qquad \text{(c)}\ \alpha_2 < 0, \quad 0 < \beta_2 < \frac{\alpha_2^2}{4}, \tag{2.3}$$

holds, then system (2.1) has at least one positive equilibrium point $E_0(u_1^*, u_2^*)$.

Let $\overline{u}_1(t) = u_1(t) - u_1^*$, $\overline{u}_2(t) = u_2(t) - u_2^*$ and still denote $\overline{u}_i(t)$ $(i = 1, 2)$ by $u_i(t)$ $(i = 1, 2)$, respectively, then (1.1) becomes

$$\begin{aligned}\dot{u}_1(t) &= m_1u_1(t) + m_2u_2(t) + m_3u_1(t-\tau_1) + F_1,\\ \dot{u}_2(t) &= n_1u_2(t) + n_2u_1(t-\tau_2) + n_3u_2(t-\tau_2) + F_2,\end{aligned} \tag{2.4}$$

where m_i, n_i $(i = 1, 2, 3)$ and F_j $(j = 1, 2)$ are defined by Appendix A.

The linearization of (2.4) at $(0, 0)$ is

$$\begin{aligned}\dot{u}_1(t) &= m_1u_1(t) + m_2u_2(t) + m_3\mathrm{u}_1(t-\tau_1),\\ \dot{u}_2(t) &= n_1u_2(t) + n_2u_1(t-\tau_2) + n_3u_2(t-\tau_2),\end{aligned} \tag{2.5}$$

whose characteristic equation is

$$\begin{aligned}&\lambda^2 - (m_1+n_1)\lambda + m_1n_1 - (n_3\lambda - m_1n_3 + m_2n_2)e^{-\lambda\tau_2}\\ &\quad - (m_3\lambda + m_3n_1)e^{-\lambda\tau_1} + m_3n_3e^{-\lambda(\tau_1+\tau_2)} = 0.\end{aligned} \tag{2.6}$$

In order to investigate the distribution of roots of the transcendental equation (2.6), the following Lemma is useful.

Lemma 2.1 (see [23]). *For the transcendental equation*

$$\begin{aligned}P\left(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m}\right) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)}\\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots\\ &\quad + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0,\end{aligned} \tag{2.7}$$

as $(\tau_1, \tau_2, \tau_3, \dots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \dots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

In the sequel, we consider three cases.

Case a. $\tau_1 = \tau_2 = 0$, (2.6) becomes

$$\lambda^2 - (m_1+m_3+n_1+n_3)\lambda + m_1n_1 + m_1n_3 + m_3n_3 - m_2n_2 - m_3n_1 = 0. \tag{2.8}$$

A set of necessary and sufficient conditions for all roots of (2.8) to have a negative real part are given in the following form:

$$\text{(H1)}\ \ (m_1+m_3+n_1+n_3) < 0, \qquad m_1n_1 + m_1n_3 + m_3n_3 - m_2n_2 - m_3n_1 > 0. \tag{2.9}$$

Then, the equilibrium point $E_0(u_1^*, u_2^*)$ is locally asymptotically stable when the condition (H1) holds.

Case b. $\tau_1 = 0,\ \tau_2 > 0$, (2.6) becomes

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau_2} = 0, \tag{2.10}$$

where

$$p = -(m_1 + m_3 + n_1), \quad r = m_1 n_1 - m_3 n_1, \quad s = -n_3, \quad q = m_1 n_3 + m_3 n_3 - m_2 n_2. \tag{2.11}$$

For $\omega > 0,\ i\omega$ be a root of (2.10), then it follows that

$$\begin{aligned} q\cos\omega\tau_2 + s\omega\sin\omega\tau_2 &= \omega^2 - r, \\ s\omega\cos\omega\tau_2 - q\sin\omega\tau_2 &= -p\omega, \end{aligned} \tag{2.12}$$

which leads to

$$\omega^4 + \left(p^2 - s^2 - 2r\right)\omega^2 + r^2 - q^2 = 0. \tag{2.13}$$

It is easy to see that if the condition

$$\text{(H2)}\ p^2 - s^2 - 2r > 0, \qquad r^2 - q^2 > 0 \tag{2.14}$$

holds, then (2.13) has no positive roots. Hence, all roots of (2.10) have negative real parts when $\tau_2 \in [0, +\infty)$ under the conditions (H1) and (H2).

If (H1) and

$$\text{(H3)}\ r^2 - q^2 < 0 \tag{2.15}$$

hold, then (2.13) has a unique positive root ω_0^2. Substituting ω_0^2 into (2.12), we obtain

$$\tau_{2_n} = \frac{1}{\omega_0}\left\{\arccos\frac{q(\omega_0^2 - r) - ps\omega_0^2}{s^2\omega_0^2 + q^2} + 2n\pi\right\}, \quad n = 0, 1, 2, \ldots. \tag{2.16}$$

If (H1) and

$$\text{(H4)}\ p^2 - s^2 - 2r < 0, \quad r^2 - q^2 > 0, \quad \left(p^2 - s^2 - 2r\right)^2 > 4\left(r^2 - q^2\right) \tag{2.17}$$

hold, then (2.10) has two positive roots ω_+^2 and ω_-^2. Substituting $\omega_\pm^2$ into (2.12), we obtain

$$\tau_{2_k}^{\pm} = \frac{1}{\omega_\pm}\left\{\arccos\frac{q(\omega_\pm^2 - r) - ps\omega_\pm^2}{s^2\omega_\pm^2 + q^2} + 2k\pi\right\}, \quad k = 0, 1, 2, \ldots. \tag{2.18}$$

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be a root of (2.10) near $\tau_2 = \tau_{2_n}$ and $\alpha(\tau_{2_n}) = 0,\ \omega(\tau_{2_n}) = \omega_0$. Due to functional differential equation theory, for every $\tau_{2_n},\ n = 0, 1, 2, \ldots$, there exists $\varepsilon > 0$ such that

$\lambda(\tau_2)$ is continuously differentiable in τ_2 for $|\tau_2 - \tau_{2_n}| < \varepsilon$. Substituting $\lambda(\tau_2)$ into the left-hand side of (2.10) and taking derivative with respect to τ_2, we have

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{(2\lambda + p)e^{\lambda\tau_2}}{\lambda(s\lambda + q)} + \frac{s}{\lambda(s\lambda + q)} - \frac{\tau_2}{\lambda}, \tag{2.19}$$

which leads to

$$\begin{aligned}
\left[\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{2_n}} &= \operatorname{Re}\left\{\frac{(2\lambda + p)e^{\lambda\tau_2}}{\lambda(s\lambda + q)}\right\}\Bigg|_{\tau_2=\tau_{2_n}} + \operatorname{Re}\left\{\frac{s}{\lambda(s\lambda + q)}\right\}\Bigg|_{\tau_2=\tau_{2_n}} \\
&= \operatorname{Re}\left\{\frac{p\cos\omega_0\tau_{2_n} - 2\omega_0\sin\omega_0\tau_{2_n} + i(2\omega_0\cos\omega_0\tau_{2_n} + p\sin\omega_0\tau_{2_n})}{-s\omega_0^2 + iq\omega_0}\right\} \\
&\quad + \operatorname{Re}\left\{\frac{s}{-s\omega_0^2 + iq\omega_0}\right\} \\
&= \frac{1}{\Lambda}\Big\{-s\omega_0^2(p\cos\omega_0\tau_{2_n} - 2\omega_0\sin\omega_0\tau_{2_n}) \\
&\qquad +q\omega_0(2\omega_0\cos\omega_0\tau_{2_n} + p\sin\omega_0\tau_{2_n}) - s^2\omega_0^2\Big\} \\
&= \frac{1}{\Lambda}\Big\{p\omega_0(q\sin\omega_0\tau_{2_n} - s\omega_0\cos\omega_0\tau_{2_n}) \\
&\qquad +2\omega_0^2(q\cos\omega_0\tau_{2_n} + s\omega_0\sin\omega_0\tau_{2_n}) - s^2\omega_0^2\Big\} \\
&= \frac{1}{\Lambda}\left\{p^2\omega_0^2 + 2\omega_0^4 - 2r\omega_0^2 - s^2\omega_0^2\right\} \\
&= \frac{\omega_0^2}{\Lambda}\left\{2\omega_0^2 + p^2 - s^2 - 2r\right\} \\
&= \frac{1}{\Lambda}\left\{-p^2 + s^2 + 2r + \sqrt{\Delta_*} + p^2 - s^2 - 2r\right\} = \frac{\omega_0^2}{\Lambda}\sqrt{\Delta_*} > 0,
\end{aligned} \tag{2.20}$$

where

$$\Lambda = s^2\omega_0^4 + q^2\omega_0^2 > 0, \qquad \sqrt{\Delta_*} = \left(s^2 - p^2 + 2r\right)^2 - 4\left(r^2 - q^2\right). \tag{2.21}$$

Noting that

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau_2}\right\}\Bigg|_{\tau_2=\tau_{2_n}} = \operatorname{sign}\left\{\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]\right\}\Bigg|_{\tau_2=\tau_{2_n}} = 1, \tag{2.22}$$

we have

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau_2}\right|_{\tau_2=\tau_{2_n}} > 0. \tag{2.23}$$

Similarly, we can obtain

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau_2}\right|_{\tau_2=\tau_{2_k}^+} > 0, \qquad \left.\frac{d(\operatorname{Re}\lambda)}{d\tau_2}\right|_{\tau_2=\tau_{2_k}^-} < 0. \tag{2.24}$$

According to above analysis the Corollary 2.4 in Ruan and Wei [23], we have the following results.

Lemma 2.2. *For $\tau_1 = 0$, assume that one of the conditions (a), (b), (c), and (d) holds and $(H1)$ is satisfied. Then, the following conclusions hold.*

(i) *If $(H2)$ holds, then the positive equilibrium $E_0(u_1^*, u_2^*)$ of system* (1.1) *is asymptotically stable for all $\tau_2 \geq 0$.*

(ii) *If $(H3)$ holds, then the positive equilibrium $E_0(u_1^*, u_2^*)$ of system* (1.1) *is asymptotically stable for $\tau_2 < \tau_{2_0}$, and unstable for $\tau_2 < \tau_{2_0}$. Furthermore, system* (1.1) *undergoes a Hopf bifurcation at the positive equilibrium $E_0(u_1^*, u_2^*)$ when $\tau_2 = \tau_{2_0}$.*

(iii) *If $(H4)$ holds, then there is a positive integer m such that the positive equilibrium $E_0(u_1^*, u_2^*)$ is stable when $\tau_2 \in [0, \tau_{2_0}^+) \cup (\tau_{2_0}^-, \tau_{2_1}^+) \cup \cdots \cup (\tau_{2_{m-1}}^-, \tau_{2_m}^+)$, and unstable when $\tau_2 \in [\tau_{2_0}^+, \tau_{2_0}^-) \cup (\tau_{2_1}^+, \tau_{2_1}^-) \cup \cdots \cup (\tau_{2_m}^+, \tau_{2_m}^-) \cup (\tau_{2_m}^+, \infty)$. Furthermore, system* (1.1) *undergoes a Hopf bifurcation at the positive equilibrium $E_0(u_1^*, u_2^*)$ when $\tau_2 = \tau_{2_k}^{\pm}$, $k = 0, 1, 2, \ldots$.*

Case c ($\tau_1 > 0$, $\tau_2 > 0$)

We consider (2.6) with τ_2 in its stable interval. Regarding τ_1 as a parameter, without loss of generality, we consider system (1.1) under the assumptions (H1) and (H3). Let $i\omega(\omega > 0)$ be a root of (2.6), then we can obtain

$$\omega^4 + k_1\omega^3 + k_2\omega^2 + k_3\omega + k_4 = 0, \tag{2.25}$$

where

$$\begin{aligned}
k_1 &= 2n_3 \sin\omega\tau_2, \\
k_2 &= 2[(m_2n_2 - m_1n_3)\cos\omega\tau_2 - m_1n_1] + (m_1 + n_1 + n_3\cos\omega\tau_2)^2 - m_3^2, \\
k_3 &= 2[(m_2n_2 - m_1n_3)\cos\omega\tau_2]n_3\sin\omega\tau_2 - 2m_3^2n_3\sin\omega\tau_2 \\
&\quad + 2(m_1 + n_1 + n_3\cos\omega\tau_2)(m_2n_2 - m_1n_3)\sin\omega\tau_2, \\
k_4 &= [(m_2n_2 - m_1n_3)\cos\omega\tau_2 - m_1n_1]^2 + [(m_2n_2 - m_1n_3)\sin\omega\tau_2]^2 \\
&\quad - (m_3n_1 - m_3n_3\cos\omega\tau_2)^2 - (m_3n_3\sin\omega\tau_2)^2.
\end{aligned} \tag{2.26}$$

Denote

$$H(\omega) = \omega^4 + k_1\omega^3 + k_2\omega^2 + k_3\omega + k_4. \tag{2.27}$$

Assume that

$$\text{(H5)}\ \ (m_2n_2 - m_1n_3 - m_1n_1)^2 < (m_3n_1 - m_3n_3)^2. \tag{2.28}$$

It is easy to check that $H(0) < 0$ if (H5) holds and $\lim_{\omega\to+\infty} H(\omega) = +\infty$. We can obtain that (2.25) has finite positive roots $\omega_1, \omega_2, \ldots, \omega_n$. For every fixed $\omega_i,\ i = 1,2,3,\ldots,k$, there exists a sequence $\{\tau_{1_i}^j \mid j = 1,2,3,\ \ldots\}$, such that (2.25) holds. Let

$$\tau_{1_0} = \min\left\{\tau_{1_i}^j \mid i = 1,2,\ldots,k; j = 1,2,\ldots\right\}. \tag{2.29}$$

When $\tau_1 = \tau_{1_0}$, (2.6) has a pair of purely imaginary roots $\pm i\omega^*$ for $\tau_2 \in [0, \tau_{2_0})$.

In the following, we assume that

$$\text{(H6)}\ \ \left[\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right]_{\lambda=i\omega^*} \neq 0. \tag{2.30}$$

Thus, by the general Hopf bifurcation theorem for FDEs in Hale [24], we have the following result on the stability and Hopf bifurcation in system (1.1).

Theorem 2.3. *For system* (1.1), *assume that one of the conditions (a), (b), (c), and (d) holds and suppose* (*H*1), (*H*3), *and* (*H*5) *are satisfied, and* $\tau_2 \in [0, \tau_{2_0})$, *then the positive equilibrium* $E_0(u_1^*, u_2^*)$ *is asymptotically stable when* $\tau_1 \in (0, \tau_{1_0})$, *and system* (1.1) *undergoes a Hopf bifurcation at the positive equilibrium* $E_0(u_1^*, u_2^*)$ *when* $\tau_1 = \tau_{1_0}$.

3. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau_1 = \tau_{1_0}$. In this section, we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the positive equilibrium $E_0(u_1^*, u_2^*)$ at this critical value of τ_1, by using techniques from normal form and center manifold theory [7]. Throughout this section, we always assume that system (1.1) undergoes Hopf bifurcation at the positive equilibrium $E_0(u_1^*, u_2^*)$ for $\tau_1 = \tau_{1_0}$, and then $\pm i\omega^*$ is corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E_0(u_1^*, u_2^*)$.

Without loss of generality, we assume that $\tau_2^* < \tau_{1_0}$, where $\tau_2^* \in (0, \tau_{2_0})$. For convenience, let $\overline{u}_i(t) = u_i(\tau t)\ (i = 1,2)$ and $\tau_1 = \tau_{1_0} + \mu$, where τ_{1_0} is defined by (2.28) and $\mu \in R$, drop the bar for the simplification of notations, then system (1.1) can be written as an FDE in $C = C([-1,0], R^2)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1(t), u_2(t))^T \in C$ and $u_t(\theta) = u(t+\theta) = (u_1(t+\theta), u_2(t+\theta))^T \in C$, and $L_\mu : C \to R,\ F : R \times C \to R$ are given by

$$L_\mu \phi = (\tau_{1_0} + \mu) B \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_{1_0} + \mu) C \begin{pmatrix} \phi_1\left(-\dfrac{\tau_2^*}{\tau_{1_0}}\right) \\ \phi_1\left(-\dfrac{\tau_2^*}{\tau_{1_0}}\right) \end{pmatrix} + (\tau_{1_0} + \mu) D \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}, \tag{3.2}$$

$$F(\mu, \phi) = (\tau_{1_0} + \mu)(f_1, f_2)^T, \tag{3.3}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C$,

$$\begin{aligned} B &= \begin{pmatrix} m_1 & m_2 \\ 0 & n_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ n_2 & n_3 \end{pmatrix}, \qquad D = \begin{pmatrix} m_3 & 0 \\ 0 & 0 \end{pmatrix}, \\ f_1 &= -\phi_1(0)\phi_1(-1) + l_1\phi_1^2(0) + l_2\phi_2^2(0) + l_3\phi_1(0)\phi_2(0) + l_4\phi_1^3(0) \\ &\quad + l_5\phi_2^3(0) + l_6\phi_1^2(0)\phi_2(0) + l_7\phi_1(0)\phi_2^2(0) + \text{h.o.t.}, \\ f_2 &= k_1\phi_1\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\phi_2(0) + k_2\phi_2(0)\phi_2\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + k_3\phi_1^2\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) \\ &\quad + k_4\phi_1\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\phi_2\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + e_1\phi_1^2\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\phi_2(0) + e_2\phi_1\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\varphi_2\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\phi_2(0) \\ &\quad + e_3\phi_1^3\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + e_4\phi_1\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\phi_2^2\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + e_5\phi_2^3\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + \text{h.o.t.} \end{aligned} \tag{3.4}$$

From the discussion in Section 2, we know that, if $\mu = 0$, then system (3.1) undergoes a Hopf bifurcation at the positive equilibrium $E_0(u_1^*, u_2^*)$ and the associated characteristic equation of system (3.1) has a pair of simple imaginary roots $\pm\omega^*\tau_{1_0}$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu),\ \theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in C. \tag{3.5}$$

In fact, we can choose

$$\eta(\theta,\mu)=\begin{cases}(\tau_{1_0}+\mu)(B+C+D), & \theta=0,\\ (\tau_{1_0}+\mu)(C+D), & \theta\in\left[-\dfrac{\tau_2^*}{\tau_{10}},0\right),\\ (\tau_{1_0}+\mu)D, & \theta\in\left(-1,-\dfrac{\tau_2^*}{\tau_{10}}\right),\\ 0, & \theta=-1.\end{cases} \tag{3.6}$$

For $\phi\in C([-1,0],R^2)$, define

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & -1\le\theta<0,\\ \displaystyle\int_{-1}^{0}d\eta(s,\mu)\phi(s), & \theta=0,\end{cases}$$
$$R\phi=\begin{cases}0, & -1\le\theta<0,\\ F(\mu,\phi), & \theta=0.\end{cases} \tag{3.7}$$

Then, (3.1) is equivalent to the abstract differential equation

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t, \tag{3.8}$$

where $u_t(\theta)=u(t+\theta),\ \theta\in[-1,0]$.

For $\psi\in C([0,1],(R^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0}d\eta^T(t,0)\psi(-t), & s=0.\end{cases} \tag{3.9}$$

For $\phi\in C([-1,0],R^2)$ and $\psi\in C([0,1],(R^2)^*)$, define the bilinear form

$$\langle\psi,\phi\rangle=\overline{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.10}$$

where $\eta(\theta)=\eta(\theta,0)$, the $A=A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega^*\tau_{1_0}$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega^*\tau_{1_0}$ and $-i\omega^*\tau_{1_0}$, respectively. By direct computation, we can obtain

$$q(\theta)=(1,\alpha)^Te^{i\omega^*\tau_{1_0}\theta},\quad q^*(s)=M(1,\alpha^*)e^{i\omega^*\tau_{1_0}s},\quad M=\frac{1}{K}, \tag{3.11}$$

where

$$\alpha = \frac{i\omega^* - m_1 - m_3 e^{-i\omega^*\tau_{1_0}}}{m_2},$$
$$\alpha^* = -\frac{i\omega^* + m_1 + m_3 e^{-i\omega^*\tau_{1_0}}}{n_2 e^{-i\omega^*\tau_2^*/\tau_{1_0}}}, \tag{3.12}$$
$$K = 1 + \overline{\alpha}\alpha^* + \tau_{1_0}\left[m_3 e^{i\omega^*\tau_{1_0}} + \frac{\tau_2^*}{\tau_{1_0}} n_2 \overline{\alpha} e^{i\omega^*(\tau_2^*/\tau_{1_0})} + \overline{\alpha}\alpha^* n_3 \frac{\tau_2^*}{\tau_{1_0}} e^{i\omega^*(\tau_2^*/\tau_{1_0})}\right].$$

Furthermore, $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

Next, we use the same notations as those in Hassard [7] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (3.1) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t,\theta) = u_t(\theta) - 2\operatorname{Re}\{z(t)q(\theta)\}, \tag{3.13}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \overline{z}(t), \theta), \tag{3.14}$$

where

$$W(z(t), \overline{z}(t), \theta) = W(z, \overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{3.15}$$

and z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (3.1),

$$\dot{z}(t) = i\omega^*\tau_{1_0}z + \overline{q}^*(\theta)F(0, W(z, \overline{z}, \theta)) + 2\operatorname{Re}\{zq(\theta)\} \stackrel{\text{def}}{=} i\omega^*\tau_{1_0}z + \overline{q}^*(0)F_0. \tag{3.16}$$

That is

$$\dot{z}(t) = i\omega^*\tau_{1_0}z + g(z, \overline{z}), \tag{3.17}$$

where

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots. \tag{3.18}$$

Hence, we have to obtain the expression of $g(z, \overline{z})$ (see Appendix B). Then, it is easy to obtain the expression of $g_{20}, g_{11}, g_{02}, g_{21}$ (see Appendix B).

For unknown $W_{20}^{(i)}(\theta), W_{11}^{(i)}(\theta), (i = 1, 2)$ in g_{21}, we still need to compute them. From (3.8), (3.13), we have

$$W' = \begin{cases} AW - 2\operatorname{Re}\left\{\overline{q}^*(0)\overline{f}q(\theta)\right\}, & -1 \le \theta < 0, \\ AW - 2\operatorname{Re}\left\{\overline{q}^*(0)\overline{f}q(\theta)\right\} + \overline{F}, & \theta = 0, \end{cases} \tag{3.19}$$
$$\stackrel{\text{def}}{=} AW + H(z, \overline{z}, \theta),$$

where

$$H(z, \overline{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.20}$$

Comparing the coefficients, we obtain

$$(A - 2i\tau_{1_0}\omega^*)W_{20} = -H_{20}(\theta), \tag{3.21}$$

$$AW_{11}(\theta) = -H_{11}(\theta), \tag{3.22}$$

and we know that, for $\theta \in [-1, 0)$,

$$H(z, \overline{z}, \theta) = -\overline{q}^*(0)f_0 q(\theta) - q^*(0)\overline{f}_0\overline{q}(\theta) = -g(z, \overline{z})q(\theta) - \overline{g}(z, \overline{z})\overline{q}(\theta). \tag{3.23}$$

Comparing the coefficients of (3.23) with (3.20) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \tag{3.24}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{3.25}$$

From (3.21), (3.24), and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega^*\tau_{1_0}W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{3.26}$$

Noting that $q(\theta) = q(0)e^{i\omega^*\tau_{1_0}\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*\tau_{1_0}}q(0)e^{i\omega^*\tau_{1_0}\theta} + \frac{i\overline{g}_{02}}{3\omega^*\tau_{1_0}}\overline{q}(0)e^{-i\omega^*\tau_{1_0}\theta} + E_1 e^{2i\omega^*\tau_{1_0}\theta}, \tag{3.27}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T \in R^2$ is a constant vector.

Similarly, from (3.22), (3.25), and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta), \tag{3.28}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^*\tau_{1_0}}q(0)e^{i\omega^*\tau_{1_0}\theta} + \frac{i\overline{g}_{11}}{\omega^*\tau_{1_0}}\overline{q}(0)e^{-i\omega^*\tau_{1_0}\theta} + E_2, \tag{3.29}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T \in R^2$ is a constant vector

In what follows, we shall seek appropriate E_1, E_2 in (3.27), (3.29), respectively. It follows from the definition of A and (3.24), (3.25) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega^* \tau_{1_0} W_{20}(0) - H_{20}(0), \tag{3.30}$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.31}$$

where $\eta(\theta) = \eta(0,\theta)$.

From (3.21), we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_{1_0}(H_1, H_2)^T, \tag{3.32}$$

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}(0)\overline{q}(0) + 2\tau_{1_0}(P_1, P_2)^T, \tag{3.33}$$

where

$$\begin{aligned}
H_1 &= -e^{-i\omega^*\tau_{1_0}} + l_1 + l_2\alpha^2 + l_3\alpha, \\
H_2 &= \left(k_1\alpha + k_2\alpha^2\right)e^{-i\omega^*\tau_2^*} + (k_3 + k_4\alpha)e^{-2i\omega^*\tau_2^*}, \\
P_1 &= -\frac{1}{2}\left(e^{-i\omega^*\tau_{1_0}} + e^{i\omega^*\tau_{1_0}}\right) + l_1 + l_2|\alpha|^2 + 2\operatorname{Re}\{\alpha\}, \\
P_2 &= k_1 \operatorname{Re}\left\{\alpha e^{i\omega^*\tau_2^*}\right\} + k_2|\alpha|^2\left(e^{-i\omega^*\tau_2^*} + e^{i\omega^*\tau_2^*}\right) + k_3 + k_4 \operatorname{Re}\{\alpha\}.
\end{aligned} \tag{3.34}$$

Noting that

$$\begin{aligned}
\left(i\omega^*\tau_{1_0} I - \int_{-1}^{0} e^{i\omega^*\tau_{1_0}\theta} d\eta(\theta)\right)q(0) &= 0, \\
\left(-i\omega^*\tau_{1_0} I - \int_{-1}^{0} e^{-i\omega^*\tau_{1_0}\theta} d\eta(\theta)\right)\overline{q}(0) &= 0,
\end{aligned} \tag{3.35}$$

and substituting (3.27) and (3.32) into (3.30), we have

$$\left(2i\omega^*\tau_{1_0} I - \int_{-1}^{0} e^{2i\omega^*\tau_{1_0}\theta} d\eta(\theta)\right)E_1 = 2\tau_{1_0}(H_1, H_2)^T. \tag{3.36}$$

That is

$$\begin{pmatrix} 2i\omega^* - m_3 e^{-2i\omega^*\tau_{1_0}} & -m_2 \\ -n_2 e^{-2i\omega^*\tau_2^*} & 2i\omega^* - n_1 e^{-2i\omega^*\tau_{1_0}} - n_3 e^{-2i\omega^*\tau_2^*} \end{pmatrix} E_1 = 2(H_1, H_2)^T. \tag{3.37}$$

It follows that

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \qquad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \tag{3.38}$$

where

$$\begin{aligned}\Delta_1 &= \left(2i\omega^* - m_3 e^{-2i\omega^*\tau_{1_0}}\right)\left(2i\omega^* - n_1 e^{-2i\omega^*\tau_{1_0}} - n_3 e^{-2i\omega^*\tau_2^*}\right) - m_2 n_2 e^{-2i\omega^*\tau_2^*},\\ \Delta_{11} &= 2H_1\left(2i\omega^* - n_1 e^{-2i\omega^*\tau_{1_0}} - n_3 e^{-2i\omega^*\tau_2^*}\right) + 2H_2 m_2,\\ \Delta_{12} &= 2H_2\left(2i\omega^* - m_3 e^{-2i\omega^*\tau_{1_0}}\right) + 2H_1 n_2 e^{-2i\omega^*\tau_2^*}.\end{aligned} \tag{3.39}$$

Similarly, substituting (3.28) and (3.33) into (3.31), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right) E_2 = 2\tau_{1_0}(P_1, P_2)^T. \tag{3.40}$$

That is

$$\begin{pmatrix} m_1 + m_3 & m_2 \\ n_2 & n_1 + n_3 \end{pmatrix} E_2 = 2(-P_1, -P_2)^T. \tag{3.41}$$

It follows that

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \qquad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \tag{3.42}$$

where

$$\begin{aligned}\Delta_2 &= (m_1 + m_3)(n_1 + n_3) - m_2 n_2,\\ \Delta_{21} &= 2m_2 P_2 - 2P_1(n_1 + n_3),\\ \Delta_{22} &= 2n_2 P_1 - 2P_2(m_1 + m_3).\end{aligned} \tag{3.43}$$

From (3.27), (3.29), (3.38), (3.42), we can calculate g_{21} and derive the following values:

$$\begin{aligned}c_1(0) &= \frac{i}{2\omega^*\tau_{1_0}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_{1_0})\}},\\ \beta_2 &= 2\operatorname{Re}(c_1(0)),\\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_{1_0})\}}{\omega^*\tau_{1_0}}.\end{aligned} \tag{3.44}$$

These formulas give a description of the Hopf bifurcation periodic solutions of (3.1) at $\tau = \tau_{1_0}$ on the center manifold. From the discussion above, we have the following result.

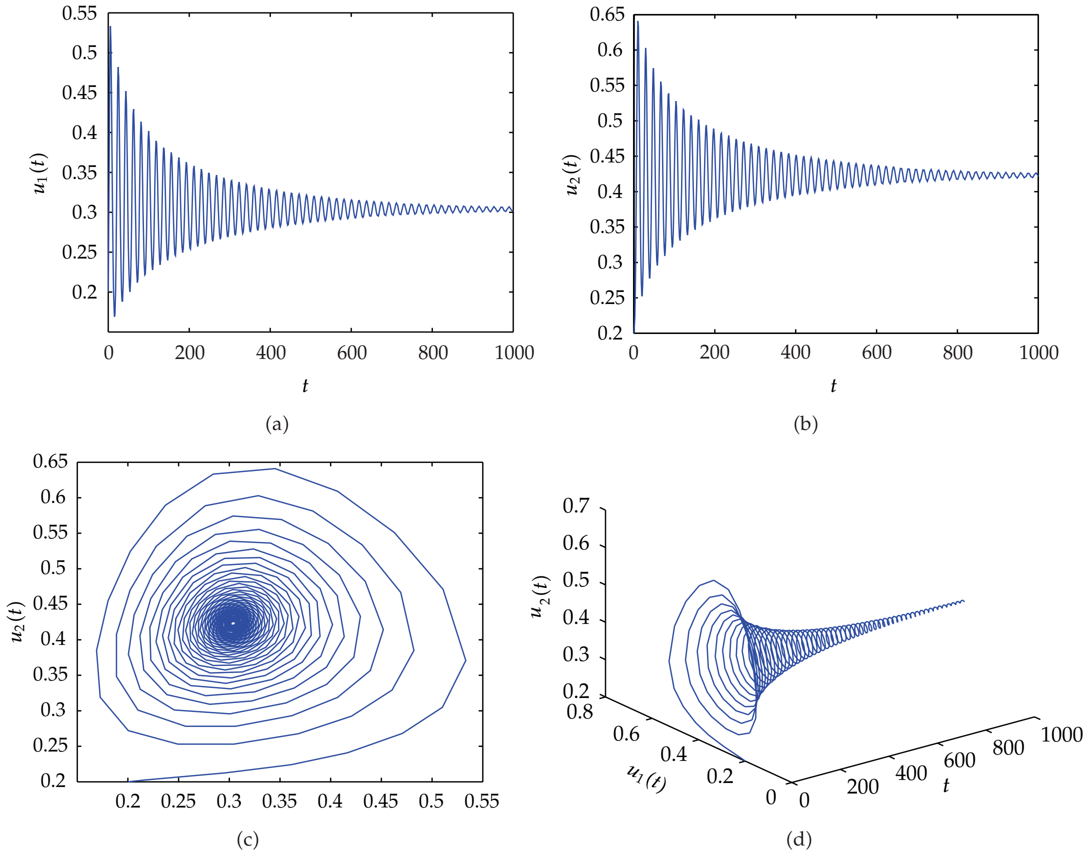


Figure 1: Trajectory portrait and phase portrait of system (4.1) with $\tau_1 = 0$, $\tau_2 = 1.5 < \tau_{2_0} \approx 1.52$. The positive equilibrium $E_0(0.3038, 0.4230)$ is asymptotically stable. The initial value is (0.2,0.2).

Theorem 3.1. *The periodic solution is forward (backward) if $\mu_2 > 0(\mu_2 < 0)$; the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0(\beta_2 > 0)$; the periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0(T_2 < 0)$.*

4. Numerical Examples

In this section, we present some numerical results of system (1.1) to verify the analytical predictions obtained in the previous section. From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following system:

$$\begin{aligned}
\dot{u}_1(t) &= u_1(t)[1 - u_1(t-\tau_1)] - \frac{u_1(t)u_2(t)}{0.05 + u_1(t) + 0.6u_2(t)}, \\
\dot{u}_2(t) &= u_2(t)\left[-0.5 + \frac{u_1(t-\tau_2)}{0.05 + u_1(t-\tau_2) + 0.6u_2(t-\tau_2)}\right],
\end{aligned} \tag{4.1}$$

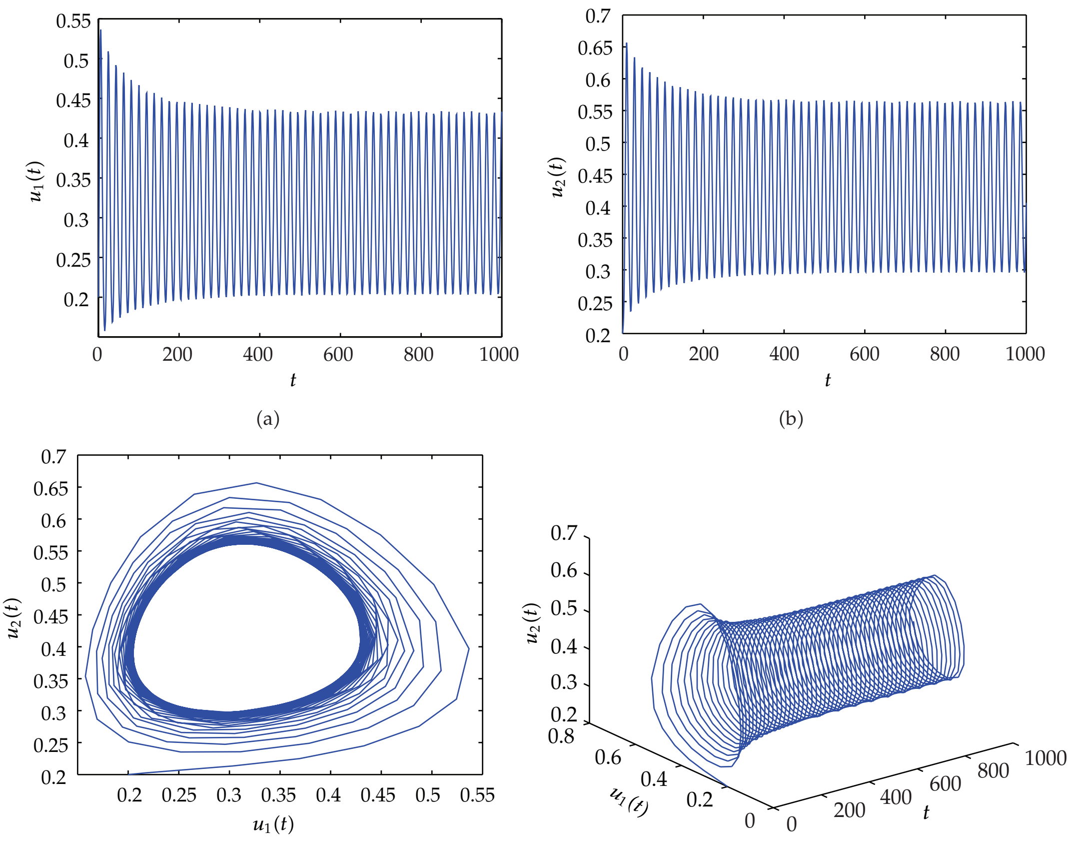


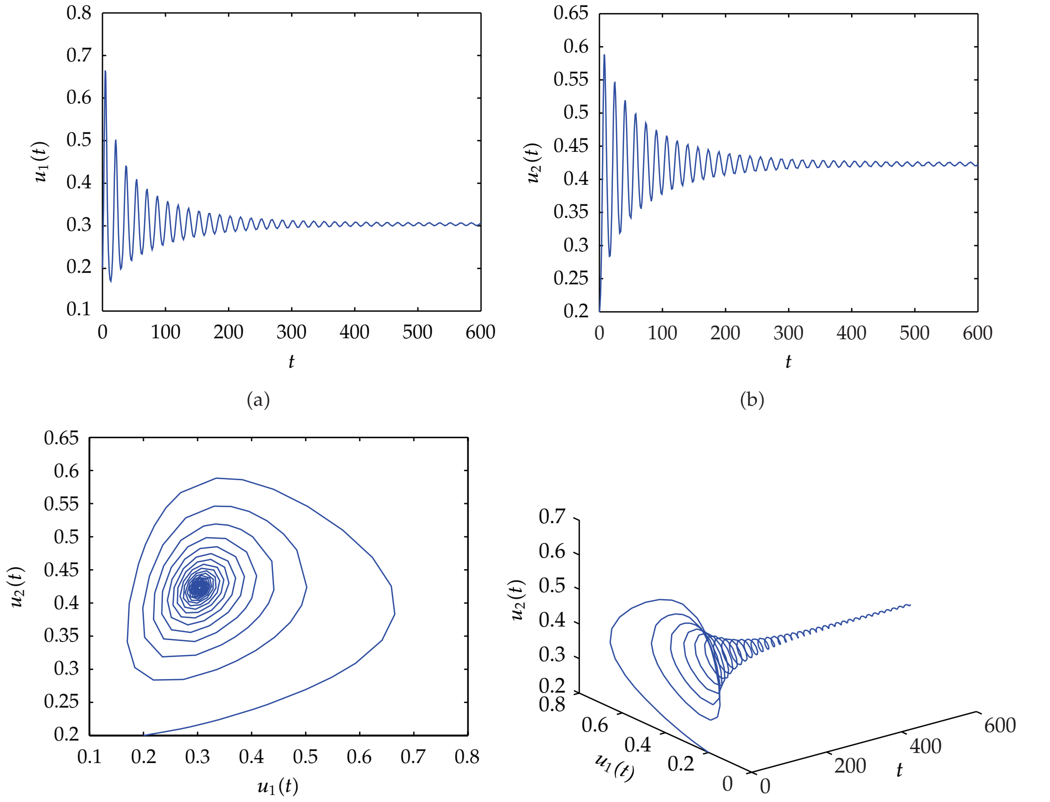
Figure 2: Trajectory portrait and phase portrait of system (4.1) with $\tau_1 = 0, \tau_2 = 1.6 > \tau_{2_0} \approx 1.52$. Hopf bifurcation occurs from the positive equilibrium $E_0(0.3038, 0.4230)$. The initial value is (0.2,0.2).

which has a positive equilibrium $E_0(u_1^*, u_2^*) \approx (0.3038, 0.4230)$. When $\tau_1 = 0$, then we can easily obtain that (H1) and (H3) are satisfied. Take $n = 0$, for example, by some computation by means of Matlab 7.0, we get $\omega_0 \approx 0.1342, \tau_{2_0} \approx 1.52$. From Lemma 2.2, we know that the transversal condition is satisfied. Thus, the positive equilibrium $E_0 \approx (0.3038, 0.4230)$ is asymptotically stable for $\tau_2 < \tau_{2_0} \approx 1.52$ and unstable for $\tau_2 > \tau_{2_0} \approx 1.52$ which is shown in Figure 1. When $\tau_2 = \tau_{2_0} \approx 1.52$, (4.1) undergoes a Hopf bifurcation at the positive equilibrium $E_0 \approx (0.3038, 0.4230)$, that is, a small amplitude periodic solution occurs around $E_0 \approx (0.3038, 0.4230)$ when $\tau_1 = 0$ and τ_2 is close to $\tau_{2_0} = 1.52$ which is shown in Figure 2.

Let $\tau_2 = 1.5 \in (0, 1.52)$ and choose τ_1 as a parameter. We have $\tau_{1_0} \approx 1.2930$. Then, the positive equilibrium is asymptotically when $\tau_1 \in [0, \tau_{1_0})$. The Hopf bifurcation value of (4.1) is $\tau_{1_0} \approx 1.2930$. By the algorithm derived in Section 3, we can obtain

$$\begin{gathered} \lambda'(\tau_{1_0}) \approx 0.5018 - 7.2021i, \qquad c_1(0) \approx -2.0231 - 3.3225i, \\ \mu_2 \approx 0.2646, \qquad \beta_2 \approx -4.2342, \qquad T_2 \approx 8.3701. \end{gathered} \tag{4.2}$$

Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the positive equilibrium $E_0 \approx (0.3038, 0.4230)$ is stable when $\tau_1 < \tau_{1_0}$ as is illustrated by the computer simulations (see Figure 3).



(a) (b)

(c) (d)

Figure 3: Trajectory portrait and phase portrait of system (4.1) with $\tau_2 = 0.5, \tau_1 = 1.2 < \tau_{1_0} \approx 1.2930$. The positive equilibrium $E_0(0.3038, 0.4230)$ is asymptotically stable. The initial value is (0.2,0.2).

When τ_1 passes through the critical value τ_{1_0}, the positive equilibrium $E_0 \approx (0.3038, 0.4230)$ loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcations from the positive equilibrium $E_0 \approx (0.3038, 0.4230)$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau_1 > \tau_{1_0}$, and these bifurcating periodic solutions from $E_0 \approx (0.3038, 0.4230)$ at τ_{1_0} are stable, which are depicted in Figure 4.

5. Conclusions

In this paper, we have investigated local stability of the positive equilibrium $E_0(u_1^*, u_2^*)$ and local Hopf bifurcation of a Beddington-DeAngelis functional response predator-prey model with two delays. We have showed that, if one of the conditions (a), (b), (c), and (d) holds and (H1), (H3), and (H5) are satisfied, and $\tau_2 \in [0, \tau_{2_0})$, then the positive equilibrium $E_0(u_1^*, u_2^*)$ is asymptotically stable when $\tau_1 \in (0, \tau_{1_0})$, as the delay τ_1 increases, the positive equilibrium $E_0(u_1^*, u_2^*)$ loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium $E_0(u_1^*, u_2^*)$, that is, a family of periodic orbits bifurcates from the the positive equilibrium $E_0(u_1^*, u_2^*)$. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold

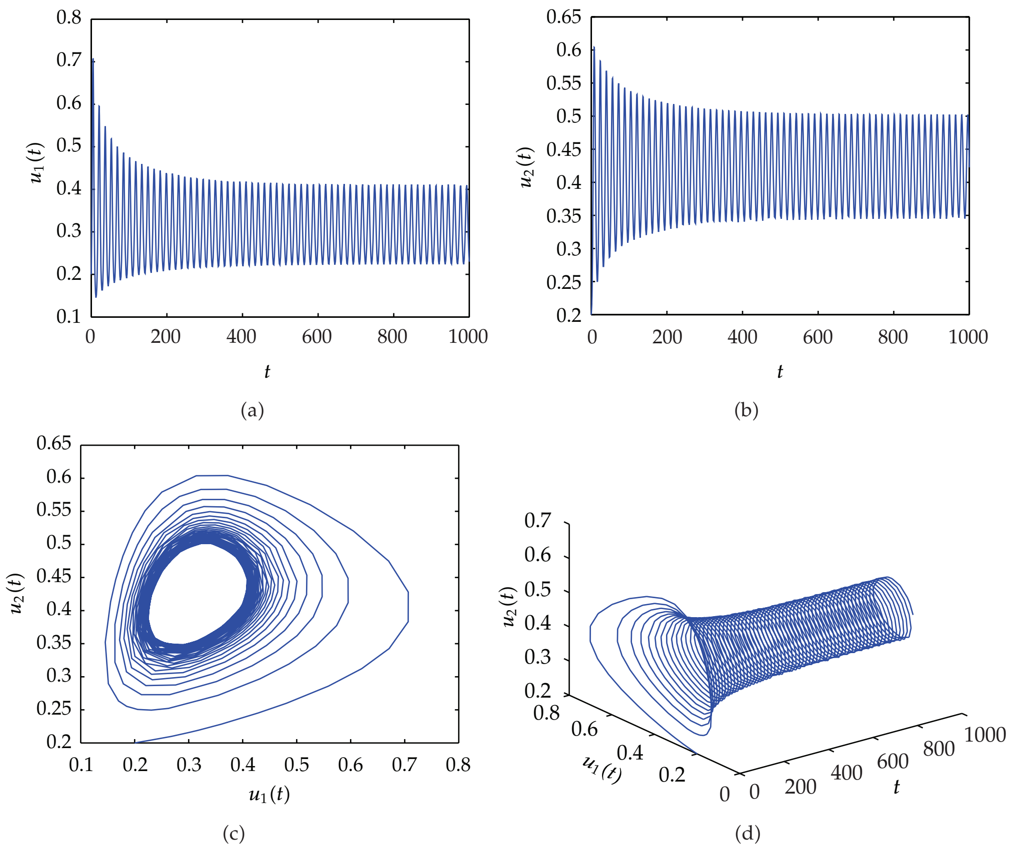


Figure 4: Trajectory portrait and phase portrait of system (4.1) with $\tau_2 = 0.5, \tau_1 = 1.35 > \tau_{1_0} \approx 1.2930$. Hopf bifurcation occurs from the positive equilibrium $E_* = (0.32, 0.96)$. The initial value is (0.2,0.2).

theorem. A numerical example verifying our theoretical results is carried out. In addition, we must point out that, although Gakkhar et al. [4] have also investigated the existence of Hopf bifurcation for system (1.1) with respect to positive equilibrium $E_0(u_1^*, u_2^*)$, it is assumed that $\tau_1 = \tau_2$. For $\tau_1 \neq \tau_2$, only numerical simulations are carried out to discuss the existence of Hopf bifurcation. In this paper, under the case $\tau_1 \neq \tau_2$, we investigate the existence of Hopf bifurcation quantitatively. Our work generalizes the known results of Sunita Gakkhar et al. [4]. Similarly, we can investigate the Hopf bifurcation of system (1.2) by choosing the delay τ_2 as bifurcation parameter. We will further investigate the topic elsewhere in the near future.

Appendices

A.

The expressions of m_i, n_i $(i = 1, 2, 3)$, and F_i $(i = 1, 2)$ are as follows:

$$m_1 = 1 - u_1^* - \frac{a_1}{a + u_1^* + bu_2^*}\left(u_2^* - \frac{u_1^* u_2^*}{a + u_1^* + bu_2^*}\right),$$

$$m_2 = -\frac{a_1}{a + u_1^* + bu_2^*}\left(u_1^* - \frac{u_1^* u_2^*}{a + u_1^* + bu_2^*}\right), \quad m_3 = -u_1^*,$$

$$n_1 = -dd_1 + \frac{d_1 u_1^*}{a + u_1^* + bu_2^*}, \qquad n_2 = \frac{d_1 u_1^* u_2^*}{a + u_1^* + bu_2^*}, \quad n_3 = -\frac{d_1 b u_1^* u_2^*}{\left(a + u_1^* + bu_2^*\right)^2},$$

$$\begin{aligned}
F_1 &= -u_1(t)u_1(t-\tau_1) + l_1 u_1^2(t) + l_2 u_2^2(t) + l_3 u_1(t)u_2(t) + l_4 u_1^3(t) \\
&\quad + l_5 u_2^3(t) + l_6 u_1^2(t)u_2(t) + l_7 u_1(t)u_2^2(t) + \text{h.o.t.}, \\
F_2 &= k_1 u_1(t-\tau_2)u_2(t) + k_2 u_2(t)u_2(t-\tau_2) + k_3 u_1^2(t-\tau_2) \\
&\quad + k_4 u_1(t-\tau_2)u_2(t-\tau_2) + e_1 u_1^2(t-\tau_2)u_2(t) + e_2 u_1(t-\tau_2)u_2(t-\tau_2)u_2(t) \\
&\quad + e_3 u_1^3(t-\tau_2) + e_4 u_1(t-\tau_2)u_2^2(t-\tau_2) + e_5 u_2^3(t-\tau_2) + \text{h.o.t.},
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
l_1 &= -\frac{a_1}{a + u_1^* + bu_2^*}\left[\frac{-u_2^*}{a + u_1^* + bu_2^*} + \frac{u_1^* u_2^*}{\left(a + u_1^* + bu_2^*\right)^2}\right], \\
l_2 &= -\frac{a_1}{a + u_1^* + bu_2^*}\left[\frac{-bu_1^*}{a + u_1^* + bu_2^*} + \frac{b^2 u_1^* u_2^*}{\left(a + u_1^* + bu_2^*\right)^2}\right], \\
l_3 &= -\frac{a_1}{a + u_1^* + bu_2^*}\left[1 - \frac{u_1^* + bu_2^*}{a + u_1^* + bu_2^*} + \frac{2bu_1^* u_2^*}{\left(a + u_1^* + bu_2^*\right)^2}\right], \\
l_4 &= -\frac{a_1}{a + u_1^* + bu_2^*}\left[\frac{u_2^*}{\left(a + u_1^* + \mathrm{b}u_2^*\right)^2} - \frac{u_1^* u_2^*}{\left(a + u_1^* + bu_2^*\right)^3}\right], \\
l_5 &= -\frac{a_1}{a + u_1^* + bu_2^*}\left[\frac{b^2 u_1^*}{\left(a + u_1^* + bu_2^*\right)^2} - \frac{b^3}{\left(a + u_1^* + bu_2^*\right)^3}\right], \\
l_6 &= -\frac{a_1}{a + u_1^* + bu_2^*}\left[\frac{-1}{a + u_1^* + bu_2^*} + \frac{u_1^* + 2bu_2^*}{\left(a + u_1^* + bu_2^*\right)^2} - \frac{3bu_1^* u_2^*}{\left(a + u_1^* + bu_2^*\right)^3}\right], \\
l_7 &= -\frac{a_1}{a + u_1^* + bu_2^*}\left[\frac{-b}{a + u_1^* + bu_2^*} + \frac{b^2 u_2^* + 2\mathrm{b}u_1^*}{\left(a + u_1^* + bu_2^*\right)^2} - \frac{3b^2 u_1^* u_2^*}{\left(a + u_1^* + bu_2^*\right)^3}\right], \\
k_1 &= d_1\left[\frac{1}{a + u_1^* + bu_2^*} - \frac{u_1^*}{\left(a + u_1^* + bu_2^*\right)^2}\right], \quad k_2 = -\frac{d_1 b}{\left(a + u_1^* + bu_2^*\right)^2}, \\
k_3 &= d_1 u_2^*\left[-\frac{1}{\left(a + u_1^* + bu_2^*\right)^2} + \frac{u_1^*}{\left(a + u_1^* + bu_2^*\right)^3}\right], \\
k_4 &= d_1 u_2^*\left[-\frac{b}{\left(a + u_1^* + bu_2^*\right)^2} + \frac{2bu_1^*}{\left(a + u_1^* + bu_2^*\right)^3}\right], \\
e_1 &= d_1 u_2^*\left[\frac{1}{\left(a + u_1^* + bu_2^*\right)^3} - \frac{u_1^*}{\left(a + u_1^* + bu_2^*\right)^4}\right],
\end{aligned}$$

$$e_2 = d_1\left[-\frac{b}{\left(a+u_1^*+bu_2^*\right)^2}+\frac{2bu_1^*}{\left(a+u_1^*+bu_2^*\right)^3}\right],$$
$$e_3 = d_1u_2^*\left[\frac{1}{\left(a+u_1^*+bu_2^*\right)^3}-\frac{u_1^*}{\left(a+u_1^*+bu_2^*\right)^4}\right],$$
$$e_4 = \frac{d_1u_2^*b^2}{\left(a+u_1^*+bu_2^*\right)^3},\quad e_5 = -\frac{d_1u_1^*u_2^*b^3}{\left(a+u_1^*+bu_2^*\right)^4}. \tag{A.2}$$

B.

The expressions of $g(z,\overline{z})$, g_{20}, g_{11}, g_{02}, and g_{21} are as follows:

$$\begin{aligned}
g(z,\overline{z}) &= \overline{q}^*(0)F_0(z,\overline{z}) = F(0,u_t)\\
&= \overline{M}\tau_{1_0}\Big[-e^{-i\omega^*\tau_{1_0}} + l_1 + l_2\alpha^2 + l_3\alpha + \overline{\alpha}^*\\
&\qquad\qquad \times\left(k_1\alpha e^{-i\omega^*\tau_2^*} + k_2\alpha^2 e^{-i\omega^*\tau_2^*} + k_3 e^{-2i\omega^*\tau_2^*} + k_4\alpha e^{-2i\omega^*\tau_2^*}\right)\Big]z^2\\
&\quad + \overline{M}\tau_{1_0}\Big\{-e^{i\omega^*\tau_{1_0}} - e^{-i\omega^*\tau_{1_0}} + 2l_1 + 2l_2|\alpha|^2 + 2l_3\operatorname{Re}\{\alpha\} + \overline{\alpha}^*\\
&\qquad\qquad \times\Big[2k_1\operatorname{Re}\left\{\alpha e^{i\omega^*\tau_2^*}\right\} + k_2|\alpha|^2\left(e^{i\omega^*\tau_2^*} + e^{-i\omega^*\tau_2^*}\right) + 2k_3 + 2k_4\operatorname{Re}\{\alpha\}\Big]\Big\}z\overline{z}\\
&\quad + \overline{M}\tau_{1_0}\Big\{-e^{i\omega^*\tau_{1_0}} + l_1 + l_2\overline{\alpha}^2 + l_3\overline{\alpha} + \overline{\alpha}^*\\
&\qquad\qquad \times\Big[k_1\overline{\alpha}e^{i\omega^*\tau_2^*} + k_2|\alpha|^2 e^{i\omega^*\tau_2^*} + k_3 e^{2i\omega^*\tau_2^*} + k_4\overline{\alpha}e^{i\omega^*\tau_2^*}\Big]\Big\}\overline{z}^2\\
&\quad + \overline{M}\tau_{1_0}\Big\{-\left(\frac{1}{2}W_{20}^{(1)}(0)e^{i\omega^*\tau_{1_0}} + \frac{1}{2}W_{20}^{(1)}(0) + W_{11}^{(1)}(0)e^{-i\omega^*\tau_{1_0}} + W_{11}^{(1)}(0)\right)\\
&\qquad\qquad + l_1\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) + l_2\left(\overline{\alpha}W_{20}^{(2)}(0) + 2\alpha W_{11}^{(2)}(0)\right)\\
&\qquad\qquad + l_3\left(W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}\overline{\alpha}W_{20}^{(1)}(0) + \alpha W_{11}^{(1)}(0)\right)\\
&\qquad\qquad + 3l_4 + 3l_5\alpha^2\overline{\alpha} + l_6(2\alpha+\overline{\alpha}) + l_7\left(2|\alpha|^2 + \alpha^2\right) + \overline{\alpha}^*\\
&\qquad\qquad \times\Big[k_1\left(W_{11}^{(2)}(0)e^{-i\omega^*\tau_2^*} + \frac{1}{2}\overline{\alpha}W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + \frac{1}{2}W_{20}^{(2)}(0)e^{i\omega^*\tau_2^*} + W_{11}^{(2)}(0)e^{-i\omega^*\tau_2^*}\right)\\
&\qquad\qquad\quad + k_2\left(\alpha W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + \frac{1}{2}\overline{\alpha}W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right) + \frac{1}{2}\overline{\alpha}W_{20}^{(2)}(0)e^{i\omega^*\tau_2^*} + \alpha W_{11}^{(2)}(0)e^{-i\omega^*\tau_2^*}\right)\\
&\qquad\qquad\quad + k_3\left(2W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{-i\omega^*\tau_2^*} + W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{i\omega^*\tau_2^*}\right)
\end{aligned}$$

$$
\begin{aligned}
&+k_4\Bigg(W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{-i\omega^*\tau_2^*}+\frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{i\omega^*\tau_2^*}\\
&\qquad+\frac{1}{2}\overline{\alpha}W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{i\omega^*\tau_2^*}+\alpha W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{-i\omega^*\tau_2^*}\Bigg)+e_1\left(2\alpha+\overline{\alpha}e^{-2i\omega^*\tau_2^*}\right)\\
&+e_2\left(\alpha^2+|\alpha|^2+|\alpha|^2e^{-2i\omega^*\tau_2^*}\right)+3e_3e^{-i\omega^*\tau_2^*}+e_4\left(\alpha^2e^{-i\omega^*\tau_2^*}+2|\alpha|^2e^{-i\omega^*\tau_2^*}\right)\\
&+e_5\left(\alpha^2e^{-i\omega^*\tau_2^*}+2|\alpha|^2\overline{\alpha}e^{-i\omega^*\tau_2^*}\right)\Bigg]\Bigg\}z^2\overline{z}+\text{h.o.t.},
\end{aligned}
$$

$$
\begin{aligned}
g_{20}&=2\overline{M}\tau_{1_0}\Big[-e^{-i\omega^*\tau_{1_0}}+l_1+l_2\alpha^2+l_3\alpha+\overline{\alpha}^*\\
&\qquad\times\left(k_1\alpha e^{-i\omega^*\tau_2^*}+k_2\alpha^2e^{-i\omega^*\tau_2^*}+k_3e^{-2i\omega^*\tau_2^*}+k_4\alpha e^{-2i\omega^*\tau_2^*}\right)\Big],\\
g_{11}&=\overline{M}\tau_{1_0}\Big\{-e^{i\omega^*\tau_{1_0}}-e^{-i\omega^*\tau_{1_0}}+2l_1+2l_2|\alpha|^2+2l_3\operatorname{Re}\{\alpha\}\\
&\qquad+\overline{\alpha}^*\Big[2k_1\operatorname{Re}\left\{\alpha e^{i\omega^*\tau_2^*}\right\}+k_2|\alpha|^2\left(e^{i\omega^*\tau_2^*}+e^{-i\omega^*\tau_2^*}\right)+2k_3+2k_4\operatorname{Re}\{\alpha\}\Big]\Big\},\\
g_{02}&=2\overline{M}\tau_{1_0}\Big\{-e^{i\omega^*\tau_{1_0}}+l_1+l_2\overline{\alpha}^2+l_3\overline{\alpha}\\
&\qquad+\overline{\alpha}^*\Big[k_1\overline{\alpha}e^{i\omega^*\tau_2^*}+k_2|\alpha|^2e^{i\omega^*\tau_2^*}+k_3e^{2i\omega^*\tau_2^*}+k_4\overline{\alpha}e^{i\omega^*\tau_2^*}\Big]\Big\},\\
g_{21}&=2\overline{M}\tau_{1_0}\Bigg\{-\left[\frac{1}{2}W_{20}^{(1)}(0)e^{i\omega^*\tau_{1_0}}+\frac{1}{2}W_{20}^{(1)}(0)+W_{11}^{(1)}(0)e^{-i\omega^*\tau_{1_0}}+W_{11}^{(1)}(0)\right]\\
&\qquad+l_1\left[W_{20}^{(1)}(0)+2W_{11}^{(1)}(0)\right]+l_2\left[\overline{\alpha}W_{20}^{(2)}(0)+2\alpha W_{11}^{(2)}(0)\right]\\
&\qquad+l_3\left[W_{11}^{(2)}(0)+\frac{1}{2}W_{20}^{(2)}(0)+\frac{1}{2}\overline{\alpha}W_{20}^{(1)}(0)+\alpha W_{11}^{(1)}(0)\right]\\
&\qquad+3l_4+3l_5\alpha^2\overline{\alpha}+l_6(2\alpha+\overline{\alpha})+l_7\left(2|\alpha|^2+\alpha^2\right)\\
&\qquad+\overline{\alpha}^*\Bigg[k_1\Bigg(W_{11}^{(2)}(0)e^{-i\omega^*\tau_2^*}+\frac{1}{2}\overline{\alpha}W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)+\frac{1}{2}W_{20}^{(2)}(0)e^{i\omega^*\tau_2^*}+W_{11}^{(2)}(0)e^{-i\omega^*\tau_2^*}\Bigg)\\
&\qquad\quad+k_2\Bigg(\alpha W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)+\frac{1}{2}\overline{\alpha}W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)+\frac{1}{2}\overline{\alpha}W_{20}^{(2)}(0)e^{i\omega^*\tau_2^*}+\alpha W_{11}^{(2)}(0)e^{-i\omega^*\tau_2^*}\Bigg)\\
&\qquad\quad+k_3\Bigg(2W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{-i\omega^*\tau_2^*}+W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{i\omega^*\tau_2^*}\Bigg)\\
&\qquad\quad+k_4\Bigg(W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{-i\omega^*\tau_2^*}+\frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{i\omega^*\tau_2^*}+\frac{1}{2}\overline{\alpha}W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{i\omega^*\tau_2^*}\\
&\qquad\qquad+\alpha W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)e^{-i\omega^*\tau_2^*}\Bigg)\\
&\qquad\quad+e_1\left(2\alpha+\overline{\alpha}e^{-2i\omega^*\tau_2^*}\right)+e_2\left(\alpha^2+|\alpha|^2+|\alpha|^2e^{-2i\omega^*\tau_2^*}\right)+3e_3e^{-i\omega^*\tau_2^*}\\
&\qquad\quad+e_4\left(\alpha^2e^{-i\omega^*\tau_2^*}+2|\alpha|^2e^{-i\omega^*\tau_2^*}\right)+e_5\left(\alpha^2e^{-i\omega^*\tau_2^*}+2|\alpha|^2\overline{\alpha}e^{-i\omega^*\tau_2^*}\right)\Bigg]\Bigg\}.
\end{aligned}
\tag{B.1}
$$

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