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Research Article

Necessary and Sufficient Condition for Stability of Generalized Expectation Value

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A class of generalized definitions of expectation value is often employed in nonequilibrium statistical mechanics for complex systems. Here, the necessary and sufficient condition is presented for such a class to be stable under small deformations of a given arbitrary probability distribution.

Given a probability distribution $\{p_i\}_{i=1,2,\ldots,W}$, that is, $0 \le p_i \le 1$ $(i = 1, 2, \ldots, W)$ and $\sum_{i=1}^{W} p_i = 1$, the ordinary expectation value of a quantity Q of a system under consideration is defined by $\sum_{i=1}^{W} p_i Q_i$, where W is the total number of accessible states and is enormously large in statistical mechanics, typically being $2^{10^{23}}$. In the field of generalized statistical mechanics for complex systems, on the other hand, discussions are often made about altering this definition. Among others, the so-called "escort average" is widely employed in the field of generalized statistical mechanics [1–3]. It is defined as follows:

$$\langle Q \rangle_{\phi}[p] = \sum_{i=1}^{W} P_i^{(\phi)} Q_i, \tag{1}$$

where $P_i^{(\phi)}$ stands for the escort probability distribution [4] given by

$$P_i^{(\phi)} = \frac{\phi(p_i)}{\sum_{j=1}^{W} \phi(p_j)}, \tag{2}$$

with a nonnegative function ϕ. In the special case when $\phi(x) = x$, $\langle Q \rangle_{\phi}$ is reduced to the ordinary expectation value mentioned above.

Consider measurements of a certain quantity of a system to obtain information about the probability distribution. Repeated measurements should be performed on the system, which is identically prepared each time. Suppose that two probability distributions, $\{p_i\}_{i=1,2,\ldots,W}$ and $\{p'_i\}_{i=1,2,\ldots,W}$, are obtained through the measurements. They may slightly be different from each other, in general. If such measurements make sense, then the expectation values, $\langle Q\rangle[p]$ and $\langle Q\rangle[p']$, calculated from these two distributions should also be close to each other. This condition, which implies "experimental robustness," is represented as follows.

Definition (stability). An expectation value $\langle Q\rangle[p]$ is said to be stable, if the following predicate holds for any pair of probability distributions, $\{p_i\}_{i=1,2,\ldots,W}$ and $\{p'_i\}_{i=1,2,\ldots,W}$:

$$(\forall\varepsilon > 0)\ (\exists\delta > 0)\ (\forall W)\quad \left(\|p - p'\|_1 < \delta \Longrightarrow \left|\langle Q\rangle[p] - \langle Q\rangle[p']\right| < \varepsilon\right). \tag{3}$$

Here, $\|p - p'\|_1 = \sum_{i=1}^{W} |p_i - p'_i|$ is the l^1-norm describing the distance between these two probability distributions. One might consider norms of other kinds, but what is physically relevant to discrete systems is the present l^1-norm [5]. Equation (3) is analogous to Lesche's stability condition on entropic functionals [5], which has recently been revisited in the literature [6–11] (note that the discussion in [8] is corrected in [9]). This concept of stability is actually equivalent to that of uniform continuity.

In recent papers [12, 13], it has been shown that the generalized expectation value in (1) with a specific class, $\phi(x) = x^q$ $(q > 0)$, (the associated expectation value being termed the q-expectation value), is not stable unless $q = 1$. This result needs the q-expectation-value formalism of nonextensive statistical mechanics [1, 2] be reconsidered. In addition, the result is supported by Boltzmann-like kinetic theory in an independent manner [14].

Here, it seems appropriate to make some comments on the latest situation of the problems concerning stabilities of entropic functionals and generalized expectation values. The authors of [15, 16] have presented discussions which aim to rescue the q-expectation values from the difficulties of their instability pointed out in [12]. Those authors insist that the q-expectation values can be stable in both the finite-W and continuous cases. Such possibilities are, however, fully refuted by the work in [13] both physically and mathematically, and the controversy seems to have been terminated with that work. The case of the continuous variables has further been carefully examined in a recent paper [17], where the so-called Tsallis q-entropies [1, 2] do not have the continuous limit in consistency with the physical principles such as the thermodynamic laws (see also [18, 19]). These controversies have led the researchers to give up the traditional form of nonextensive statistical mechanics based on the q-entropies and q-expectation values and to examine other entropic functionals combined with the ordinary definition of expectation values [20] (see also [21, 22]). Thus, it seems that nonextensive statistical mechanics has to be fully reexamined, theoretically.

In this paper, we present the necessary and sufficient condition for $\langle Q\rangle_\phi[p]$ in (1) to be stable.

Our main result is as follows.

Theorem. *Let ϕ be nonnegative and continuous on $[0,1]$, differentiable on $(0,1)$, and satisfy the condition that $\phi(x) = 0 \Leftrightarrow x = 0$. And, let $Q = \{Q_i\}_{i=1,2,\ldots,W}$ be a random variable. Then, $\langle Q\rangle_\phi[p]$ in* (1) *is stable, if and only if* $\lim_{x\to+0}\phi(x)/x \in (0,\infty)$.

Proof. First, assume that $\lim_{x\to+0}\phi(x)/x = a > 0$. Then, there exists $\delta_1 > 0$ such that

$$a - \frac{a}{2} < \frac{\phi(x)}{x} < a + \frac{a}{2} \quad (\forall x \in (0, \delta_1]). \tag{4}$$

$\phi(x)/x$ does not vanish because of the condition $\phi(x) = 0 \Leftrightarrow x = 0$. Therefore, there exists $b > 0$ such that

$$\frac{\phi(x)}{x} \geq b \quad (\forall x \in (\delta_1, 1]). \tag{5}$$

Putting $c = \min\{a/2, b\}$ we have

$$cx \leq \phi(x) \quad (\forall x \in [0, 1]). \tag{6}$$

Consequently, for an arbitrarily large W and an arbitrary probability distribution $\{p_i\}_{i=1,2,\ldots,W}$, we obtain

$$\frac{1}{\sum_{i=1}^{W} \phi(p_i)} \leq c. \tag{7}$$

From the mean value theorem, it follows that

$$\left|\phi(p_i) - \phi(p_i')\right| \leq \left|p_i - p_i'\right| \cdot \sup_{x\in(0,1)} \left|\phi'(x)\right|, \tag{8}$$

where $\phi'(x)$ is the derivative of $\phi(x)$ with respect to x. For $\varepsilon > 0$, we put

$$\delta = \inf\left(\delta_1, \frac{c\varepsilon}{2|Q_{\max}| \cdot \left(\sup_{x\in(0,1)}\left|\phi'(x)\right|\right)}\right), \tag{9}$$

where $Q_{\max} = \max\{Q_i\}_{i=1,2,\ldots,W}$. Now, for $\|p - p'\|_1 < \delta$, we have

$$\begin{aligned}
&\left|\langle Q\rangle_\phi[p] - \langle Q\rangle_\phi[p']\right| \\
&\quad = \frac{1}{\sum_{i=1}^{W}\phi(p_i)\sum_{j=1}^{W}\phi\left(p_j'\right)}\left|\sum_{i=1}^{W} Q_i\left\{\phi(p_i)\sum_{j=1}^{W}\phi\left(p_j'\right) - \phi(p_i')\sum_{j=1}^{W}\phi(p_j)\right\}\right| \\
&\quad \leq \frac{1}{\sum_{i=1}^{W}\phi(p_i)\sum_{j=1}^{W}\phi\left(p_j'\right)} \\
&\qquad \times \left[\sum_{i=1}^{W}|Q_i|\left\{\left|\phi(p_i) - \phi(p_i')\right|\sum_{j=1}^{W}\phi\left(p_j'\right) + \phi(p_i')\left|\sum_{j=1}^{W}\phi\left(p_j\right) - \sum_{j=1}^{W}\phi\left(p_j'\right)\right|\right\}\right]
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{\sum_{j=1}^{W}\phi(p_j)}\sum_{i=1}^{W}|Q_i|\left|\phi(p_i)-\phi(p_i')\right| \\
&\quad + \frac{\sum_{j=1}^{W}\left|\phi(p_j)-\phi\left(p_j'\right)\right|}{\sum_{i=1}^{W}\phi(p_i)\sum_{j=1}^{W}\phi\left(p_j'\right)}\sum_{i=1}^{W}|Q_i|\phi(p_i') \\
&\le \frac{2|Q_{\max}|}{\sum_{j=1}^{W}\phi(p_j)}\sum_{i=1}^{W}\left|\phi(p_i)-\phi(p_i')\right| \\
&\le \frac{2|Q_{\max}|}{\sum_{j=1}^{W}\phi(p_j)}\left\|p-p'\right\|_1\cdot\sup_{x\in(0,1)}\left|\phi'(x)\right| \\
&\le \frac{2|Q_{\max}|}{c}\left\|p-p'\right\|_1\cdot\sup_{x\in(0,1)}\left|\phi'(x)\right| \\
&< \varepsilon.
\end{aligned} \tag{10}$$

Therefore, $\langle Q\rangle_\phi[p]$ is stable.

On the other hand, suppose that $\lim_{x\to+0}\phi(x)/x\notin(0,\infty)$. That is, (i) $\lim_{x\to+0}\phi(x)/x=0$ or (ii) $\lim_{x\to+0}\phi(x)/x=\infty$. Below, we will examine these cases separately.

(i) Consider the following deformation:

$$\begin{aligned}
p_i &= \frac{1}{W-1}(1-\delta_{i1}), \\
p_i' &= \left(1-\frac{\delta}{2}\right)p_i+\frac{\delta}{2}\delta_{i1},
\end{aligned} \tag{11}$$

which are normalized and satisfy $\|p-p'\|_1=\delta$. We have

$$\begin{aligned}
\sum_{i=1}^{W}\phi(p_i) &= (W-1)\phi\left(\frac{1}{W-1}\right), \\
\sum_{i=1}^{W}\phi(p_i') &= \phi\left(\frac{\delta}{2}\right)+(W-1)\phi\left(\frac{1}{W-1}\left(1-\frac{\delta}{2}\right)\right).
\end{aligned} \tag{12}$$

Difference of the expectation values is calculated as follows:

$$\begin{aligned}
&\langle Q\rangle_\phi[p]-\langle Q\rangle_\phi[p'] \\
&\quad = -\frac{Q_1\phi(\delta/2)}{\phi(\delta/2)+(W-1)\phi((1/(W-1))(1-\delta/2))} \\
&\qquad + \left(\sum_{i=2}^{W}Q_i\right)\left\{\frac{1}{W-1}-\frac{\phi((1/(W-1))(1-\delta/2))}{\phi(\delta/2)+(W-1)\phi((1/(W-1))(1-\delta/2))}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{W}{W-1}\left(\overline{Q}-Q_1\right)\\
&\quad\times \frac{\phi(\delta/2)/(1-\delta/2)}{\phi(\delta/2)/(1-\delta/2)+\phi((1/(W-1))(1-\delta/2))/[(1/(W-1))(1-\delta/2)]}\\
&\xrightarrow{W\to\infty} \overline{Q}-Q_1,
\end{aligned}
\tag{13}
$$

since $\lim_{x\to+0}\phi(x)/x = 0$, where $\overline{Q}$ is the arithmetic mean, $\overline{Q} = \sum_{i=1}^{W} Q_i/W$. Therefore, $\langle Q\rangle_\phi[p]$ is not stable.

(ii) Consider the following deformation:

$$
\begin{aligned}
p_i &= \delta_{i1},\\
p_i' &= \left(1-\frac{\delta}{2}\frac{W}{W-1}\right)p_i+\frac{\delta}{2}\frac{1}{W-1},
\end{aligned}
\tag{14}
$$

which are also normalized and satisfy $\|p-p'\|_1 = \delta$. We have

$$
\begin{aligned}
\sum_{i=1}^{W}\phi(p_i) &= \phi(1),\\
\sum_{i=1}^{W}\phi(p_i') &= \phi\left(1-\frac{\delta}{2}\right)+(W-1)\phi\left(\frac{\delta}{2}\frac{1}{W-1}\right).
\end{aligned}
\tag{15}
$$

Difference of the expectation values is calculated as follows:

$$
\begin{aligned}
\langle Q\rangle_\phi[p]-\langle Q\rangle_\phi[p'] &= Q_1\left\{1-\frac{\phi(1-\delta/2)}{\phi(1-\delta/2)+(W-1)\phi((\delta/2)(1/(W-1)))}\right\}\\
&\quad-\left(\sum_{i=2}^{W}Q_i\right)\frac{\phi((\delta/2)(1/(W-1)))}{\phi(1-\delta/2)+(W-1)\phi((\delta/2)(1/(W-1)))}\\
&= \frac{W}{W-1}\left(Q_1-\overline{Q}\right)\\
&\quad\times\frac{\phi((\delta/2)(1/(W-1)))/[(\delta/2)(1/(W-1))]}{\phi(1-\delta/2)/(\delta/2)+\phi((\delta/2)(1/(W-1)))/[(\delta/2)(1/(W-1))]}\\
&\xrightarrow{W\to\infty} Q_1-\overline{Q},
\end{aligned}
\tag{16}
$$

since $\lim_{x\to+0}\phi(x)/x = \infty$. Therefore, $\langle Q\rangle_\phi[p]$ is not stable. □

In the above proof, we have employed the specific deformations of the probability distributions as the counterexamples, which are considered in [5]. It is pointed out in [13] that these deformed distributions may experimentally be generated.

Finally, we mention a couple of simple stable examples.

Example 1.

$$\phi(x) = e^x - 1. \tag{17}$$

Example 2.

$$\phi(x) = \ln(1 + x^\alpha), \tag{18}$$

which yields a stable generalized expectation value, if and only if $\alpha = 1$.

On the other hand, as mentioned earlier, the q-expectation value is not stable, since $\phi(x) = x^q$ $(q > 0,\ q \neq 1)$ does not satisfy the condition $\lim_{x \to +0} \phi(x)/x \in (0, \infty)$.

In conclusion, we have considered a class of generalized definitions of expectation value that are often employed in nonequilibrium statistical mechanics for complex systems, and have presented the necessary and sufficient condition for such a class to be stable under small deformations of a given arbitrary probability distribution.

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