Finite Difference Schemes as Algebraic Correspondences between Layers

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Abstract. For some differential equations, especially for Riccati equation, new finite difference schemes are suggested. These schemes define protective correspondences between the layers. Calculation using these schemes can be extended to the area beyond movable singularities of exact solution without any error accumulation.

Introduction

Modern development of computer science revived the old investigations on the solvability of differential equations in the finite terms. There are remarkable differential equations which can be integrated in computer algebra systems (CAS). In our work we want to talk about remarkable differential equations in another sense: for these equations there are finite difference schemes that correctly describe singularities of exact solutions.

1 Finite difference method and singularities

Consider an ordinary differential equation (ODE)

$$\frac{dy}{dx} = f(x,y), \qquad f \in \mathbb{Q}(x,y). \tag{1}$$

If a singular point of the solution to the Cauchy problem

$$\frac{dy}{dx} = f(x,y), \qquad y|_{x=a} = y_0$$

depends on the initial data then it is called movable singularity of ODE. By the Painlevé theorem these singularities are always algebraic, that is, the equation in the neighborhood of such singularity can be expanded into Puiseux series

$$y = \sum_{s=n}^{\infty} k_s (x-c)^{s/m},$$

where $m > 0$ and n are integers and $k_n \neq 0$ [1]. The power $p = n/m$ is called the order of the singularity.

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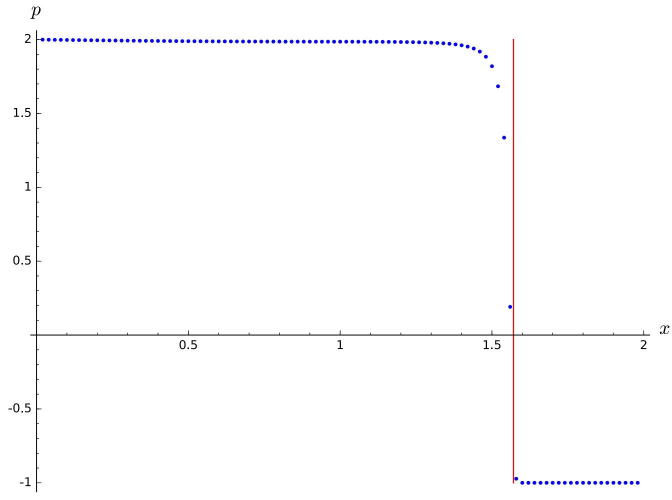

Figure 1. The graph of the effective order of the finite difference scheme of the example 1

In general we don't know the exact solution $y = \varphi(x)$ of the given Cauchy problem but we can calculate an approximate solution by finite difference approach. This suggests replacing the original differential equation with an algebraic equation (scheme) of the form

$$F(y, \hat{y}; x, \hat{x}) = 0$$

in commonly used notations [2]. This equation defines a correspondence between neighboring layers y and $\hat{y}$, which are usually investigated as points on two affine straight lines.

Problem 1 *Given a Cauchy problem and an interval $a < x < b$, we want to detect mobile singularities on this interval by analysis of one or several approximate solutions.*

Many authors think that this problem can't be solved because finite difference method describes mistakenly the solution in the neighborhood of a singularity. However a practical solution was given by E. Alshina et al. in 2005 [3], applications and development of these ideas see in [4–7]. There is such a scheme (ex. Complex Rosenbrock scheme, CROS), within which the approximate solution goes to a finite value when the exact solution has a pole. For CROS the ratio

$$\frac{y(x_n, \Delta x) - y(x_n, \Delta x/2)}{y(x_n, \Delta x/2) - y(x_n, \Delta x/2^2)}$$

is equal approximately to 2^2 in regular points and 2^p in the neighborhood of a singularity of order p. The logarithm of this ratio is called the effective order of scheme: its jump indicates the position of the singularity and its value after the jump is equal to the order of the singularity.

Example 1 *In fig. 1 we can see that the numerical solution of the Cauchy problem*

$$\frac{dy}{dx} = 1 + y^2, \qquad y|_{x=0} = 0$$

has a pole at $x \simeq 1.6$. It is true because the exact solution $y = \tan x$ has a pole at $x = \pi/2$. The calculations were made in our package for Sagemath [8].

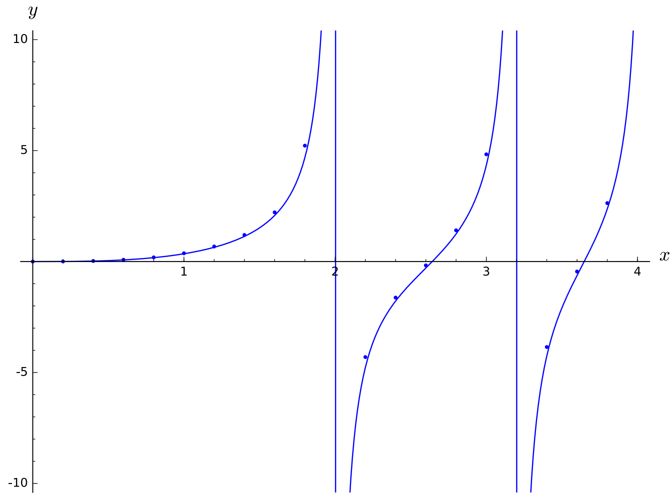

Figure 2. The graph of the solution for example 2: the solid line shows the graph of the exact solution, points mark the graph of the approximate solution

2 Riccati equation

All the commonly used schemes describe mistakenly the solutions of Cauchy problem in the neighborhood of singularities. However, the Riccati equation

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2 \tag{2}$$

can be approximated by the scheme

$$\frac{\Delta y}{\Delta x} = p + qy + ry\hat{y}. \tag{3}$$

The calculations by this scheme can be extended to the area beyond the pole without any error accumulation, a feature noticed in [9].

Example 2 *The initial value problem*

$$\frac{dy}{dx} = x^2 + y^2, \qquad y|_{x=0} = 0,$$

can be integrated in Bessel functions. We have calculated the exact solution in Maple and compared it to the approximate solution computed by our scheme (3) *in Sagemath. In fig. 2 we can see that points of the numerical solution give good approximation of the solution everywhere.*

Theorem 1 *Let p, q, r denote rational functions which have not any pole on the interval $[a, b]$, and let $y = g(x)$ be the exact solution of Cauchy problem for the Riccati equation* (2) *on this interval. For any $\delta > 0$ there exist the finite constants M_x and M_y such that any approximate solution $\{y_1, \dots, y_N\}$ found by the scheme* (3) *satisfies the estimate*

$$|y_n - g(x_n)| \leq M_y \Delta x,$$

if the step size Δx satisfies the condition $\Delta x \leq M_x$ and distance between x_n and the poles is larger than δ.

3 Finite difference method and algebraic geometry

When we write difference scheme we map differential equation to a pair (V, F), where 1) V is a layer, that is, an algebraic variety which can depend on x; 2) F is a difference scheme, that is, an algebraic correspondence between $V(x)$ and $V(\hat{x})$. We can define the notion of approximation and approximate solutions in the frame of the Weierstrass Vorbereitungssatz. The simplest, in the geometric sense, correspondences aren't used in the standard theory of the finite differences. In the simplest case the layer is a projective straight line $\mathbb{P}$ and the correspondence is a projective one-to-one transformation. Such schemes do not exist for all the differential equations (1) but only for the Riccati equation (2). One of these schemes is the scheme (3) the remarkable properties of which were described above.

Natural generalization of the projective one-to-one transformation is the n-to-n projective correspondence on algebraic varieties. We can prove that there is a projective n-to-n scheme for a given differential equation iff the exact general solution of the differential equation depends on constant algebraically. This class was investigated in early works by Painlevé [10, 11]. Thus, the classical transcendental functions are good functions not only from the point of view of the power series expansions. The evaluation of these functions by finite difference schemes can be extended to the area beyond singularities without any error accumulation.

Conclusions

Given the Riccati equation (2), the existence of a pole inside the investigated interval does not necessarily results in the accumulation of the discretization errors in a finite difference method. The reason is a bad choice of finite difference scheme. Projective n-to-n finite difference schemes always exist for the differential equations which can be integrated in classical (in Painlevé sense) transcendental functions.

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References

[1] W.W. Golubev, *Vorlesungen über Differentialgleichungen im Komplexen* (Berlin: VEB Deutscher Verlag der Wissenschaften, 1958)
[2] A.A. Samarskii, *The Theory of Difference Schemes* (NY, 2001)
[3] E.A. Al'shina, N.N. Kalitkin, and P.V. Koryakin, Comp. Math. and Math. Phys. **45** (10), 1769–1779 (2005)
[4] A.A. Belov, Doklady Mathematics **93** (3), 334–338 (2016)
[5] A.A. Belov, Computational Mathematics and Mathematical Physics **57** (1), 122–132 (2017)
[6] M.O. Korpusov, D.V. Lukyanenko, A.A. Panin, and E.V. Yushkov, Journal of Mathematical Analysis and Applications **442** (2), 451–468 (2016)
[7] M.O. Korpusov, D.V. Lukyanenko, A.A. Panin, and E.V. Yushkov, Mathematical Methods in the Applied Sciences **40** (7), 2336–2346 (2017)
[8] M.D. Malykh, Cros for Sagemath (2017). Available from: https://malykhmd.neocities.org
[9] M.D. Malykh, RUDN Journal of Mathematics, Information Sciences and Physics **3**, 5–9 (2015)
[10] H. Umemura, Nagoya Math. J. **119**, 1–80 (1990)
[11] M.D. Malykh, Journal of Mathematical Sciences **209** (6), 935–952 (2015)
[12] SageMath [Internet]. Available from: http://www.sagemath.org