Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 380784, 14 pages
doi:10.1155/2011/380784

Research Article

Contractive Mapping in Generalized, Ordered Metric Spaces with Application in Integral Equations

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Received 22 June 2011; Revised 1 October 2011; Accepted 3 October 2011

Academic Editor: Cristian Toma

We consider the concept of Ω-distance on a complete, partially ordered G-metric space and prove some fixed point theorems. Then, we present some applications in integral equations of our obtained results.

1. Introduction

The Banach fixed point theorem for contraction mapping has been generalized and extended in many directions [1–11]. Nieto and Rodríguez-López [10], Ran and Reurings [12], and Petrusel and Rus [13] presented some new results for contractions in partially ordered metric spaces. The main idea in [10, 12, 14] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. Also, Mustafa and Sims [15] introduced the concept of G-metric. Some authors [16, 17] have proved some fixed point theorems in these spaces. Recently, Saadati et al. [18], using the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of w-distance due to Kada et al. [19].

In this paper, we extend some recent fixed point theorems by using this concept and prove various fixed point theorems in generalized partially ordered G-metric spaces.

At first we recall some definitions and lemmas. For more information see [15–18, 20–23].

Definition 1 (see [15]). Let X be a nonempty set. A function $G : X \times X \times X \to [0, \infty)$ is called a G-metric if the following conditions are satisfied:

(i) $G(x, y, z) = 0$ if $x = y = z$ (coincidence),

(ii) $G(x, x, y) > 0$ for all $x, y \in X$, where $x \neq y$,

(iii) $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,

(iv) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2. Let $(X,\ G)$ be a G-metric space,

(1) a sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(2) a sequence $\{x_n\}$ in X is said to be G-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(x_m, x_n, x) < \varepsilon$.

Definition 3 (see [15]). Let $(X,\ G)$ be a G-metric space. Then a function $\Omega : X \times X \times X \to [0, \infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$,

(b) for any $x, y \in X$, $\Omega(x, y, \cdot), \Omega(x, \cdot, y) : X \to [0, \infty)$ are lower semicontinuous,

(c) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \varepsilon$.

Example 1 (see [18]). Let (X, d) be a metric space and $G : X^3 \to [0, \infty)$ defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \tag{1.1}$$

for all $x, y, z \in X$. Then $\Omega = G$ is an Ω-distance on X.

Example 2 (see [18]). In $X = \mathbb{R}$ we consider the G-metric G defined by

$$G(x, y, z) = \frac{1}{3}\left(|x - y| + |y - z| + |x - z|\right), \tag{1.2}$$

for all $x, y, z \in \mathbb{R}$. Then $\Omega : \mathbb{R}^3 \to [0, \infty)$ defined by

$$\Omega(x, y, z) = \frac{1}{3}\left(|z - x| + |x - y|\right), \tag{1.3}$$

for all $x, y, z \in \mathbb{R}$ is an Ω-distance on $\mathbb{R}$.

For more example see [18].

Lemma 1.1 (see [18]). *Let X be a metric space with metric G and Ω be an Ω-distance on X. Let x_n, y_n be sequences in X, α_n, β_n be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then one has the following.*

(1) *If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$.*

(2) *If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$ then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$.*

(3) *If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then x_n is a G-Cauchy sequence.*

(4) *If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$ then x_n is a G-Cauchy sequence.*

Definition 4 (see [18]). G-metric space X is said to be Ω-bounded if there is a constant $M > 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, z \in X$.

2. Fixed Point Theorems on Partially Ordered G-Metric Spaces

Definition 5. Suppose $(X, \leq)$ is a partially ordered space and $T : X \to X$ is a mapping of X into itself. We say that T is nondecreasing if for $x, y \in X$,

$$x \leq y \Longrightarrow T(x) \leq T(y). \tag{2.1}$$

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X such that X is Ω-bounded. Let $f : X \to X$ and $g : X \to X$ weakly compatible and f, g be non-decreasing mapping such that*

(a) *$g(X) \subseteq f(X)$;*

(b) *$\Omega(gx, gy, gz) \leq k \max\{\Omega(fx, fy, fz), \Omega(fx, gx, fz), \Omega(fy, gy, fz), \Omega(fx, gy, fz), \Omega(fy, gx, fz)\}$; for all $x, y, z \in X$ and $0 \leq k < 1$,*

(c) *for every $x \in X$ and $y \in X$ with $f(y) \neq g(y)$, $\inf\{\Omega(fx, y, fx) + \Omega(fx, y, gx) + \Omega(fx, gx, y) : f(x) \leq g(x)\} > 0$;*

(d) *there exist $x_0 \in X$ that $f(x_0) \leq g(x_0)$; then f and g have a unique common fixed point u in X and $\Omega(u, u, u) = 0$.*

Proof. Let $x_0 \in X$ that $f(x_0) \leq g(x_0)$. By part (a), we can choose $x_1 \in X$ such that $f(x_1) = g(x_0)$. Again from part (a), we can choose $x_2 \in X$ such that $f(x_2) = g(x_1)$. Continuing this process we can construct sequences $\{x_n\}$ in X such that,

$$\begin{gathered} y_n = gx_n = fx_{n+1}, \quad \forall\, n \geq 0, \\ x_n \leq x_{n+1}. \end{gathered} \tag{2.2}$$

Now, since g is non-decreasing mapping then,

$$gx_n \leq gx_{n+1}, \quad \forall\, n \geq 0, \tag{2.3}$$

so, for all $s \geq 0$,

$$\begin{aligned}
\Omega(y_n, y_{n+1}, y_{n+s}) &= \Omega(gx_n, gx_{n+1}, gx_{n+s}) \\
&\leq k \max\{\Omega(fx_n, fx_{n+1}, fx_{n+s}), \Omega(fx_n, gx_n, fx_{n+s}), \Omega(fx_{n+1}, gx_{n+1}, fx_{n+s}), \\
&\qquad \Omega(fx_n, gx_{n+1}, fx_{n+s}), \Omega(fx_{n+1}, gx_n, fx_{n+s})\} \\
&= k \max\{\Omega(y_{n-1}, y_n, y_{n+s-1}), \Omega(y_{n-1}, y_n, y_{n+s-1}), \Omega(y_n, y_{n+1}, y_{n+s-1}), \\
&\qquad \Omega(y_{n-1}, y_{n+1}, y_{n+s-1}), \Omega(y_n, y_n, y_{n+s-1})\}.
\end{aligned} \tag{2.4}$$

Then,

$$\begin{aligned}
\Omega(y_n, y_{n+1}, y_{n+s}) \leq k \max\{&\Omega(y_{n-1}, y_n, y_{n+s-1}), \Omega(y_n, y_{n+1}, y_{n+s-1}), \\
&\Omega(y_{n-1}, y_{n+1}, y_{n+s-1}), \Omega(y_n, y_n, y_{n+s-1})\}.
\end{aligned} \tag{2.5}$$

Now since,

$$\begin{aligned}
\Omega(y_{n-1}, y_{n+1}, y_{n+s-1}) &\leq k \max\{\Omega(y_{n-2}, y_n, y_{n+s-2}), \Omega(y_{n-2}, y_{n-1}, y_{n+s-2}), \Omega(y_n, y_{n+1}, y_{n+s-2}), \\
&\qquad \Omega(y_{n-2}, y_{n+1}, y_{n+s-2}), \Omega(y_n, y_{n-1}, y_{n+s-2})\} \\
\Omega(y_n, y_n, y_{n+s-1}) &\leq k \max\{\Omega(y_{n-1}, y_{n-1}, y_{n+s-2}), \Omega(y_{n-1}, y_n, y_{n+s-2}), \Omega(y_{n-1}, y_n, y_{n+s-2}), \\
&\qquad \Omega(y_{n-1}, y_n, y_{n+s-2}), \Omega(y_{n-1}, y_n, y_{n+s-2})\},
\end{aligned} \tag{2.6}$$

thus,

$$\begin{aligned}
\Omega(y_n, y_{n+1}, y_{n+s}) &\leq k^2 \max\{\Omega(y_i, y_j, y_t), \quad n-2 \leq i \leq n, n-1 \leq j \leq n+1, n+s-2 \leq t \leq n+s-1\} \\
&\vdots \\
&\leq k^{n-1} \max\{\Omega(y_i, y_j, y_t); \quad 1 \leq i \leq n, 2 \leq j \leq n+1, s+1 \leq t \leq n+s-1\}.
\end{aligned} \tag{2.7}$$

So $\Omega(y_n, y_{n+1}, y_{n+s}) \leq k^{n-1} M_{n,s}$ where

$$M_{n,s} := \max\{\Omega(y_i, y_j, y_t), \quad 1 \leq i \leq n, 2 \leq j \leq n+1, s+1 \leq t \leq n+s-1\}. \tag{2.8}$$

Now, for any $l > m > n$ with $m = n + k$ and $l = m + t$ $(k, t \in \mathbb{N})$, we have,

$$\lim_{m,n,l \to \infty} \Omega(y_n, y_m, y_l) = 0. \tag{2.9}$$

Since X is Ω-bounded and

$$\begin{aligned}\Omega(y_n, y_m, y_l) &\leq \Omega(y_n, y_{n+1}, y_{n+1}) + \Omega(y_{n+1}, y_m, y_l) \\ &\leq \Omega(y_n, y_{n+1}, y_{n+1}) + \Omega(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + \Omega(y_{m-1}, y_m, y_l) \\ &\leq k^{n-1} M_{n,1} + k^n M_{n+1,2} + \cdots + k^{m-2} M_{m-1,t+1} \\ &\leq \sum_{j=1}^{n-m+2} k^{n-j} M \leq \frac{k^{n-1}}{1-k} M,\end{aligned} \tag{2.10}$$

so, by Part (3) of Lemma 1.1, $\{y_n\}$ is a G-Cauchy sequence. Since X is G-complete, $\{y_n\}$ converges to a point $y \in X$. Thus, for $\varepsilon > 0$ and by the lower semicontinuity of Ω, we have

$$\begin{aligned}\Omega(y_n, y_m, y) &\leq \liminf_{p\to\infty} \Omega(y_n, y_m, y_p) \leq \varepsilon, \quad m \geq n \\ \Omega(y_n, y, y_l) &\leq \liminf_{p\to\infty} \Omega(y_n, y_p, y_l) \leq \varepsilon, \quad l \geq n.\end{aligned} \tag{2.11}$$

Assume that $fy \neq gy$. Since,

$$y_n = f x_{n+1} = g x_n \leq g x_{n+1} = f x_{n+2} = y_{n+1}, \tag{2.12}$$

so, $y_n \leq y_{n+1}$, and,

$$0 < \inf\{\Omega(y_n, y, y_n) + \Omega(y_n, y_{n+1}, y) + \Omega(y_n, y, y_{n+1})\} \leq 3\varepsilon, \tag{2.13}$$

for every $\varepsilon > 0$, that is a contraction. So, we have $fy = gy$. Then, by (b),

$$\Omega(gy, gy, gy) \leq k\Omega(gy, gy, gy), \tag{2.14}$$

so, $\Omega(gy, gy, gy) = 0$. Similarly, $\Omega(g^2y, g^2y, gy) = 0$.
Now,

$$\begin{aligned}\Omega\left(gy, g^2y, gy\right) &\leq k \max\left\{\Omega\left(gy, g^2y, gy\right), \Omega\left(g^2y, gy, gy\right),\right. \\ &\qquad \left.\Omega\left(g^2y, g^2y, gy\right), \Omega(gy, gy, gy)\right\} \\ &= k \max\left\{\Omega\left(gy, g^2y, gy\right), \Omega\left(g^2y, gy, gy\right)\right\} \\ \Omega\left(g^2y, gy, gy\right) &\leq k \max\left\{\Omega\left(gy, g^2y, gy\right), \Omega\left(g^2y, gy, gy\right)\right\}.\end{aligned} \tag{2.15}$$

Thus,

$$\Omega\left(gy, g^2y, gy\right) = 0, \qquad \Omega\left(g^2y, gy, gy\right) = 0. \tag{2.16}$$

By Part (c) of Definition 3, $G(g^2y, g^2y, gy) = 0$ and consequently $g^2y = gy$ which implies that gy is a fixed point for g. Now,

$$f(gy) = g(fy) = g^2y = gy. \tag{2.17}$$

So, it is enough to put $gy = u$, then u is a common fixed point of f and g.

Uniqueness: Assume that there exist $v \in X$ such that $fv = gv = v$. Hence, we have,

$$\Omega(v, v, v) \leq k\Omega(v, v, v), \tag{2.18}$$

and so $\Omega(v, v, v) = 0$. Also, $\Omega(v, v, u) = 0$. On the other hand,

$$\begin{aligned} \Omega(v, u, u) &\leq k \max\{\Omega(v, u, u), \Omega(u, v, u)\}, \\ \Omega(u, v, u) &\leq k \max\{\Omega(u, v, u), \Omega(v, u, u)\}, \end{aligned} \tag{2.19}$$

which follows that, $\Omega(v, u, u) = \Omega(u, v, u) = 0$. Then by Part (c) of Definition 3, $u = v$ and $\Omega(u, u, u) = 0$. □

The following corollary is a generalization of [24, Theorem 2.1].

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a G-metric space and Ω is an Ω-distance on X such that X be Ω-bounded. Let $f : X \to X$ and $g : X \to X$ be weakly compatible and f, g be a non-decreasing mapping such that*

(a) *$g(X) \subseteq f(X)$ and either $f(X)$ or $g(X)$ is complete;*

(b) *for all $x, y, z \in X$ and $0 \leq k < 1$, $\Omega(gx, gy, gz) \leq k\Omega(fx, fy, fz)$;*

(c) *for every $x \in X$ and $y \in X$ with $f(y) \neq g(y)$, $\inf\{\Omega(fx, y, fx) + \Omega(fx, y, gx) + \Omega(fx, gx, y) : f(x) \leq g(x)\} > 0$;*

(d) *there exist $x_0 \in X$ that $f(x_0) \leq g(x_0)$;*

then f and g have a unique common fixed point y in X and $\Omega(y, y, y) = 0$.

Definition 6 (see [25]). Let Φ be the set of all functions φ such that $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\varphi(t) < t$ for all $t \in \mathbb{R}^+$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t \in \mathbb{R}^+$. The function φ is called a growth or control function of $T : X \to X$.

It is clear that

$$\lim_{n \to \infty} \varphi^n(t) = 0, \quad \forall t \in \mathbb{R}^+, \varphi^n(0) = 0. \tag{2.20}$$

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Let X be Ω-bounded. Suppose that $\varphi \in \Phi$ and*

$$\Omega\left(Tx, T^2x, Tw\right) \leq \varphi(\Omega(x, Tx, w)) \quad \forall x \leq Tx, \ w \in X. \tag{2.21}$$

Also, for every $x \in X$

$$\inf\Big\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega\Big(x,T^2x,y\Big) : x \le Tx\Big\} > 0, \tag{2.22}$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \le Tx_0$, then T has a unique fixed point. Moreover, if $v = Tv$, then $\Omega(v,v,v) = 0$.

Proof. If $x_0 = Tx_0$, then the proof is finished. Suppose that $Tx_0 \neq x_0$. since $x_0 \le Tx_0$ and T is non-decreasing, we obtain

$$x_0 \le Tx_0 \le T^2x_0 \le \cdots \le T^{n+1}x_0 \le \cdots \tag{2.23}$$

For all $n \in \mathbb{N}$ and $t \ge 0$,

$$\begin{aligned}\Omega\Big(T^n x_0, T^{n+1}x_0, T^{n+t}x_0\Big) &\le \varphi\Big(\Omega\Big(T^{n-1}x_0, T^n x_0, T^{n+t-1}x_0\Big)\Big)\\ &\le \varphi^2\Big(\Omega\Big(T^{n-2}x_0, T^{n-1}x_0, T^{n+t-2}x_0\Big)\Big)\\ &\;\;\vdots\\ &\le \varphi^n\left(\Omega\left(x_0, Tx_0, T^t x_0\right)\right).\end{aligned} \tag{2.24}$$

We claim that for $m = n + k$ and $l = m + t$ $(k, t \in \mathbb{N})$ with $l > m > n$,

$$\lim_{m,n,l\to\infty} \Omega\Big(T^n x_0, T^m x_0, T^l x_0\Big) = 0. \tag{2.25}$$

We prove by,

$$\begin{aligned}\Omega\Big(T^n x_0, T^m x_0, T^l x_0\Big) &\le \Omega\Big(T^n x_0, T^{n+1}x_0, T^{n+1}x_0\Big) + \Omega\Big(T^{n+1}x_0, T^m x_0, T^l x_0\Big)\\ &\le \Omega\Big(T^n x_0, T^{n+1}x_0, T^{n+1}x_0\Big) + \Omega\Big(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0\Big)\\ &\quad + \cdots + \Omega\Big(T^{m-1}x_0, T^m x_0, T^l x_0\Big)\\ &\le \varphi^n(\Omega(x_0, Tx_0, Tx_0)) + \varphi^{n+1}(\Omega(x_0, Tx_0, Tx_0))\\ &\quad + \cdots + \varphi^{m-2}(\Omega(x_0, Tx_0, Tx_0)) + \varphi^{m-1}\Big(\Omega\Big(x_0, Tx_0, T^{t+1}x_0\Big)\Big)\\ &\le \varphi^{n-1}(M)\left(\sum_{n=1}^{\infty}\varphi^n(M)\right).\end{aligned} \tag{2.26}$$

Since $\sum_{n=1}^{\infty}\varphi^n(M) < \infty$, so,

$$\lim_{m,n\to\infty} \Omega\Big(T^n x_0, T^m x_0, T^l x_0\Big) = 0. \tag{2.27}$$

By Part (c) of Lemma 1.1$\{T^n x_0\}$ is a G-Cauchy sequence. Since X is G-complete, $\{T^n x_0\}$ converges to a point $u \in X$. Let $n \in \mathbb{N}$ be fixed. By lower semicontinuity of Ω,

$$\begin{aligned}\Omega(T^n x_0, T^m x_0, u) &\leq \liminf_{p\to\infty} \Omega(T^n x_0, T^m x_0, T^p x_0) \leq \varepsilon, \quad m > n,\\ \Omega\left(T^n x_0, u, T^l x_0\right) &\leq \liminf_{p\to\infty} \Omega(T^n x_0, T^p x_0, T^m x_0) \leq \varepsilon, \quad l \geq n.\end{aligned} \tag{2.28}$$

Assume that $u \neq Tu$. Since $T^n x_0 \leq T^{n+1} x_0$,

$$0 < \inf\left\{\Omega(T^n x_0, u, T^n x_0) + \Omega\left(T^n x_0, u, T^{n+1} x_0\right) + \Omega\left(T^n x_0, T^{n+2} x_0, u\right) : n \in \mathbb{N}\right\} \leq 3\varepsilon, \tag{2.29}$$

for every $\varepsilon > 0$, which is a contraction. Therefore, we have $u = Tu$.

Uniqueness: let v be another fixed point of T, then

$$\Omega(u, u, v) = \Omega\left(Tu, T^2 u, Tv\right) \leq \varphi(\Omega(u, Tu, v)) < \Omega(u, u, v), \tag{2.30}$$

which is a contraction. Therefore, fixed point u is unique. Now, if $v = Tv$, we have,

$$\Omega(v, v, v) = \Omega\left(Tv, T^2 v, T^3 v\right) \leq \varphi\left(\Omega\left(v, Tv, T^2 v\right)\right) = \varphi(\Omega(v, v, v)). \tag{2.31}$$

So $\Omega(v, v, v) = 0$. □

Corollary 2.4. *Let the assumptions of Theorem 2.3 hold and*

$$\Omega\left(T^m x, T^{m+1} x, T^m w\right) \leq \varphi(\Omega(x, Tx, w)) \quad \forall m \in \mathbb{N},\ x \leq Tx,\ w \in X, \tag{2.32}$$

then T has a unique fixed point.

Proof. From Theorem 2.3, T^m has a unique fixed point u. However,

$$Tu = T(T^m u) = T^{m+1} u = T^m Tu, \tag{2.33}$$

so Tu is also a fixed point of T^m. Since the fixed point of T^m is unique, it must be the case that $Tu = u$. □

Corollary 2.5. *Let the assumptions of Theorem 2.3 hold and $T : X \to X$ satisfies,*

$$\Omega\left(Tx, T^2 x, Tx\right) \leq \varphi(\Omega(x, Tx, x)) \quad \forall x \leq Tx. \tag{2.34}$$

Then T has a unique fixed point.

Proof. Take $w = x$, and apply Theorem 2.3. □

Theorem 2.6. *Let $(X, \preceq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space, Ω is an Ω-distance on X, and T is a non-decreasing mapping from X into itself. Let X be Ω-bounded. Suppose that*

$$\Omega\left(Tx, T^2x, Tw\right) \leq k\left(\Omega\left(x, T^2x, Tw\right) + \Omega(x, Tx, Tx)\right), \tag{2.35}$$

where $x \preceq Tx, w \in X, k \in [0, 1/3)$. Also for every $x \in X$,

$$\inf\left\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega\left(x, T^2x, y\right) : x \preceq Tx\right\} > 0, \tag{2.36}$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a unique fixed point say u and $\Omega(u, u, u) = 0$.

Proof. Let $x_0 \in X$ be an arbitrary point, and define the sequence x_n by $x_n = T^n x_0$. By (2.35) and for all $t \geq 0$,

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq k(\Omega(x_{n-1}, x_{n+1}, x_{n+t}) + \Omega(x_{n-1}, x_n, x_n)). \tag{2.37}$$

But by Part (a) of Definition 3,

$$\Omega(x_{n-1}, x_{n+1}, x_{n+t}) \leq \Omega(x_{n-1}, x_n, x_n) + \Omega(x_n, x_{n+1}, x_{n+t}). \tag{2.38}$$

Hence,

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq k[2\,\Omega(x_{n-1}, x_n, x_n) + \Omega(x_n, x_{n+1}, x_{n+t})], \tag{2.39}$$

which implies,

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq \frac{2k}{1-k}\Omega(x_{n-1}, x_n, x_n). \tag{2.40}$$

Let $r = 2k/(1-k)$, then $r < 1$ and by repeated application of (2.40), we have

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq r^n \Omega(x_0, x_1, x_1). \tag{2.41}$$

Now, for any $l > m > n$ with $m = n + k$ and $l = m + t$ $(k, t \in \mathbb{N})$, we have,

$$\begin{aligned} \Omega(x_n, x_m, x_l) &\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \Omega(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \Omega(x_{m-1}, x_m, x_l) \\ &\leq \left(r^n + r^{n+1} + \cdots + r^{m-1}\right)\Omega(x_0, x_1, x_1) \\ &\leq \frac{r^n}{1-r}\Omega(x_0, x_1, x_1). \end{aligned} \tag{2.42}$$

So,

$$\lim_{m,n,l\to\infty} \Omega(x_n, x_m, x_l) = 0. \tag{2.43}$$

By Part (3) of Lemma 1.1, x_n is a G-Cauchy sequence. Since X is G-complete, x_n converges to a point $u \in X$. Now, similar to proving Theorem 2.1, T has a unique fixed point and $\Omega(u,u,u) = 0$. □

Corollary 2.7. *Let the assumptions of Theorem 2.6 hold and*

$$\Omega\left(T^m x, T^{m+2}x, T^m w\right) \le k\left(\Omega\left(x, T^{m+2}x, T^m w\right) + \Omega(x, T^m x, T^m x)\right) \tag{2.44}$$

where $k \in [0, 1/3)$, then T has a unique fixed point.

Proof. The argument is similar to that used in the proof of Corollary 2.4. □

3. Applications

In this section, we give an existence theorem for a solution of a class of integral equations. Denote by Λ the set of all functions $\lambda : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(i) λ is a Lebesgue-integrable mapping on each compact of $[0,+\infty)$,

(ii) for every $\epsilon > 0$, we have $\int_0^\epsilon \lambda(s)ds > 0$,

(iii) $\|\lambda\| < 1$, where $\|\lambda\|$ denotes to the norm of λ.

Now, we have the following results.

Theorem 3.1. *Let $(X, \le)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Let X be Ω-bounded. Suppose that*

$$\Omega\left(Tx, T^2x, Tw\right) \le \int_0^{\Omega(x,Tx,w)} \alpha(s)ds, \tag{3.1}$$

where $\alpha \in \Lambda$. Also, suppose that for every $x \in X$

$$\inf\left\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega\left(x, T^2x, y\right) : x \le Tx\right\} > 0, \tag{3.2}$$

for every $y \in X$ with $y \ne Ty$. If there exists an $x_0 \in X$ with $x_0 \le Tx_0$, then T has a unique fixed point.

Proof. Define $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = \int_0^t \alpha(s)ds$. It is clear that ϕ is nondecreasing and continuous. From (iii), we have

$$\phi(t) = |\phi(t)| = \left|\int_0^t \lambda(s)ds\right| \le \int_0^t |\lambda(s)|ds \le \|\lambda\|t < t. \tag{3.3}$$

Also, note that

$$\phi^2(t) = \phi(\phi(t)) \leq \|\lambda\|\phi(t) \leq \|\lambda\|^2 t. \tag{3.4}$$

In general, we have $\phi^n(t) \leq \|\lambda\|^n t$. Thus, we have

$$\sum_{n=1}^{\infty}\phi^n(t) \leq \sum_{n=1}^{\infty}\|\lambda\|^n t = \frac{\|\lambda\|t}{1-\|\lambda\|} < +\infty. \tag{3.5}$$

Therefore ϕ satisfies all the hypotheses of Definition 6. By inequality (3.1), we have $\Omega(Tx, T^2x, Tw) \leq \phi(\Omega(x, Tx, w)$. Therefore by Theorem 2.3, T has a unique fixed point. □

Now, our aim is to give an existence theorem for a solution of the following integral equation:

$$u(t) = \int_0^1 K(t, s, u(s))ds + g(t), \quad t \in [0,1]. \tag{3.6}$$

Let $X = C([0,1])$ be the set of all continuous functions defined on $[0,1]$. Define

$$G : X \times X \times X \longrightarrow \mathbb{R}^+ \tag{3.7}$$

by

$$G(x, y, z) = \max\{\|x-y\|, \|x-z\|, \|y-z\|\}, \tag{3.8}$$

where $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Then (X, G) is a complete G-metric space. Let $\Omega = G$. Then Ω is an Ω-distance on X.

Define an ordered relation $\leq$ on X by

$$x \leq y \quad \text{iff } x(t) \leq y(t), \quad \forall t \in [0,1]. \tag{3.9}$$

Then $(X, \leq)$ is a partially ordered set. Now, we prove the following result.

Theorem 3.2. *Suppose the following hypotheses hold.*

(a) $K : [0,1] \times [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous.*

(b) K *is nondecreasing in its first coordinate and* g *is nondecreasing.*

(c) *There exist a continuous function* $G : [0,1] \times [0,1] \to [0, +\infty]$ *such that*

$$|K(t, s, u) - K(t, s, v)| \leq G(t, s)|u - v|, \tag{3.10}$$

for each comparable $u, v \in \mathbb{R}^+$ *and each* $t, s \in [0,1]$.

(d) $\sup_{t\in[0,1]} \int_0^1 G(t, s)ds \leq r$ *for some* $r < 1$.

Then the integral equation (3.6) *has a solution* $u \in C([0,1])$.

Proof. Define $T : C([0,1]) \to C([0,1])$ by

$$Tx(t) = \int_0^1 K(t,s,x(s))ds + g(t), \quad t \in [0,1]. \tag{3.11}$$

By hypothesis (b), we have that T is nondecreasing.

Now, if

$$\inf\left\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega\left(x,T^2x,y\right) : x \le Tx\right\} = 0, \tag{3.12}$$

for $y \in C([0,1])$ with $y \neq Ty$, then for each $n \in \mathbb{N}$ there exists $x_n \in C([0,1])$ with $x_n \le Tx_n$ such that

$$\Omega(x_n,y,x_n) + \Omega(x_n,y,Tx_n) + \Omega\left(x_n,T^2x_n,y\right) \le \frac{1}{n}. \tag{3.13}$$

So, we have

$$\Omega(x_n,y,Tx_n) = \max\{\|x_n - y\|, \|x_n - Tx_n\|, \|y - Tx_n\|\} \le \frac{1}{n}. \tag{3.14}$$

Therefore, for each $t \in [0,1]$, we have

$$\begin{aligned} \lim_{n\to+\infty} x_n(t) &= y(t), \\ \lim_{n\to+\infty} Tx_n(t) &= y(t). \end{aligned} \tag{3.15}$$

By the continuity of K, we have

$$\begin{aligned} y(t) &= \lim_{n\to+\infty} Tx_n(t) \\ &= \int_0^1 K\left(t,s,\lim_{n\to+\infty} x_n(s)\right)ds + g(t) \\ &= \int_0^1 K(t,s,y(s))ds + g(t) = Ty(t). \end{aligned} \tag{3.16}$$

Thus, we have $y = Ty$, a contradiction. Thus,

$$\inf\left\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega\left(x,T^2x,y\right) : x \le Tx\right\} > 0. \tag{3.17}$$

Define $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = rt$. For $x \in C([0,T])$ with $x \leq Tx$, we have

$$\begin{aligned}
\Omega\left(Tx, T^2x, Tx\right) &= \sup_{t\in[0,1]} \left|Tx(t) - T^2x(t)\right| \\
&= \sup_{t\in[0,1]} \left|\int_0^1 K(t,s,x(s)) - K(t,s,Tx(s))ds\right| \\
&\leq \sup_{t\in[0,1]} \int_0^1 |K(t,s,x(s)) - K(t,s,Tx(s))|ds \\
&\leq \sup_{t\in[0,1]} \int_0^1 G(t,s)|x(s) - Tx(s)|ds \\
&\leq \sup_{t\in[0,1]} |x(t) - Tx(t)| \sup_{t\in[0,1]} \int_0^T G(t,s)ds \\
&= \Omega(x,Tx,x) \sup_{t\in[0,1]} \int_0^1 G(t,s)ds \\
&\leq r\Omega(x,Tx,x) \\
&= \phi(\Omega(x,Tx,x)).
\end{aligned} \tag{3.18}$$

Moreover, take $x_0 = 0$, then $x_0 \leq Tx_0$. Thus all the required hypotheses of Corollary 2.5 are satisfied. Thus there exists a solution $u \in C([0,T])$ of the integral equation (3.6). □

Acknowledgment

The authors would like to thank the referee and area editor Professor Cristian Toma for providing useful suggestions and comments for the improvement of this paper.

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