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Research Article

Necessary and Sufficient Condition for Stability of Generalized Expectation Value

Aziz El Kaabouchi¹ and Sumiyoshi Abe^{1,2}

¹ Institut Supérieur des Matériaux et Mécaniques Avancés, 44 F. A. Bartholdi, 72000 Le Mans, France

Correspondence should be addressed to Sumiyoshi Abe, suabe@sf6.so-net.ne.jp

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A class of generalized definitions of expectation value is often employed in nonequilibrium statistical mechanics for complex systems. Here, the necessary and sufficient condition is presented for such a class to be stable under small deformations of a given arbitrary probability distribution.

Given a probability distribution $\{p_i\}_{i=1,2,\dots,W}$, that is, $0 \le p_i \le 1$ $(i=1,2,\dots,W)$ and $\sum_{i=1}^W p_i = 1$, the ordinary expectation value of a quantity Q of a system under consideration is defined by $\sum_{i=1}^W p_i Q_i$, where W is the total number of accessible states and is enormously large in statistical mechanics, typically being $2^{10^{23}}$. In the field of generalized statistical mechanics for complex systems, on the other hand, discussions are often made about altering this definition. Among others, the so-called "escort average" is widely employed in the field of generalized statistical mechanics [1-3]. It is defined as follows:

$$\langle Q \rangle_{\phi} [p] = \sum_{i=1}^{W} P_i^{(\phi)} Q_i, \tag{1}$$

where $P_i^{(\phi)}$ stands for the escort probability distribution [4] given by

$$P_i^{(\phi)} = \frac{\phi(p_i)}{\sum_{j=1}^{W} \phi(p_j)},$$
 (2)

with a nonnegative function ϕ . In the special case when $\phi(x) = x$, $\langle Q \rangle_{\phi}$ is reduced to the ordinary expectation value mentioned above.

² Department of Physical Engineering, Mie University, Mie 514-8507, Japan

Consider measurements of a certain quantity of a system to obtain information about the probability distribution. Repeated measurements should be performed on the system, which is identically prepared each time. Suppose that two probability distributions, $\{p_i\}_{i=1,2,\dots,W}$ and $\{p_i'\}_{i=1,2,\dots,W}$, are obtained through the measurements. They may slightly be different from each other, in general. If such measurements make sense, then the expectation values, $\langle Q \rangle[p]$ and $\langle Q \rangle[p']$, calculated from these two distributions should also be close to each other. This condition, which implies "experimental robustness," is represented as follows.

Definition (stability). An expectation value $\langle Q \rangle[p]$ is said to be stable, if the following predicate holds for any pair of probability distributions, $\{p_i\}_{i=1,2,...,W}$ and $\{p_i'\}_{i=1,2,...,W}$:

$$(\forall \varepsilon > 0) \ (\exists \delta > 0) \ (\forall W) \quad (\|p - p'\|_1 < \delta \Longrightarrow |\langle Q \rangle [p] - \langle Q \rangle [p']| < \varepsilon). \tag{3}$$

Here, $||p - p'||_1 = \sum_{i=1}^W |p_i - p'_i|$ is the l^1 -norm describing the distance between these two probability distributions. One might consider norms of other kinds, but what is physically relevant to discrete systems is the present l^1 -norm [5]. Equation (3) is analogous to Lesche's stability condition on entropic functionals [5], which has recently been revisited in the literature [6–11] (note that the discussion in [8] is corrected in [9]). This concept of stability is actually equivalent to that of uniform continuity.

In recent papers [12, 13], it has been shown that the generalized expectation value in (1) with a specific class, $\phi(x) = x^q$ (q > 0), (the associated expectation value being termed the q-expectation value), is not stable unless q = 1. This result needs the q-expectation-value formalism of nonextensive statistical mechanics [1, 2] be reconsidered. In addition, the result is supported by Boltzmann-like kinetic theory in an independent manner [14].

Here, it seems appropriate to make some comments on the latest situation of the problems concerning stabilities of entropic functionals and generalized expectation values. The authors of [15, 16] have presented discussions which aim to rescue the q-expectation values from the difficulties of their instability pointed out in [12]. Those authors insist that the q-expectation values can be stable in both the finite-W and continuous cases. Such possibilities are, however, fully refuted by the work in [13] both physically and mathematically, and the controversy seems to have been terminated with that work. The case of the continuous variables has further been carefully examined in a recent paper [17], where the so-called Tsallis q-entropies [1, 2] do not have the continuous limit in consistency with the physical principles such as the thermodynamic laws (see also [18, 19]). These controversies have led the researchers to give up the traditional form of nonextensive statistical mechanics based on the q-entropies and q-expectation values and to examine other entropic functionals combined with the ordinary definition of expectation values [20] (see also [21, 22]). Thus, it seems that nonextensive statistical mechanics has to be fully reexamined, theoretically.

In this paper, we present the necessary and sufficient condition for $\langle Q \rangle_{\phi}[p]$ in (1) to be stable.

Our main result is as follows.

Theorem. Let ϕ be nonnegative and continuous on [0,1], differentiable on (0,1), and satisfy the condition that $\phi(x) = 0 \Leftrightarrow x = 0$. And, let $Q = \{Q_i\}_{i=1,2,\dots,W}$ be a random variable. Then, $\langle Q \rangle_{\phi}[p]$ in (1) is stable, if and only if $\lim_{x\to +0} \phi(x)/x \in (0,\infty)$.

Proof. First, assume that $\lim_{x\to+0}\phi(x)/x=a>0$. Then, there exists $\delta_1>0$ such that

$$a - \frac{a}{2} < \frac{\phi(x)}{x} < a + \frac{a}{2} \quad (\forall x \in (0, \delta_1]). \tag{4}$$

 $\phi(x)/x$ does not vanish because of the condition $\phi(x) = 0 \Leftrightarrow x = 0$. Therefore, there exists b > 0 such that

$$\frac{\phi(x)}{x} \ge b \quad (\forall x \in (\delta_1, 1]). \tag{5}$$

Putting $c = \min\{a/2, b\}$ we have

$$cx \le \phi(x) \quad (\forall x \in [0,1]).$$
 (6)

Consequently, for an arbitrarily large W and an arbitrary probability distribution $\{p_i\}_{i=1,2,\dots,W}$, we obtain

$$\frac{1}{\sum_{i=1}^{W} \phi(p_i)} \le c. \tag{7}$$

From the mean value theorem, it follows that

$$\left|\phi(p_i) - \phi(p_i')\right| \le \left|p_i - p_i'\right| \cdot \sup_{x \in (0,1)} \left|\phi'(x)\right|,\tag{8}$$

where $\phi'(x)$ is the derivative of $\phi(x)$ with respect to x. For $\varepsilon > 0$, we put

$$\delta = \inf \left(\delta_1, \frac{c\varepsilon}{2|Q_{\text{max}}| \cdot \left(\sup_{x \in (0,1)} |\phi'(x)| \right)} \right), \tag{9}$$

where $Q_{\max} = \max \{Q_i\}_{i=1,2,\dots,W}$. Now, for $\|p - p'\|_1 < \delta$, we have

$$\begin{split} \left| \langle Q \rangle_{\phi} [p] - \langle Q \rangle_{\phi} [p'] \right| \\ &= \frac{1}{\sum_{i=1}^{W} \phi(p_{i}) \sum_{j=1}^{W} \phi(p'_{j})} \left| \sum_{i=1}^{W} Q_{i} \left\{ \phi(p_{i}) \sum_{j=1}^{W} \phi(p'_{j}) - \phi(p'_{i}) \sum_{j=1}^{W} \phi(p_{j}) \right\} \right| \\ &\leq \frac{1}{\sum_{i=1}^{W} \phi(p_{i}) \sum_{j=1}^{W} \phi(p'_{j})} \\ &\times \left[\sum_{i=1}^{W} |Q_{i}| \left\{ \left| \phi(p_{i}) - \phi(p'_{i}) \right| \sum_{j=1}^{W} \phi(p'_{j}) + \phi(p'_{i}) \right| \sum_{j=1}^{W} \phi(p_{j}) - \sum_{j=1}^{W} \phi(p'_{j}) \right| \right\} \right] \end{split}$$

$$+ \frac{\sum_{j=1}^{W} \left| \phi(p_{j}) - \phi(p'_{j}) \right|}{\sum_{i=1}^{W} \phi(p_{i}) \sum_{j=1}^{W} \phi(p'_{j})} \sum_{i=1}^{W} |Q_{i}| \phi(p'_{i})$$

$$\leq \frac{2|Q_{\max}|}{\sum_{j=1}^{W} \phi(p_{j})} \sum_{i=1}^{W} |\phi(p_{i}) - \phi(p'_{i})|$$

$$\leq \frac{2|Q_{\max}|}{\sum_{j=1}^{W} \phi(p_{j})} ||p - p'||_{1} \cdot \sup_{x \in (0,1)} |\phi'(x)|$$

$$\leq \frac{2|Q_{\max}|}{c} ||p - p'||_{1} \cdot \sup_{x \in (0,1)} |\phi'(x)|$$

$$\leq \frac{2|Q_{\max}|}{c} ||p - p'||_{1} \cdot \sup_{x \in (0,1)} |\phi'(x)|$$

$$< \varepsilon.$$
(10)

Therefore, $\langle Q \rangle_{\phi}[p]$ is stable.

On the other hand, suppose that $\lim_{x\to +0}\phi(x)/x \notin (0,\infty)$. That is, (i) $\lim_{x\to +0}\phi(x)/x = 0$ or (ii) $\lim_{x\to +0}\phi(x)/x = \infty$. Below, we will examine these cases separately.

(i) Consider the following deformation:

 $\leq \frac{1}{\sum_{i=1}^{W} \phi(p_i)} \sum_{i=1}^{W} |Q_i| \left| \phi(p_i) - \phi(p_i') \right|$

$$p_{i} = \frac{1}{W-1}(1-\delta_{i1}),$$

$$p'_{i} = \left(1-\frac{\delta}{2}\right)p_{i} + \frac{\delta}{2}\delta_{i1},$$
(11)

which are normalized and satisfy $\|p - p'\|_1 = \delta$. We have

$$\sum_{i=1}^{W} \phi(p_i) = (W-1)\phi\left(\frac{1}{W-1}\right),$$

$$\sum_{i=1}^{W} \phi(p_i') = \phi\left(\frac{\delta}{2}\right) + (W-1)\phi\left(\frac{1}{W-1}\left(1 - \frac{\delta}{2}\right)\right).$$
(12)

Difference of the expectation values is calculated as follows:

$$\begin{split} \langle Q \rangle_{\phi} \big[p \big] - \langle Q \rangle_{\phi} \big[p' \big] \\ &= - \frac{Q_1 \phi(\delta/2)}{\phi(\delta/2) + (W-1)\phi((1/(W-1))(1-\delta/2))} \\ &+ \left(\sum_{i=2}^W Q_i \right) \Big\{ \frac{1}{W-1} - \frac{\phi((1/(W-1))(1-\delta/2))}{\phi(\delta/2) + (W-1)\phi((1/(W-1))(1-\delta/2))} \Big\} \end{split}$$

$$= \frac{W}{W-1} (\overline{Q} - Q_1)$$

$$\times \frac{\phi(\delta/2)/(1-\delta/2)}{\phi(\delta/2)/(1-\delta/2) + \phi((1/(W-1))(1-\delta/2))/[(1/(W-1))(1-\delta/2)]}$$

$$\xrightarrow{W \to \infty} \overline{Q} - Q_1,$$
(13)

since $\lim_{x\to +0}\phi(x)/x=0$, where \overline{Q} is the arithmetic mean, $\overline{Q}=\sum_{i=1}^WQ_i/W$. Therefore, $\langle Q\rangle_{\phi}[p]$ is not stable.

(ii) Consider the following deformation:

$$p_{i} = \delta_{i1},$$

$$p'_{i} = \left(1 - \frac{\delta}{2} \frac{W}{W - 1}\right) p_{i} + \frac{\delta}{2} \frac{1}{W - 1},$$
(14)

which are also normalized and satisfy $\|p - p'\|_1 = \delta$. We have

$$\sum_{i=1}^{W} \phi(p_i) = \phi(1),$$

$$\sum_{i=1}^{W} \phi(p_i') = \phi\left(1 - \frac{\delta}{2}\right) + (W - 1)\phi\left(\frac{\delta}{2} \frac{1}{W - 1}\right).$$
(15)

Difference of the expectation values is calculated as follows:

$$\langle Q \rangle_{\phi}[p] - \langle Q \rangle_{\phi}[p'] = Q_{1} \left\{ 1 - \frac{\phi(1 - \delta/2)}{\phi(1 - \delta/2) + (W - 1)\phi((\delta/2)(1/(W - 1)))} \right\}$$

$$- \left(\sum_{i=2}^{W} Q_{i} \right) \frac{\phi((\delta/2)(1/(W - 1)))}{\phi(1 - \delta/2) + (W - 1)\phi((\delta/2)(1/(W - 1)))}$$

$$= \frac{W}{W - 1} \left(Q_{1} - \overline{Q} \right)$$

$$\times \frac{\phi((\delta/2)(1/(W - 1)))/[(\delta/2)(1/(W - 1))]}{\phi(1 - \delta/2)/(\delta/2) + \phi((\delta/2)(1/(W - 1)))/[(\delta/2)(1/(W - 1))]}$$

$$\xrightarrow{W \to \infty} Q_{1} - \overline{Q}, \tag{16}$$

since $\lim_{x\to +0} \phi(x)/x = \infty$. Therefore, $\langle Q \rangle_{\phi}[p]$ is not stable.

In the above proof, we have employed the specific deformations of the probability distributions as the counterexamples, which are considered in [5]. It is pointed out in [13] that these deformed distributions may experimentally be generated.

Finally, we mention a couple of simple stable examples.

Example 1.

$$\phi(x) = e^x - 1. \tag{17}$$

Example 2.

$$\phi(x) = \ln(1 + x^{\alpha}),\tag{18}$$

which yields a stable generalized expectation value, if and only if $\alpha = 1$.

On the other hand, as mentioned earlier, the *q*-expectation value is not stable, since $\phi(x) = x^q \ (q > 0, \ q \neq 1)$ does not satisfy the condition $\lim_{x \to +0} \phi(x)/x \in (0, \infty)$.

In conclusion, we have considered a class of generalized definitions of expectation value that are often employed in nonequilibrium statistical mechanics for complex systems, and have presented the necessary and sufficient condition for such a class to be stable under small deformations of a given arbitrary probability distribution.

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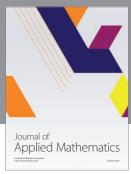
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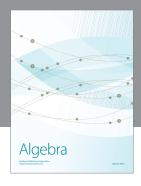
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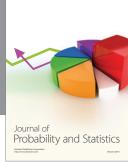
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