

Rainbow's stars

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Abstract In recent years, a growing interest in the equilibrium of compact astrophysical objects like white dwarf and neutron stars has been manifested. In particular, various modifications due to Planck-scale energy effects have been considered. In this paper we analyze the modification induced by gravity's rainbow on the equilibrium configurations described by the Tolman–Oppenheimer–Volkoff (TOV) equation. Our purpose is to explore the possibility that the rainbow Planck-scale deformation of space-time could support the existence of different compact stars.

1 Introduction

Compact stars, exotic stars, wormholes, and black holes are astrophysical objects described by the Einstein field equations. For a perfect fluid and in the case of spherical symmetry, these objects obey the Tolman–Oppenheimer–Volkoff (TOV) equation (in c.g.s. units) [1, 2]

$$\frac{dp_r(r)}{dr} = - \left(\rho(r) + \frac{p_r(r)}{c^2} \right) \frac{4\pi Gr^3 p_r(r)/c^2 + Gm(r)}{r^2[1 - 2Gm(r)/rc^2]} + \frac{2}{r}(p_t(r) - p_r(r)) \quad (1)$$

and

$$\frac{dm}{dr} = 4\pi\rho(r)r^2, \quad (2)$$

where c is the speed of light, G is the gravitational constant, $\rho(r)$ is the macroscopic energy density measured in proper coordinates, and $p_r(r)$ and $p_t(r)$ are the radial pressure and the transverse pressure, respectively. It is clear that the knowledge of $\rho(r)$ allows one to understand the astrophysical structure under examination. If we fix our attention on compact stars, ordinary general relativity offers two kinds of exact solutions for the isotropic TOV equation:

- (a) the constant energy-density solution,
- (b) the Misner–Zapolsky energy-density solution [3].

Of course, (a) and (b) can be combined to give a new profile, which has been considered by Dev and Gleiser [4]. Case (b) is satisfied with the help of an equation of state of the form $p_r = \omega\rho$ with $\omega = 1/3$

$$p_r = \omega\rho(r) = \omega \frac{3c^2}{56\pi Gr^2} = \frac{c^2}{56\pi Gr^2} \quad (3)$$

and

$$m(r) = \frac{3c^2 r}{14G}. \quad (4)$$

Other different solutions can be found introducing anisotropy [4, 5] and/or polytropic transformations [6] or other forms of modification of gravity like $f(r)$ gravity [7, 8] and generalized uncertainty principle (GUP) [9]. The GUP distortion is only one of the different examples involving Planckian or trans-Planckian modifications due to quantum gravitational effects coming into play. Indeed, a number of recent studies have already focused on the effects of Planck-scale physics on the equilibrium configuration of compact astrophysical objects (see e.g. [10–18]). Usually Planck-scale physics is considered to affect equilibrium configuration via the modification of the energy-momentum dispersion relation that implies deformed equation of state (EoS) for the fluid composing the star. This is for example the approach followed in Refs. [10–13]. However, there are Planck-scale scenarios in which the deformation occurs by means of the metric deformation as well. This is the case of the so called gravity's rainbow [19, 20]. Gravity's rainbow is a distortion of space-time induced by two arbitrary functions, $g_1(E/E_{Pl})$ and $g_2(E/E_{Pl})$, which have the following property¹

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¹ Applications and implications of gravity's rainbow in astrophysics and cosmology can be found in [21–26].

$$\lim_{E/E_{Pl} \rightarrow 0} g_1(E/E_{Pl}) = 1 \quad \text{and} \quad \lim_{E/E_{Pl} \rightarrow 0} g_2(E/E_{Pl}) = 1. \tag{5}$$

It has been introduced for the first time by Magueijo and Smolin [19,20], who proposed that the energy-momentum tensor and the Einstein field equations were modified with the introduction of a one parameter family of equations

$$G_{\mu\nu}(E/E_{Pl}) = 8\pi G(E/E_{Pl}) T_{\mu\nu}(E/E_{Pl}) + g_{\mu\nu} \Lambda(E/E_{Pl}), \tag{6}$$

where $G(E/E_{Pl})$ is an energy dependent Newton’s constant and $\Lambda(E/E_{Pl})$ is an energy dependent cosmological constant, defined so that $G(0)$ is the low-energy Newton’s constant and $\Lambda(0)$ is the low-energy cosmological constant. It is clear that the modified Einstein’s field equations (6) give rise to a class of solutions which are dependent on $g_1(E/E_{Pl})$ and $g_2(E/E_{Pl})$. For instance, the rainbow version of the Schwarzschild line element is

$$ds^2 = - \left(1 - \frac{2MG(0)}{r} \right) \frac{d\tilde{t}^2}{g_1^2(E/E_{Pl})} + \frac{d\tilde{r}^2}{\left(1 - \frac{2MG(0)}{r} \right) g_2^2(E/E_{Pl})} + \frac{\tilde{r}^2}{g_2^2(E/E_{Pl})} \times (d\theta^2 + \sin^2 \theta d\phi^2). \tag{7}$$

As shown in Refs. [23,24], one of the effects of the functions $g_1(E/E_P)$ and $g_2(E/E_P)$ is to keep under control UV divergences allowing therefore the computation of quantum corrections to classical quantities, at least to one loop. As a result, the computation of zero point energy (ZPE) in gravity’s rainbow is well defined for appropriate choices of $g_1(E/E_P)$ and $g_2(E/E_P)$. In this paper, we would like to consider the effect of gravity’s rainbow on the TOV equations to explore the possibility of finding new forms of compact stars. The paper is organized as follows. In Sect. 2 we consider the TOV modified by gravity’s rainbow, in Sect. 3 we examine the constant energy-density case and its consequence on the redshift factor, in Sect. 4 we examine the variable energy-density case and its consequence on the redshift factor including the Dev–Gleiser case. We summarize and conclude in Sect. 5.

2 TOV equation in Gravity’s Rainbow

To see how gravity’s rainbow affects the TOV equations, we need to define the following line element:

$$ds^2 = - \frac{e^{2\Phi(r)}}{g_1^2(E/E_{Pl})} c^2 dt^2 + \frac{dr^2}{g_2^2(E/E_{Pl}) \left(1 - \frac{2Gm(r)}{rc^2} \right)} + \frac{r^2}{g_2^2(E/E_{Pl})} (d\theta^2 + \sin^2 \theta d\phi^2). \tag{8}$$

From Appendix A, we can see that only G_{00} modifies:

$$G_{00} = 2G \frac{e^{2\Phi(r)}}{r^2} \frac{g_2^2(E/E_{Pl})}{g_1^2(E/E_{Pl})} m'(r). \tag{9}$$

For the energy-momentum stress tensor describing a perfect fluid, we assume the following form:

$$T_{\mu\nu} = (\rho c^2 + p_t) u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t) n_\mu n_\nu, \tag{10}$$

where u^μ is the four-velocity normalized in such a way that $g_{\mu\nu} u^\mu u^\nu = -1$, n_μ is the unit spacelike vector in the radial direction, i.e. $g_{\mu\nu} n^\mu n^\nu = 1$ with $n^\mu = \sqrt{1 - 2Gm(r)/rc^2} \delta_r^\mu$. $\rho(r)$ is the energy density, $p(r)$ is the radial pressure measured in the direction of n^μ , and $p_t(r)$ is the transverse pressure measured in the orthogonal direction to n^μ . From the results of Appendix A, we can see that the equilibrium equation

$$\frac{dp}{dr} + (\rho c^2 + p)\Phi'(r) = 0 \tag{11}$$

must hold also in gravity’s rainbow. From this equation follows that

$$\frac{dp_r}{dr} = - \left(\rho + \frac{p_r}{c^2} \right) \frac{\kappa r^3 p_r / c^4 g_2^2(E/E_{Pl}) + 2Gm(r)/c^2}{2r^2 [1 - 2Gm(r)/rc^2]} + \frac{2}{r} (p_t - p_r) \tag{12}$$

and

$$\frac{dm}{dr} = \frac{4\pi \rho(r) r^2}{g_2^2(E/E_{Pl})}, \tag{13}$$

where ρ is the mass density. Equation (12) is the anisotropic TOV equation modified by gravity’s rainbow. As a first simplification, we will assume that the star is isotropic. Then we will consider the constant energy-density case (I) and the Misner–Zapolsky energy-density case (II). We begin to consider the case (I).

3 Isotropic pressure and the constant energy-density case

With the assumption of an isotropic star, the pressure in Eq. (12) becomes

$$\frac{dp_r}{dr} = - \left(\rho + \frac{p_r(r)}{c^2} \right) \frac{4\pi Gr^3 p_r(r)/c^2 g_2^2(E/E_{Pl}) + Gm(r)}{r^2 [1 - 2Gm(r)/rc^2]} \tag{14}$$

The constant energy-density assumption allows an easy solution of Eq. (13). Indeed, one gets

$$m(r) = \frac{4\pi\rho}{3g_2^2(E/E_{Pl})} r^3, \tag{15}$$

where we have used the boundary conditions $m(0) = 0$. Nevertheless, Eqs. (14) and (15) are referred to the whole star included the external boundary R . To account for different scenarios we discuss two fundamental cases:

- (a) The star is divided in two regions: the inner region or the core, where gravity’s rainbow is relevant, and the outer region, where gravity’s rainbow is negligible.
- (b) The whole star is modified by gravity’s rainbow.

1. Case (a)

In this case, the star is divided in two parts: the external part of the star without gravity’s rainbow and the core with gravity’s rainbow. Basically, we can write

$$dm = \begin{cases} 4\pi\rho r^2 dr/g_2^2(E/E_{Pl}) & \bar{r} \geq r > 0 \\ 4\pi\rho r^2 dr & R > r > \bar{r} \end{cases} \tag{16}$$

The transition between the distorted and the undistorted mass is represented by introducing an intermediate radius \bar{r} , assuming that

$$R \gg \bar{r} > l_{Pl}. \tag{17}$$

In this first approach, the transition between the distorted and the undistorted mass is very sharp, but we cannot exclude the possibility of describing a smoothed variation between the external part of the star and the core in a next future. After an integration, we can write

$$m(r) = \begin{cases} m_1(r) = \frac{4\pi\rho r^3}{3g_2^2(E/E_{Pl})} = Mr^3/(\bar{R}^3 g_2^2(E/E_{Pl})) & \bar{r} \geq r > 0 \\ m_2(r) = \frac{4\pi\rho}{3}(r^3 + \bar{r}^3 A(E/E_{Pl})) = M(r^3 + \bar{r}^3 A(E/E_{Pl}))/\bar{R}^3 & R > r > \bar{r} \end{cases} \tag{18}$$

In Eq. (18) we have used the total mass density

$$\rho = M \left[\frac{4\pi}{3} (R^3 + A(E/E_{Pl})\bar{r}^3) \right]^{-1} = M/\tilde{V} \tag{19}$$

and we have defined

$$\tilde{V} = \frac{4\pi}{3} (R^3 + A(E/E_{Pl})\bar{r}^3) = \frac{4\pi}{3} \tilde{R}^3, \tag{20}$$

with

$$\tilde{R}^3 = R^3 + A(E/E_{Pl})\bar{r}^3 \tag{21}$$

and

$$A(E/E_{Pl}) = (g_2^2(E/E_{Pl})^{-1} - 1). \tag{22}$$

We indicate with ρ_0 , the mass density (19) with $g_2(E/E_{Pl}) = 1$. Note that the volume distorted by gravity’s rainbow, for a sphere of radius R , is

$$V = \int d^3x \sqrt{g} = \frac{4\pi}{3g_2^3(E/E_{Pl})} \int_0^R \frac{r^2 dr}{\sqrt{1 - 2m(r)/r}}. \tag{23}$$

Therefore the mass density in (19) does not coincide with the ratio M/V . To calculate the pressure, we divide the radius of the star into two sectors exactly like in Eq. (16). We begin to consider the range $R \geq r \geq \bar{r}$. This is the sector where the TOV equation is undeformed. From Eq. (14), with $g_2(E/E_{Pl}) = 1$, one gets

$$p_r(r) = \rho_0 c^2 \frac{(\sqrt{3c^2 - \kappa\rho_0 r^2} - \sqrt{3c^2 - \kappa\rho_0 R^2})}{(3\sqrt{3c^2 - \kappa\rho_0 R^2} - \sqrt{3c^2 - \kappa\rho_0 r^2})}, \tag{24}$$

where $\kappa = 8\pi G$ and where we have used the boundary condition $p(R) = 0$. It is immediate to recognize that in this region of the star, to avoid a singularity in the denominator, we have to impose

$$R < \sqrt{\frac{c^2}{3\pi G\rho_0}}. \tag{25}$$

When we use Eq. (15) with $g_2(E/E_{Pl}) = 1$, then we recover the Buchdahl–Bondi bound [27–30],

$$M < \frac{4}{9} \frac{c^2}{G} R. \tag{26}$$

However, because of the distortion introduced by gravity’s rainbow in Eq. (15) and in the mass density ρ (19), the inequality (26) becomes

$$M < \frac{4c^2}{9GR^2} (R^3 + A(E/E_{Pl})\bar{r}^3) \tag{27}$$

and the Buchdahl–Bondi bound is modified. It is useful to consider the limit in which $E/E_{Pl} \rightarrow 0$. In this limit, we find

that Eq. (27) reduces to

$$M < \frac{4c^2}{9GR^2} (R^3 + ((1 + h(E/E_{Pl}))^{-1} - 1)\bar{r}^3) \simeq \frac{4c^2 R}{9G} - h(E/E_{Pl}) \frac{4c^2 \bar{r}^3}{9GR^2}, \tag{28}$$

where $h(E/E_{Pl}) \rightarrow 0$, when $E/E_{Pl} \rightarrow 0$. Note that $h(E/E_{Pl}) \geq 0$ depending on the form of the rainbow's function. To complete the analysis, we have to examine the core of the star $\bar{r} \geq r \geq 0$ where gravity's rainbow is switched on, leading to the following TOV equation:

$$\frac{dp_r}{dr} = -\frac{\kappa r(\rho c^2 + p(r))(3p(r) + \rho c^2)}{2c^2[3c^2 g_2^2(E/E_{Pl}) - \kappa \rho r^2]}, \tag{29}$$

whose solution is

$$p_r(r) = \rho c^2 \frac{CB(r, E) - 1}{3 - CB(r, E)}, \tag{30}$$

where A is a constant to be determined by an appropriate choice of the boundary conditions and where

$$B(r, E) = \sqrt{3c^2 g_2^2(E/E_{Pl}) - \kappa \rho r^2}. \tag{31}$$

Since $p_r(r)$ must be continuous, we have to impose

$$\lim_{r \rightarrow \bar{r}_-} p_r(r) = \lim_{r \rightarrow \bar{r}_+} p_r(r) \tag{32}$$

which implies

$$\frac{CB(\bar{r}, E) - 1}{3 - CB(\bar{r}, E)} = \frac{\rho_0 (\sqrt{3c^2 - \kappa \rho_0 \bar{r}^2} - \sqrt{3c^2 - \kappa \rho_0 R^2})}{\rho (3\sqrt{3c^2 - \kappa \rho_0 R^2} - \sqrt{3c^2 - \kappa \rho_0 \bar{r}^2})} = D(\bar{r}, R). \tag{33}$$

Thus C is no longer a constant but it has become a function of \bar{r} , R and E and it is determined to find

$$C \equiv C(\bar{r}, R, E) = \frac{3D(\bar{r}, R) + 1}{B(\bar{r}, E)(1 + D(\bar{r}, R))}. \tag{34}$$

Plugging the value of $C(\bar{r}, R, E)$ into (30), we obtain

$$p_r(r) = \rho c^2 \frac{(3D(\bar{r}, R) + 1)B(r, E) - B(\bar{r}, E)(1 + D(\bar{r}, R))}{3B(\bar{r}, E)(1 + D(\bar{r}, R)) - (3D(\bar{r}, R) + 1)B(r, E)} \tag{35}$$

and the radial pressure for the whole star is

$$p_r(r) = \rho c^2 \begin{cases} \frac{(3D(\bar{r}, R) + 1)B(r, E) - B(\bar{r}, E)(1 + D(\bar{r}, R))}{3B(\bar{r}, E)(1 + D(\bar{r}, R)) - (3D(\bar{r}, R) + 1)B(r, E)} & \bar{r} > r \geq 0 \\ D(\bar{r}, R) & r = \bar{r} \\ \frac{\sqrt{3c^2 - \kappa \rho r^2} - \sqrt{3c^2 - \kappa \rho R^2}}{3\sqrt{3c^2 - \kappa \rho R^2} - \sqrt{3c^2 - \kappa \rho r^2}} & r > \bar{r} \end{cases}, \tag{36}$$

from which is possible to compute the pressure at the center of the star. One finds

$$p_r(0) = p_c = \rho c^2 \frac{(3D(\bar{r}, R) + 1)\sqrt{3c^2 g_2^2(E/E_{Pl})} - B(\bar{r}, E)(1 + D(\bar{r}, R))}{3B(\bar{r}, E)(1 + D(\bar{r}, R)) - (3D(\bar{r}, R) + 1)\sqrt{3c^2 g_2^2(E/E_{Pl})}} \tag{37}$$

and in order to have a finite p_c , we have to impose the requirement that the denominator of (37) be not naught, namely

$$\frac{24c^4 g_2^2(E/E_{Pl}) - 9\kappa \rho c^2 g_2^2(E/E_{Pl})R^2}{9c^2 \kappa \rho - 3\kappa^2 \rho^2 R^2 - \kappa \rho c^2 g_2^2(E/E_{Pl})} \neq \bar{r}^2. \tag{38}$$

Due to the complexity of Eq. (35), it is useful to discuss the following limiting cases:

(1) $g_2(E/E_{Pl}) \rightarrow 0$. Although the central pressure p_c approaches a finite and real limit

$$p_c \simeq -\frac{\rho c^2}{3}, \tag{39}$$

the constant A in (34) becomes imaginary. Moreover, the inequality (27) becomes dominated by the $A(E/E_{Pl})$ function which is divergent allowing the underlying mass to assume any value. For this reason, this limit will be discarded.

(2) $g_2(E/E_{Pl}) \rightarrow \infty$. In this case, Eq. (38) becomes

$$\frac{9\kappa \rho c^2 R^2 - 24c^4}{\kappa \rho c^2} \neq \bar{r}^2 \tag{40}$$

and by imposing

$$R < \sqrt{\frac{c^2}{3\pi G \rho}}, \tag{41}$$

we obtain a Buchdahl–Bondi-like bound, because the mass density becomes

$$\rho = M \left[\frac{4\pi}{3} (R^3 - \bar{r}^3) \right]^{-1}. \tag{42}$$

In this limit, the central pressure becomes

$$p_c \simeq \rho c^2 D(\bar{r}, R) = \rho c^2 \frac{(\sqrt{3c^2 - \kappa \rho_0 \bar{r}^2} - \sqrt{3c^2 - \kappa \rho_0 R^2})}{(3\sqrt{3c^2 - \kappa \rho_0 R^2} - \sqrt{3c^2 - \kappa \rho_0 \bar{r}^2})}, \tag{43}$$

from which is possible to obtain information on the radius of the star

$$R = \sqrt{\frac{1}{\kappa\rho_0} \frac{\sqrt{(24c^2 + \kappa\rho_0\bar{r}^2)p_c^2 + (12\rho_0c^4 + 2c^2\kappa\rho_0^2\bar{r}^2)p_c + c^4\kappa\rho_0^3\bar{r}^2}}{\rho_0c^2 + 3p_c}}. \tag{44}$$

Note that when $\bar{r} \rightarrow 0$, we recover the usual Buchdahl–Bondi bound. On the other hand, it is possible to have the expression of the intermediate radius \bar{r} as a function of R , ρ_0 , ρ and p_c

$$\bar{r} = \sqrt{\frac{1}{\kappa\rho_0} \frac{\sqrt{(9\kappa R^2\rho_0 - 24c^2)p_c^2 + 6c^2\rho_0(\kappa R^2\rho_0 - 2c^2)p_c + R^2c^4\rho_0^3\kappa}}{\rho_0c^2 + p_c}}. \tag{45}$$

2. Case (b)

In this case, the whole star is distorted by gravity’s rainbow and the boundary is set very close to the core. The integration of Eq. (29) with the condition $p_r(R) = 0$, leads to

$$p_r(r) = \rho c^2 \frac{(\sqrt{3c^2g_2^2(E/E_{Pl}) - \kappa\rho r^2} - \sqrt{3c^2g_2^2(E/E_{Pl}) - \kappa\rho R^2})}{(3\sqrt{3c^2g_2^2(E/E_{Pl}) - \kappa\rho R^2} - \sqrt{3c^2g_2^2(E/E_{Pl}) - \kappa\rho r^2})}. \tag{46}$$

Because of Eq. (15) at the boundary R , we find

$$\rho = g_2^2(E/E_{Pl}) \frac{3M}{4\pi R^3} = g_2^2(E/E_{Pl}) \tilde{\rho}, \tag{47}$$

where $\tilde{\rho}$ is the mass density in ordinary GR. Thus Eq. (46) becomes

$$p_r(r) = g_2^2(E/E_{Pl}) \tilde{\rho} c^2 \frac{(\sqrt{3c^2 - \kappa\tilde{\rho}r^2} - \sqrt{3c^2 - \kappa\tilde{\rho}R^2})}{(3\sqrt{3c^2 - \kappa\tilde{\rho}R^2} - \sqrt{3c^2 - \kappa\tilde{\rho}r^2})} = g_2^2(E/E_{Pl}) \tilde{p}_r(r). \tag{48}$$

It is immediate to recognize that all the properties obtained in ordinary GR are here valid, except for the pressure which scales with $g_2^2(E/E_{Pl})$. The same behavior appears of course, when we describe the pressure in terms of the mass M and the radius R . Indeed, always with the help of Eq. (15), one gets

$$p_r(r) = \frac{3Mg_2^2(E/E_{Pl})}{4\pi R^3} c^2 \frac{\sqrt{c^2 - 2MGr^2/R^3} - \sqrt{c^2 - 2MG/R}}{3\sqrt{c^2 - 2MG/R} - \sqrt{c^2 - 2MGr^2/R^3}} = g_2^2(E/E_{Pl}) \tilde{p}_r(r) \tag{49}$$

and the Buchdahl–Bondi bound is preserved. We can now compute the pressure at the center of the star to obtain

$$p_r(0) = p_c = g_2^2(E/E_{Pl}) \tilde{p}_r(0) = g_2^2(E/E_{Pl}) \tilde{\rho} c^2 \frac{(\sqrt{3c^2} - \sqrt{3c^2 - \kappa\tilde{\rho}R^2})}{(3\sqrt{3c^2 - \kappa\tilde{\rho}R^2} - \sqrt{3c^2})}$$

$$= g_2^2(E/E_{Pl}) \tilde{p}_c \tag{50}$$

while in terms of the mass M , we obtain

$$p_r(0) = p_c = \frac{3Mg_2^2(E/E_{Pl})}{4\pi R^3} c^2 \frac{c - \sqrt{c^2 - 2MG/R}}{3\sqrt{c^2 - 2MG/R} - c} = g_2^2(E/E_{Pl}) \tilde{p}_c. \tag{51}$$

Because of the pressure scaling, we find that the radius of the star can be computed in the same way of the undeformed case. Indeed, in terms of the rescaled density we find

$$R = \sqrt{\frac{3c^2}{8\tilde{\rho}\pi G} \left[1 - \frac{(\tilde{\rho}c^2 + \tilde{p}_c)^2}{(\tilde{\rho}c^2 + 3\tilde{p}_c)^2} \right]}. \tag{52}$$

The same undeformed result is obtained in terms of the mass M

$$R = \frac{2MG}{c^2 \sqrt{1 - \frac{(\tilde{\rho}c^2 + \tilde{p}_c)^2}{(\tilde{\rho}c^2 + 3\tilde{p}_c)^2}}}, \tag{53}$$

where we have used the Schwarzschild form on the boundary of the star. However, when we go back to the deformed pressure and energy density, we find that the undeformed radius R described by (52), becomes²

$$\frac{R}{g_2(E/E_{Pl})} = \sqrt{\frac{3c^2}{8\rho\pi G} \left[1 - \frac{(\rho c^2 + p_c)^2}{(\rho c^2 + 3p_c)^2} \right]}. \tag{55}$$

When $g_2(E/E_{Pl}) \gg 1$, to obtain the shrinking of the radius of the star R , necessarily we need $\tilde{\rho}c^2 \gg 3\tilde{p}_c$, since the central pressure can be large but finite. When R is small, we find

$$p_c \simeq g_2^2(E/E_{Pl}) \frac{2\pi G \tilde{\rho}^2 R^2}{3} + O(R^4) \tag{56}$$

or, in terms of the mass M ,

$$p_c \simeq g_2^2(E/E_{Pl}) \frac{3M^2G}{8\pi R^4} + O(R^4). \tag{57}$$

This also means that from (53), M must be small. Notice that in terms of ρ the equilibrium condition becomes

$$R < \frac{cg_2(E/E_{Pl})}{\sqrt{3\pi G\rho}}. \tag{58}$$

² Note that the relation between the undeformed star radius R and the deformed \tilde{R} is

$$R_d = R/g_2^{2/3}(E/E_{Pl}) \tag{54}$$

as suggested by Eq. (47).

In the standard framework $g_2(E/E_{Pl}) = 1$ and Eq. (58) imply that when Planckian densities are approached, $\rho \approx \rho_{Pl}$, one gets

$$R \lesssim l_{Pl}, \tag{59}$$

i.e. only stars smaller than the Planck size can satisfy the TOV equilibrium equation. Instead, in our Rainbow scenario, at Planckian densities we get

$$R \lesssim g_2(E/E_{Pl})l_{Pl}, \tag{60}$$

suggesting that macroscopic stars are also allowed, if the function $g_2(E/E_{Pl})$ is very large.

3.1 The redshift function for the constant energy-density case

In the case of a constant density, the redshift function becomes

$$\Phi(r) + K = - \int_0^r \frac{dp/dr'}{\rho c^2 + p(r')} dr'. \tag{61}$$

Because of the modification due to gravity's rainbow, we are forced to separate the discussion of the redshift function into two cases. We begin with case a.

1. Case a

In this case the computation of the redshift function separates into two pieces

$$\begin{aligned} \Phi(r) + K &= - \int_0^{\bar{r}} \frac{dp/dr'}{\rho c^2 + p(r')} dr' - \int_{\bar{r}}^r \frac{dp/dr'}{\rho c^2 + p(r')} dr' \\ &= I_1 + I_2, \end{aligned} \tag{62}$$

where \bar{r} has been defined in (16) and the related range in (17). Plugging Eq. (29) into the first integral one finds

$$\begin{aligned} I_1 &= - \int_0^{\bar{r}} \frac{dp/dr'}{\rho c^2 + p(r')} dr' \\ &= - \int_0^{\bar{r}} \frac{dp/dr'}{g_2^2(E/E_{Pl})\tilde{\rho}c^2 + p(r')} dr' \\ &= \frac{\kappa}{2c^2} \int_0^{\bar{r}} \frac{r' (3p(r') + g_2^2(E/E_{Pl})\tilde{\rho}c^2)}{[3c^2g_2^2(E/E_{Pl}) - \kappa g_2^2(E/E_{Pl})\tilde{\rho}r'^2]} dr' \\ &= I_{1a} + I_{1b}, \end{aligned} \tag{63}$$

where

$$I_{1a} = \frac{3\kappa}{2c^2} \int_0^{\bar{r}} \frac{r' p(r')}{[3c^2g_2^2(E/E_{Pl}) - \kappa g_2^2(E/E_{Pl})\tilde{\rho}r'^2]} dr' \tag{64}$$

and

$$I_{1b} = \frac{\kappa\tilde{\rho}}{2} \int_0^{\bar{r}} \frac{r' dr'}{[3c^2 - \kappa\tilde{\rho}r'^2]} = -\frac{1}{4} \ln \left(\frac{3c^2 - \kappa\tilde{\rho}\bar{r}^2}{3c^2} \right). \tag{65}$$

Plugging (35) into the integral I_{1a} , one gets

$$\begin{aligned} I_{1a} &= \frac{3\kappa\tilde{\rho}}{2} \int_0^{\bar{r}} \\ &\times \frac{r' (3C(\bar{r}, R) + 1) \tilde{B}(r', E) - \tilde{B}(\bar{r}, E) (1 + C(\bar{r}, R))}{[3c^2 - \kappa\tilde{\rho}r'^2][3\tilde{B}(\bar{r}, E) (1 + C(\bar{r}, R)) - (3C(\bar{r}, R) + 1) \tilde{B}(r', E)]} dr', \end{aligned} \tag{66}$$

where we have used the following relationship:

$$\begin{aligned} B(r, E) &= \sqrt{3c^2g_2^2(E/E_{Pl}) - \kappa\rho r^2} \\ &= g_2(E/E_{Pl})\sqrt{3c^2 - \kappa\tilde{\rho}r^2} \\ &= g_2(E/E_{Pl})\tilde{B}(r, E). \end{aligned} \tag{67}$$

Define the new variable

$$3c^2 - \kappa\tilde{\rho}r'^2 = y^2 \implies -\kappa\tilde{\rho}r' dr' = y dy, \tag{68}$$

then I_1 becomes

$$\begin{aligned} I_{1a} &= -\frac{3}{2} \int_{\sqrt{3c^2}}^{y(\bar{r})} \frac{(3C(\bar{r}, R) + 1) y - \tilde{B}(\bar{r}, E) (1 + C(\bar{r}, R))}{y[3\tilde{B}(\bar{r}, E) (1 + C(\bar{r}, R)) - (3C(\bar{r}, R) + 1) y]} dy \\ &= -\frac{3}{2} \int_{\sqrt{3c^2}}^{y(\bar{r})} \frac{C_1 y - C_2}{y[3C_2 - C_1 y]} dy, \end{aligned} \tag{69}$$

where

$$\begin{aligned} C_1 &= 3C(\bar{r}, R) + 1 \\ C_2 &= \tilde{B}(\bar{r}, E) (1 + C(\bar{r}, R)). \end{aligned} \tag{70}$$

Now I_{1a} can easily be integrated to give

$$\begin{aligned} I_{1a} &= -\frac{3}{2} \int_{\sqrt{3c^2}}^{y(\bar{r})} \frac{C_1 y - C_2}{y[3C_2 - C_1 y]} dy \\ &= \ln \left(\frac{3C_2 - C_1 y(\bar{r})}{3C_2 - C_1 \sqrt{3c^2}} \right) + \frac{1}{2} \ln \left(\frac{y(\bar{r})}{\sqrt{3c^2}} \right) \end{aligned} \tag{71}$$

and

$$I_1 = \ln \left(\frac{3C_2 - C_1 y(\bar{r})}{3C_2 - C_1 \sqrt{3c^2}} \right). \tag{72}$$

Following the same procedure for I_2 , one gets

$$\begin{aligned} I_2 &= - \int_{\bar{r}}^r \frac{dp/dr'}{\rho c^2 + p(r')} dr' = \frac{\kappa}{2c^2} \int_{\bar{r}}^r \frac{r' (3p(r') + \rho c^2)}{[3c^2 - \kappa\rho r'^2]} dr' \\ &= \ln \left(\frac{3z(R) - z(r)}{3z(R) - z(\bar{r})} \right). \end{aligned} \tag{73}$$

Therefore (62) becomes

$$\begin{aligned} \Phi(r) + K &= \ln \left(\frac{3C_2 - C_1 y(\bar{r})}{3C_2 - C_1 \sqrt{3c^2}} \right) + \ln \left(\frac{3z(R) - z(r)}{3z(r) - z(\bar{r})} \right) \\ &= \ln \left(\frac{(3C_2 - C_1 y(\bar{r})) (3z(R) - z(r))}{(3C_2 - C_1 \sqrt{3c^2}) (3z(R) - z(\bar{r}))} \right). \end{aligned} \tag{74}$$

At the boundary of the star we obtain

$$\begin{aligned} \exp 2(\Phi(R) + K) &= \left(\frac{(3C_2 - C_1 y(\bar{r})) (2z(R))}{(3C_2 - C_1 \sqrt{3c^2}) (3z(R) - z(\bar{r}))} \right)^2 \\ \implies \exp 2K &= \frac{1}{\exp 2\Phi(R)} \left(\frac{(3C_2 - C_1 y(\bar{r})) (2z(R))}{(3C_2 - C_1 \sqrt{3c^2}) (3z(R) - z(\bar{r}))} \right)^2, \end{aligned} \tag{75}$$

then

$$\exp 2(\Phi(R) + K) = \left(\frac{(3C_2 - C_1 y(\bar{r})) (2z(R))}{(3C_2 - C_1 \sqrt{3c^2}) (3z(R) - z(\bar{r}))} \right)^2 \tag{76}$$

and

$$\exp 2\Phi(r) = \exp 2\Phi(R) \left(\frac{(3z(R) - z(r))}{(2\sqrt{3c^2 - \kappa\rho R^2})} \right)^2. \tag{77}$$

However, because of the Schwarzschild boundary condition, namely

$$\exp 2\Phi(R) = 1 - \frac{2MG}{c^2 R}, \tag{78}$$

and because of the (19), one finds that the redshift surface becomes

$$\begin{aligned} \exp 2\Phi(r) &= \left(1 - \frac{2MG}{c^2 R} \right) \\ &\times \left(\frac{\left(3\sqrt{1 - \frac{2MGr^2}{c^2 R^3}} - \sqrt{1 - \frac{2MGr^2}{c^2 R^3}} \right)^2}{\left(2\sqrt{1 - \frac{2MGr^2}{c^2 R^3}} \right)} \right)^2, \end{aligned} \tag{79}$$

where we have used (21). In any case, on the star surface the redshift factor reduces to

$$z = \frac{\Delta\lambda}{\lambda} = \frac{g_1(E/E_{Pl})}{\exp[\Phi(R)]} - 1 = \frac{g_1(E/E_{Pl})}{\sqrt{1 - \frac{2MG}{Rc^2}}} - 1. \tag{80}$$

The rainbow upper bound on the redshift factor

$$z \leq z_{\max} = 3g_1(E/E_{Pl}) - 1 \tag{81}$$

becomes $z_{\max} = 2$ in the undeformed limit $g_1(E/E_P) = 1$, as expected. It is clear that, for energies comparable with

E_{Pl} , one can have deviations from the usual redshift factor. Indeed, from

$$g_1(E/E_P) \simeq 1 + \alpha \frac{E}{E_P} + O \left(\left(\frac{E}{E_P} \right)^2 \right), \tag{82}$$

where

$$\alpha = \left(\frac{dg_1(E/E_P)}{dE} \right)_{|E=0} \frac{1}{E_P}, \tag{83}$$

we have

$$z_{\max} = 2 + 3\alpha \frac{E}{E_P} + O \left(\frac{E}{E_P} \right)^2, \tag{84}$$

with $\alpha \leq 0$.

2. Case b

In the case of a constant density one can also calculate the redshift function explicitly. Indeed, from Eq. (11), we find

$$\Phi(r) + K = - \int_0^r \frac{dp/dr'}{g_2^2(E/E_{Pl}) \tilde{\rho} c^2 + p(r')} dr' \tag{85}$$

and with the help of (49), one can write

$$\begin{aligned} p_r(y') \Big|_{y'^2=c^2-2MGr'^2/R^3} &= g_2^2(E/E_{Pl}) \tilde{\rho} c^2 \\ &\times \frac{\left(y' - \sqrt{c^2 - 2MG/R} \right)}{\left(3\sqrt{c^2 - 2MG/R} - y' \right)}. \end{aligned} \tag{86}$$

Thus

$$\frac{dp_r}{dy'} = g_2^2(E/E_{Pl}) \tilde{\rho} c^2 \frac{2\sqrt{c^2 - 2MG/R}}{\left(3\sqrt{c^2 - 2MG/R} - y' \right)^2} \tag{87}$$

and (85) becomes

$$\begin{aligned} \Phi(y) + K &= - \int_{c^2}^y \frac{dp/dy'}{g_2^2(E/E_{Pl}) \tilde{\rho} c^2 + p(y')} dy' \\ &= - \int_{c^2}^y \frac{dy'}{\left(3\sqrt{c^2 - 2MG/R} - y' \right)} \\ &= \ln \left(\frac{3\sqrt{c^2 - 2MG/R} - y}{3\sqrt{c^2 - 2MG/R} - c^2} \right). \end{aligned} \tag{88}$$

At the boundary of the star, we obtain $\exp 2\Phi(R) = 1 - 2MG/c^2 R$, thus

$$\begin{aligned} \exp 2(\Phi(R) + K) &= \left(\frac{3\sqrt{c^2 - 2MG/R} - y(R)}{3\sqrt{c^2 - 2MG/R} - c^2} \right)^2 \\ \implies \exp 2K &= \frac{1}{\exp 2\Phi(R)} \left(\frac{3\sqrt{c^2 - 2MG/R} - y(R)}{3\sqrt{c^2 - 2MG/R} - c^2} \right)^2, \end{aligned} \tag{89}$$

then

$$\begin{aligned} \exp 2\Phi(r) &= \exp 2\Phi(R) \left(\frac{3\sqrt{c^2 - 2MG/R} - \sqrt{c^2 - 2MGr^2/R^3}}{2\sqrt{c^2 - 2MG/R}} \right)^2 \end{aligned} \tag{90}$$

$$= \exp \left(\frac{1}{4c^2} \left(3\sqrt{c^2 - 2MG/R} - \sqrt{c^2 - 2MGr^2/R^3} \right)^2 \right). \tag{91}$$

Explicitly

$$\begin{aligned} \Phi(r) &= \ln \left[\frac{1}{4} \left(3\sqrt{1 - 2MG/c^2R} - \sqrt{1 - 2MGr^2/c^2R^3} \right) \right] \\ r &\in [0, R]. \end{aligned} \tag{92}$$

It is immediate to recognize that the behavior of the surface redshift is the same of the case a), except for the range which here is related to the whole star.

4 The isotropic TOV equation and the EoS: variable energy-density case

In this section, we will consider an energy-density profile of the following form:

$$\rho = Ar^\alpha, \tag{93}$$

where A is a constant with dimensions of an energy density divided by a (length) $^\alpha$ with $\alpha \in \mathbb{R}$ to be determined. Solving (13) leads to

$$m(r) = \int_0^r \frac{4\pi A}{g_2^2(E/E_{Pl})} r'^{2+\alpha} dr' = \frac{4\pi A}{g_2^2(E/E_{Pl})} (3 + \alpha) r^{3+\alpha}. \tag{94}$$

Plugging (93) and (94) into (1), one finds

$$\begin{aligned} \omega \frac{d\rho(r)}{dr} &= -\rho(r) \left(\frac{c^2 + \omega}{c^2} \right) \\ &\times \frac{4\pi Gr^3 \omega \rho(r) + Gm(r)c^2 g_2^2(E/E_{Pl})}{r^2 [1 - 2Gm(r)/rc^2] c^2 g_2^2(E/E_{Pl})} \end{aligned} \tag{95}$$

↓

$$\begin{aligned} \alpha &= - \left(\frac{c^2 + \omega}{\omega c^2} \right) \\ &\times \frac{4\pi GAr^{2+\alpha} ((3 + \alpha)\omega + c^2)}{[c^2 g_2^2(E/E_{Pl}) (3 + \alpha) - 8\pi GAr^{2+\alpha}]}. \end{aligned} \tag{96}$$

It is immediate to see that $\forall \alpha \neq -2$, there is a singularity into the TOV equation and a dependence on r still persists. Therefore if we fix $\alpha = -2$, one gets the relationship

$$1 = \frac{3(c^2 + \omega)^2}{4\omega [7c^2 g_2^2(E/E_{Pl}) - 3]}, \tag{97}$$

where we have set $A = 3c^2 / (56\pi G)$. We find an identity when $\omega = 1/3, \omega = 3, c = 1$ and $g_2(E/E_{Pl}) = 1$. Therefore in ordinary GR, TOV is satisfied for

$$p_r = \omega \rho(r) = \omega \frac{3c^2}{56\pi Gr^2} \tag{98}$$

and

$$m(r) = \frac{3c^2 r}{14G}. \tag{99}$$

The energy density in (3) has been found for the first time by Misner and Zapolsky [3]. When gravity's rainbow comes into play, one can find the values of ω satisfying the constraint (97). One finds

$$\omega_{\pm} = \frac{14}{3} c^2 g_2^2(E/E_{Pl}) - c^2 - 2 \pm \frac{2}{3} \sqrt{\Delta}, \tag{100}$$

where

$$\begin{aligned} \Delta &= 49c^4 g_2^4(E/E_{Pl}) - 21c^4 g_2^2(E/E_{Pl}) \\ &\quad - 42c^2 g_2^2(E/E_{Pl}) + 9c^2 + 9. \end{aligned} \tag{101}$$

When $g_2(E/E_{Pl}) \gg 1$, the asymptotic form of ω_{\pm} is

$$\begin{aligned} \omega_+ &\simeq \frac{28}{3} c^2 g_2^2(E/E_{Pl}) - 2c^2 - 4 - \frac{3c^2}{28g_2^2(E/E_{Pl})} \\ &+ O\left(\frac{1}{g^4}\right) \simeq \frac{28}{3} c^2 g_2^2(E/E_{Pl}) \end{aligned} \tag{102}$$

and

$$\omega_- \simeq \frac{3c^2}{28g_2^2(E/E_{Pl})} + O\left(\frac{1}{g^4}\right). \tag{103}$$

It is immediate to see that both solutions acquire a dependence on $g_2(E/E_{Pl})$ which is decreasing for ω_- and increasing for ω_+ . Note that at this stage, E acts as a parameter independent on the radial coordinate r . Of course, it is always possible to consider the situation in which $g_1(E/E_{Pl}) \equiv g_1(E(r)/E_{Pl})$ and $g_2(E/E_{Pl}) \equiv g_2(E(r)/E_{Pl})$ [31]. However, this goes beyond the purpose of this paper and it will be investigated elsewhere. Note that as in the original model of Dev and Gleiser, $p_r(R) = 0$, only if we allow anisotropy. However, if we take under consideration the relation with ω_- , one can consider the situation in which

$$p_r(R) = \omega_- \rho(R) = \frac{9c^4}{1568\pi G g_2^2(E/E_{Pl}) R^2} \rightarrow 0 \quad (104)$$

when $g_2(E/E_{Pl}) \gg 1$ without invoking a boundary that goes to infinity. As we can see, in this regime, the star seems to behave as dust, because $\omega_- \rightarrow 0$. For completeness, we present also the expansion for small energies where $g_1(E/E_{Pl}) \simeq g_2(E/E_{Pl}) \simeq 1$. For example we can write for ω_+

$$\begin{aligned} \omega_+ &\simeq -c^2 + \frac{8}{3} + \frac{4}{3}\sqrt{4-3c^2} \\ &\quad + \frac{7((3c^2-8)\sqrt{4-3c^2} + 12c^2 - 16)}{9c^2 - 12} (g_2(E/E_{Pl}) - 1) \\ &\quad + O((g-1)^2) \\ &= -c^2 + \frac{8}{3} + \frac{4}{3}\sqrt{4-3c^2} \\ &\quad + \frac{7((3c^2-8)\sqrt{4-3c^2} + 12c^2 - 16)}{9c^2 - 12} \beta + O(\beta^2) \\ &\stackrel{c^2 \rightarrow 1}{=} 3 + 21\beta + O(\beta^2) \end{aligned} \quad (105)$$

and for ω_-

$$\begin{aligned} \omega_- &\simeq -c^2 + \frac{8}{3} - \frac{4}{3}\sqrt{4-3c^2} \\ &\quad - \frac{7((3c^2-8)\sqrt{4-3c^2} - 12c^2 + 16)}{9c^2 - 12} (g_2(E/E_{Pl}) - 1) \\ &\quad + O((g-1)^2) \\ &= -c^2 + \frac{8}{3} - \frac{4}{3}\sqrt{4-3c^2} \\ &\quad - \frac{7((3c^2-8)\sqrt{4-3c^2} - 12c^2 + 16)}{9c^2 - 12} \beta + O(\beta^2) \\ &\stackrel{c^2 \rightarrow 1}{=} \frac{1}{3} - \frac{7}{3}\beta + O(\beta^2), \end{aligned} \quad (106)$$

where we have defined

$$\beta = \left(\frac{dg_2(E/E_{Pl})}{dE} \right)_{|E=0} \frac{1}{E_p}, \quad (107)$$

in analogy with definition (83). As regards the star mass, one can easily verify that

$$m(r) = \frac{3c^2 r}{g_2^2(E/E_{Pl}) 14G}, \quad (108)$$

and at the boundary R one gets

$$M = m(R) = \frac{3c^2 R}{g_2^2(E/E_{Pl}) 14G}. \quad (109)$$

4.1 The redshift function for the variable energy-density case

The mass of the star at the boundary R , Eq. (109), is useful also to determine the redshift factor. Indeed, if we define the compactness of the star as

$$\frac{MG}{Rc^2} = \frac{3}{g_2^2(E/E_{Pl}) 14}, \quad (110)$$

then the surface redshift z corresponding to the above compactness factor is obtained

$$\begin{aligned} z &= \frac{g_1(E/E_{Pl})}{\sqrt{1 - 2MG/Rc^2}} - 1 \\ &= g_1(E/E_{Pl}) \left(1 - \frac{3}{7g_2^2(E/E_{Pl})} \right)^{-\frac{1}{2}} - 1. \end{aligned} \quad (111)$$

It is immediate to see that only the case in which $g_2(E/E_{Pl}) > \sqrt{3/7}$ is allowed, otherwise z would become imaginary. This means that, for an energy-density profile of the form (93), the case in which $g_2(E/E_{Pl}) \leq \sqrt{3/7}$ is automatically excluded. Moreover, if $g_2(E/E_{Pl})$ is very large, we get

$$z \simeq \frac{3g_1(E/E_{Pl})}{14g_2^2(E/E_{Pl})}. \quad (112)$$

Note that when $g_1(E/E_{Pl}) \propto g_2^2(E/E_{Pl})$, then z is approximately a constant. On the other hand, when we consider the situation in which $E \ll E_{Pl}$, one can have small deviations from the undeformed redshift factor $z^* = \sqrt{7}/2 - 1 \simeq 0.32288$. Indeed one finds

$$\begin{aligned} z &\simeq g_1(E/E_{Pl}) \frac{\sqrt{7}}{2} \left(1 - \frac{3}{8}\beta \frac{E}{E_{Pl}} \right) \\ -1 &\simeq \left(1 + \alpha \frac{E}{E_p} \right) \frac{\sqrt{7}}{2} \left(1 - \frac{3}{8}\beta \frac{E}{E_{Pl}} \right) \\ -1 &\simeq z^* + \frac{\sqrt{7}E}{2E_p} \left(\alpha - \frac{3}{8}\beta \right), \end{aligned} \quad (113)$$

with $(\alpha - \frac{3}{8}\beta) \leq 0$, where we have used definitions (83) and (107).

4.2 The redshift function for the Dev–Gleiser energy-density case

The combination of the constant and variable energy-density profile considered in Sects. 3 and 4, is known as the Dev–Gleiser [4] energy-density profile whose expression is

$$\rho(r) = \rho + \frac{A}{r^2}, \quad (114)$$

where we have set $A = 3c^2 / (56\pi G)$. We know that in ordinary GR, Dev–Gleiser solved the TOV equation in the presence of anisotropy showing that the pressureless condition on the boundary could be satisfied. However, in the isotropic case, it is not trivial to find solutions for the TOV equation. Nevertheless, it is again possible to discuss the behavior of the redshift for such a configuration. Indeed, it is immediate to see that Eq. (13) can easily be solved to give

$$m(r) = \int_0^r \frac{4\pi\rho(r')r'^2}{g_2^2(E/E_{Pl})} dr' = \frac{4\pi}{g_2^2(E/E_{Pl})} \left(\frac{\rho r^3}{3} + Ar \right) \tag{115}$$

and the total mass M for a star of radius R is simply

$$M = \frac{4\pi}{g_2^2(E/E_{Pl})} \left(\frac{\rho R^3}{3} + AR \right). \tag{116}$$

To simplify the computation we have considered the case (b) of Sect. 3 where $R \simeq \alpha l_{Pl}$. Then we can define the compactness of the star as

$$\frac{MG}{Rc^2} = \frac{4\pi}{g_2^2(E/E_{Pl})c^2} \left(\frac{\rho R^2}{3} + A \right), \tag{117}$$

the surface redshift z corresponding to the above compactness factor is obtained:

$$\begin{aligned} z &= \frac{g_1(E/E_{Pl})}{\sqrt{1 - 2MG/Rc^2}} - 1 \\ &= g_1(E/E_{Pl}) \left(1 - \frac{8\pi}{g_2^2(E/E_{Pl})c^2} \left(\frac{\rho R^2}{3} + A \right) \right)^{-\frac{1}{2}} - 1. \end{aligned} \tag{118}$$

Even in the Dev–Gleiser profile only the case in which $g_2(E/E_{Pl}) \gg 1$ is allowed, otherwise z would become imaginary. This means that, for an energy-density profile of the form (114), the case in which $g_2(E/E_{Pl}) \ll 1$ is automatically excluded. Instead, if $g_2(E/E_{Pl})$ is very large, we get

$$z \simeq \frac{4\pi g_1(E/E_{Pl})}{g_2^2(E/E_{Pl})c^2} \left(\frac{\rho R^2}{3} + A \right). \tag{119}$$

It is immediate to see that even if $g_1(E/E_{Pl}) \propto g_2^2(E/E_{Pl})$, then z cannot be approximated by a constant as in the previous subsection, because a dependence on the radius of the star R still persists, not having found, for the Dev–Gleiser energy-density profile, a simple analytical expression analogous to (25).

5 Conclusions

In this paper we have considered the effects of gravity’s rainbow on the TOV equations. After having derived the deformed TOV equations, we have focused our attention on two particular simple cases: the constant energy-density profile and the variable energy-density profile, respectively. Since the deformation induced by Gravity’s Rainbow is expected to become more relevant when Planckian energy density is approached, we have considered two specific situations for the constant energy-density profile: the first one deals with a star which has a deformed core and an undeformed external region, that is to say, a two-fluid model. The second one considers a star which is deformed everywhere. Even if it is possible to compute a pressure for the whole star in both situations, due to the complexity of the analytical expressions, we have considered two limiting cases: $g_2(E/E_{Pl}) \rightarrow \infty$ and $g_2(E/E_{Pl}) \rightarrow 0$. For the two-fluid model or case (a) of Sect. 3, only the $g_2(E/E_{Pl}) \rightarrow \infty$ limit has been considered to avoid complex pressures and infinite masses. In this extreme limit, one finds that the central pressure depends on the undeformed mass density and on the boundary \bar{r} where gravity’s rainbow switches off, namely the core is cut off as shown in (42). It is clear that this is the result of a crude approximation and the addition of a dependence on the radius r from $g_1(E/E_{Pl}) \equiv g_1(E(r)/E_{Pl})$ and $g_2(E/E_{Pl}) \equiv g_2(E(r)/E_{Pl})$ [31] could give light to this result. On the other hand, when gravity’s rainbow is applied to the whole star or case (b), we find that the star can survive in the TOV sense and that, due to the g_2 factor, the size on the star does not necessarily become Planckian (60). Even in this case, we do not know if some corrections due to a full quantum gravitational theory can corroborate or destroy the picture. Regarding the redshift factor for both cases (a) and (b), we find that the deformation is induced by $g_1(E/E_{Pl})$ only and there is a deviation that could be detected in principle, even for small values of E . As regards, the variable energy-density profile, we have found that the parameter of the EoS ω cannot be considered as constant but acquires a dependence on E/E_{Pl} . Even for the variable case, we have considered the $g_2(E/E_{Pl}) \rightarrow \infty$ limit, to avoid infinite masses. In this regime, we have found two solutions ω_{\pm} : ω_+ is divergent when $g_2(E/E_{Pl}) \rightarrow \infty$, while $\omega_- \rightarrow 0$, when in the same limit. While ω_+ must be discarded, we can see that ω_- can represent a form of “gravity’s rainbow dust”. It is interesting to note that the vanishing of the pressure at the boundary R is here reached as a limit procedure. Indeed as shown by Dev and Gleiser [4], only if we introduce anisotropy, we can have the exact vanishing of the pressure at the boundary. Regarding the redshift we here find that z depends on both Rainbow functions. As a particular case, one can fix the ideas where $g_1(E/E_{Pl}) \propto g_2^2(E/E_{Pl})$. With this choice, one finds that the redshift factor is almost constant. Almost because, the exact

value $z = 3/14$ is reached when $g_1(E/E_{Pl}) = g_2^2(E/E_{Pl})$ and not simply proportional. The same situation appears also for the Dev–Gleiser potential, where we have only considered the redshift problem since the pressure computation needs a more elaborate scheme. In summary, it seems that the distortion created by gravity’s rainbow on the TOV equation is able to create stars that are really Planckian in density without necessarily being Planckian in size. These “Planck stars” seem to be completely different by the Planck stars proposed by Rovelli and Vidotto [16]. Indeed, for an appropriate choice of the function $g_2(E/E_{Pl})$, the Buchdahl–Bondi bound is satisfied and the collapse never appears. It is clear that the correction due to a dependence on the radial coordinate of the form $g_1(E/E_{Pl}) \equiv g_1(E(r)/E_{Pl})$ and $g_2(E/E_{Pl}) \equiv g_2(E(r)/E_{Pl})$ or a correction induced by a quantum gravitational calculation could considerably improve the present stage of the computation.

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Appendix A: Derivation of the TOV equations in Gravity’s Rainbow

For a static fluid, we can define

$$u^1 = \frac{dr}{d\tau} = 0 \quad u^2 = \frac{d\theta}{d\tau} = 0 \quad u^3 = \frac{d\phi}{d\tau} = 0 \tag{A1}$$

and with the help of the normalization $u_\mu u^\mu = -1$, we can write

$$-1 = -\frac{e^{2\Phi(r)}}{g_1^2(E/E_{Pl})} u^0 u^0 \rightarrow u^0 = \frac{dt}{d\tau} = g_1(E/E_{Pl}) e^{-\Phi(r)}. \tag{A2}$$

For the energy-momentum stress tensor, one finds

$$\begin{aligned} T^{00} &= \rho(r)c^2 g_1^2(E/E_{Pl}) c^{-2} e^{-2\Phi(r)} \\ T^{11} &= g_2^2(E/E_{Pl}) p (1 - b(r)/r) \\ T^{22} &= g_2^2(E/E_{Pl}) p r^{-2} \\ T^{33} &= g_2^2(E/E_{Pl}) p r^{-2} \sin^{-2} \theta, \end{aligned} \tag{A3}$$

and in terms of the mixed tensor, one gets

$$T^0_0 = -\rho(r)c^2 \quad T^1_1 = T^2_2 = T^3_3 = p(r). \tag{A4}$$

Thus from Einstein’s equations ($\kappa = 8\pi G$) we obtain

$$G_{00} = \kappa T_{00} \rightarrow b'(r) = \frac{\kappa \rho(r)c^2 r^2}{c^4 g_2^2(E/E_{Pl})} \tag{A5}$$

and

$$\begin{aligned} G_{11} &= \kappa T_{11} \rightarrow \Phi'(r) \\ &= \frac{\kappa r^3 p(r)/c^2 g_2^2(E/E_{Pl}) + 2Gm(r)}{2r^2 c^2 \left[1 - \frac{2Gm(r)}{rc^2} \right]}. \end{aligned} \tag{A6}$$

From the conservation of the stress-energy tensor $T^{\mu\nu}_{;\nu} = 0$ follows

$$T^{\mu\nu}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\beta\nu} T^{\beta\nu} + \Gamma^\nu_{\nu\beta} T^{\mu\beta} = 0.$$

However, for practical purposes, it is convenient to adopt the mixed stress-energy tensor leading to

$$\begin{aligned} \mu = 0 &\implies \frac{\partial T^0_0(t, r, \theta, \phi)}{\partial t} = 0, \\ \mu = 2 &\implies \frac{\partial T^0_0(t, r, \theta, \phi)}{\partial \theta} = 0 \\ \mu = 3 &\implies \frac{\partial T^0_0(t, r, \theta, \phi)}{\partial \phi} = 0 \end{aligned} \tag{A7}$$

and

$$\mu = 1 \implies \frac{\partial p(r)}{\partial r} + \Phi'(r)(\rho(r)c^2 + p(r)) = 0. \tag{A8}$$

Appendix B: The Dev–Gleiser energy-density profile induced by the ZPE in a Gravity’s Rainbow context

In this section we shall consider the formalism outlined in detail in Refs. [23, 24], where the graviton one loop contribution to a fixed background is used. The latter contribution is evaluated through a variational approach with Gaussian trial wave functionals, and the divergences are taken under control with the help of gravity’s rainbow. We refer the reader to Refs. [23, 24] for details. In ordinary gravity the computation of ZPE for quantum fluctuations of the *pure gravitational field* can be extracted by rewriting the Wheeler–DeWitt equation (WDW) [32] in a form which looks like an expectation value computation [33–35]. We remind the reader that the WDW equation is the quantum version of the classical constraint which guarantees the invariance under time reparametrization. Its original form with the cosmological term included is described by

$$N(r) \rightarrow e^{2\Phi(r)} \quad \text{and} \quad b(r) \rightarrow \frac{2Gm(r)}{c^2}. \tag{B1}$$

$$\mathcal{H}\Psi = \left[(2\kappa) G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa} ({}^3R - 2\Lambda) \right] \Psi = 0. \tag{B2}$$

Note that $\mathcal{H} = 0$ represents one of the classical constraints. The other one is the invariance by spatial diffeomorphism. If we multiply (B2) by $\Psi^* [g_{ij}]$ and functionally integrate over the three spatial metric g_{ij} , we can write³ [33–35]

$$\frac{1}{V} \frac{\int \mathcal{D} [g_{ij}] \Psi^* [g_{ij}] \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} \Psi [g_{ij}]}{\int \mathcal{D} [g_{ij}] \Psi^* [g_{ij}] \Psi [g_{ij}]} = \frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}, \tag{B3}$$

where we have also integrated over the hypersurface Σ and we have defined

$$V = \int_{\Sigma} d^3x \sqrt{g} \tag{B4}$$

as the volume of the hypersurface Σ with

$$\hat{\Lambda}_{\Sigma} = (2\kappa) G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g} {}^3R / (2\kappa). \tag{B5}$$

In this form, (B3) can be used to compute ZPE provided that Λ/κ be considered as an eigenvalue of $\hat{\Lambda}_{\Sigma}$, namely the WDW equation is transformed into an expectation value computation. In Eq. (B2), G_{ijkl} is the super-metric, π^{ij} is the super-momentum,³ R is the scalar curvature in three dimensions and Λ is the cosmological constant, while $\kappa = 8\pi G$ with G the Newton constant. Nevertheless, solving (B3) is a quite impossible task, therefore we are oriented to use a variational approach with trial wave functionals. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case, are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that ZPE should be described by a good candidate of the “vacuum state”. To fix the ideas, a variant of the line element (7) will be considered

$$ds^2 = -N^2(r) \frac{dt^2}{g_1^2(E/E_{Pl})} + \frac{dr^2}{\left(1 - \frac{b(r)}{r}\right) g_2^2(E/E_{Pl})} + \frac{r^2}{g_2^2(E/E_{Pl})} \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \tag{B6}$$

where N is the lapse function and $b(r)$ is subject to the only condition $b(r_t) = r_t$. For instance, For the Schwarzschild case, we find $b(r) = 2MG = r_t$. For the de Sitter case (dS), ons gets $b(r) = \Lambda_{dS} r^3/3$ and for the anti-de Sitter (AdS) case

one gets $b(r) = -\Lambda_{AdS} r^3/3$. The graviton contribution of (B3) is

$$\frac{\Lambda}{8\pi G} = -\frac{1}{3\pi^2} \sum_{i=1}^2 \int_{E^*}^{+\infty} E_i g_1(E/E_{Pl}) g_2(E/E_{Pl}) \times \frac{d}{dE_i} \sqrt{\left(\frac{E_i^2}{g_2^2(E/E_{Pl})} - m_i^2(r) \right)^3} dE_i, \tag{B7}$$

where E^* is the value which annihilates the argument of the root and where we have defined two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{3}{2r^2} b'(r) - \frac{3}{2r^3} b(r) \\ m_2^2(r) = \frac{6}{r^2} \left(1 - \frac{b(r)}{r} \right) + \frac{1}{2r^2} b'(r) + \frac{3}{2r^3} b(r) \end{cases} \quad (r \equiv r(x)). \tag{B8}$$

We refer the reader to Refs. [23, 24] for the deduction of these expressions. It is immediate to recognize that the induced cosmological constant is no longer a constant but is induced by quantum fluctuations with the help of Eq. (B7). Therefore, if we make the following identification:

$$\rho(r) = \frac{\Lambda(r)}{8\pi G} \tag{B9}$$

we have the possibility to probe different energy-density profiles induced by quantum fluctuations of the gravitational field itself. To be more explicit, we choose [24]:

$$g_1(E/E_{Pl}) = \left(1 + \beta \frac{E}{E_{Pl}} + \delta \frac{E^2}{E_{Pl}^2} + \gamma \frac{E^3}{E_{Pl}^3} \right) \exp\left(-\alpha \frac{E^2}{E_{Pl}^2} \right) \\ g_2(E/E_{Pl}) = 1. \tag{B10}$$

We can recognize two relevant cases:

- (a) $m_1^2(r) = -m_2^2(r) = m_0^2(r)$,
- (b) $m_1^2(r) = m_2^2(r) = m_0^2(r)$.

When condition (a) is satisfied, this means that we are describing the Schwarzschild, Schwarzschild–de Sitter and Schwarzschild–anti de Sitter cases in proximity of the throat. On the other hand, when condition (b) is satisfied, we are describing the Minkowski, de Sitter and anti-de Sitter cases. For our purposes, the case (b) is the most significant, especially if we fix our attention to the de Sitter case which, in static coordinates is simply described by $b(r) = \Lambda_{dS} r^3/3$. In this situation the effective masses of (B8) take the form

$$m_1^2(r) = m_2^2(r) = \frac{6}{r^2} - \Lambda_{dS}, \quad r \in (0, r_C] \tag{B11}$$

³ See also Ref. [36] for an application of the method to a $f(R)$ theory.

with $r_C = \sqrt{3/\Lambda_{dS}}$. Defining the dimensionless variable

$$x = \frac{L_P}{r} \sqrt{6 - \Lambda_{dS} r^2}, \tag{B12}$$

we can use the following expression:

$$\frac{\Lambda}{8\pi G E_P^4} = C_1 + C_2 x^2 + \left[C_3 - \frac{1}{8\pi^2} \ln(x^2 \alpha/4) \right] x^4 + O(x^5), \tag{B13}$$

which is valid for $x \ll 1$. Assuming $r \gg L_P$ and $\Lambda r^2 = O(1)$, one gets at the leading order

$$\begin{aligned} \frac{\Lambda}{8\pi G E_P^4} &= C_1 + C_2 \left(\frac{L_P}{r} \right)^2 (6 - \Lambda_{dS} r^2) \\ &= C_1 - 6C_2 \Lambda_{dS} L_P^2 + 6C_2 \frac{L_P^2}{r^2}, \end{aligned} \tag{B14}$$

where

$$C_1 = \frac{-8\alpha^{3/2} - 6\sqrt{\pi}\alpha\beta - 15\sqrt{\pi}\gamma - 16\sqrt{\alpha}\delta}{8\pi^2\alpha^{7/2}}, \tag{B15}$$

$$C_2 = + \frac{4\alpha^{3/2} + 2\sqrt{\pi}\alpha\beta + 3\sqrt{\pi}\gamma + 4\sqrt{\alpha}\delta}{8\pi^2\alpha^{5/2}} \tag{B16}$$

and

$$C_3 = \frac{-\alpha^{3/2} - 2\gamma_E\alpha^{3/2} + 2\sqrt{\pi}\alpha\beta + \sqrt{\pi}\gamma + 2\sqrt{\alpha}\delta}{16\pi^2\alpha^{3/2}}. \tag{B17}$$

Because of the identification (B9), we have obtained a Dev–Gleiser-like energy-density profile.

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