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Well posedness and regularity for heat equation with the initial condition in weighted Orlicz–Slobodetskii space subordinated to Orlicz space like $\lambda(\log\lambda)^{\alpha}$ and the logarithmic weight

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Abstract We consider the initial-value problem $\tilde{u}_t = \Delta_x \tilde{u}(x,t)$, $\tilde{u}(x,0) = u(x)$, where $x \in \mathbb{R}^{n-1}$, $t \in (0,T)$ and u belongs to certain weighted Orlicz–Slobodetskii space $Y^{\Phi,\Phi}_{log}(\mathbb{R}^{n-1})$ subordinated to the logarithmic weight. We prove that under certain assumptions on Orlicz function Φ, the solution $\tilde{u}$ belongs to Orlicz–Sobolev space $W^{1,\Psi}(\Omega \times (0,T))$ for certain function Ψ which in general dominates Φ. The typical representants are $\Phi(\lambda) = \lambda(\log(2+\lambda))^{\alpha}$, $\Psi(\lambda) = \lambda(\log(2+\lambda))^{\alpha+1}$ where $\alpha > 0$.

Keywords Orlicz–Sobolev spaces · Orlicz–Slobodetskii spaces · Heat equation · Evolution problems

Mathematics Subject Classification Primary 35K05 · 35K15; Secondary 46E35 · 26D10

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1 Introduction

The purpose of this paper is to study the initial-value problem for the heat equation:

$$\begin{cases} \tilde{u}_t(x,t) = \Delta_x \tilde{u}(x,t) & \text{in } \mathbb{R}^{n-1} \times (0,T), \\ \tilde{u}(x,0) = u & \text{for } x \in \mathbb{R}^{n-1}, \end{cases} \tag{1.1}$$

where the initial function u lies in the completion of Lipschitz functions in certain weighted Orlicz–Slobodetski type space denoted by $Y_{log}^{\Phi,\Phi}(\Omega)$, Φ is a N-function. It consists of all $v \in L^{\Phi}(\Omega)$ (the Orlicz space generated by Φ), for which the seminorm

$$I_{log}^{\Phi}(v) := \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \Phi\left(\frac{|v(x)-v(y)|}{|x-y|}\right) \frac{\ln(|x-y|)}{|x-y|^{n-2}} dxdy \tag{1.2}$$

is finite.

As our main result formulated in Theorem 7.2, we prove that if Φ satisfies certain assumptions (Assumption B in Definition 6.1), then the solution $\tilde{u}$ of (1.1) lies in the Orlicz–Sobolev space $W^{1,\Psi}(\mathbb{R}^{n-1} \times (0,T))$, i. e. $\tilde{u}$, together with its all first order partial derivatives belongs to $L^{\Psi}(\mathbb{R}^{n-1} \times (0,T))$, where Ψ is in a sense conjugate to Φ (see Definition 2.1). The natural representative pair of admitted functions would be functions generating the logarithmic Zygmund spaces: $\Phi(\lambda) = \lambda(\ln(2+\lambda))^{\alpha}$ and $\Psi(\lambda) = \lambda(\ln(2+\lambda))^{\alpha+1}$, where $\alpha > 0$. They cannot grow to fast. The conjugate of Φ is equivalent to $\exp t^{1/\alpha}$ near infinity and does not satisfy the Δ_2-condition. Logarithmic Orlicz spaces are of particular interest in functional analysis, see, e.g. [4,14,19,21].

For regularity results dealing with the initial data in the classical Besov spaces $B_p^{\alpha,q}(\Omega)$ we refer to papers: [20,43,51,58–60] and to their references.

Our motivation to ask about regularity in the Orlicz setting comes from the fact that many mathematical models in the nonlinear elliptic and parabolic PDEs arising from the mathematical physics seem to have a good interpretation only when stated in Orlicz framework, see e.g. [3,6,15,16,37,54]. Moreover, not much is known about regularity established for the heat equation with initial data in the Orlicz–Slobodetskii-type spaces, even in the nonweighted ones, where N-function Φ is essentially different than λ^p. We focus on the paper [30] for an approach with an initial condition in the Orlicz space. For another result in this direction we refer to our recent paper [24], where we have proven that if Orlicz function R satisfies certain assumptions (Assumption B from Definition 6.1), then the solution $\tilde{u}$ of (1.1) lies in the Orlicz–Sobolev space $W^{1,R}(\Omega \times (0,T))$. The difference between an approach presented here and our previous one is that now we provide the estimates between Orlicz–Sobolev-type spaces and Orlicz–Slobodetskii-type spaces defined by the possibly different Orlicz functions Φ and Ψ. This motivated us to consider weighted Orlicz–Slobodetskii setting, while our previous analysis did not require weights.

Weighted Sobolev–Slobodetskii spaces have appeared in many issues, stating from classical literature mostly by the Russian school: [42] (which appeared before fundamental paper by Slobodetskii [50]), [31, Section 9] [35,39,47,56,57]. Those works were involving measures being the power of the distance from the boundary: $\mathrm{dist}(x, \partial\Omega)^\alpha$ See also books [32,38,53] and later works [1,22,26,36,40,41,46,55].

Weighted Orlicz–Slobodetskii spaces involving general weights have appeared in old papers by Lacroix [34] and Palmieri [45]. Recently first author and Dhara [11,12] were investigating properties of the extension operator from Orlicz–Slobodetski type space to Orlicz–Sobolev space in the weighted setting. See also [7,8,29,32,38], for some interesting related results.

Except the standard arguments based on Young and Jensen's inequalities and the a priori estimates, we propose an approach based on obtained here pointwise estimates for the time-maximal functions of $\tilde{u}$ and its first order derivatives (see Lemmas 4.4, 5.3 and 6.4) and Stein type theorem due to Kita, see Theorem 4.1 obtained in [28].

The estimates for solutions of the heat equation are of interest to many mathematicians form various branches of mathematics including the probability theory and analysis on metric spaces, see e.g. [2,5,10,13,17,18,44,48], and the references therein.

2 Notation and preliminaries

2.1 Basic notation

Let $\Omega \subset \mathbb{R}^n$ be an open set. By $C^\infty(\bar{\Omega})$ we mean set of functions which have smooth extension to certain open neighborhood of $\bar{\Omega}$. If f is defined on a set $A \subseteq \mathbb{R}^n$, by $f\chi_A$ we mean the function f extended by 0 outside A. Let $Mh(t_0) := \sup_{t>0} \frac{1}{2r}\int_{(t_0-r,t_0+r)} h(t)\,dt$ be the Hardy–Littlewood maximal function of $h \in L^1_{loc}(\mathbb{R})$ [52]. We will be also dealing with the time-directional maximal function of function $w \in L^1_{loc}(\mathbb{R}^{n-1} \times [0,\infty))$, the function

$$M^2 w(x, t_0) := \sup_{t>0} \frac{1}{2r} \int_{(t_0-r,t_0+r)} w(x,t)\chi_{\mathbb{R}^{n-1}\times(0,\infty)}\,dt \tag{2.1}$$

(defined for almost every (x, t_0)). Having to norms $\|\cdot\|$ and $\|\cdot\|_1$ defined on a Banach space X, we will write $\|\cdot\| \sim \|\cdot\|_1$ if norm $\|\cdot\|$ is equivalent to $\|\cdot\|_1$ on X. Having two functions Φ, Ψ defined on $[0,\infty)$ we will say that Ψ dominates Φ ($\Phi \prec \Psi$) if there exist constants $C_1, C_2 > 0$ such that $\Phi(x) \le C_1\Psi(C_2 x)$ for every $x > 0$. Functions Φ, Ψ are called equivalent if $\Psi \prec \Phi$ and $\Phi \prec \Psi$.

The notation "$\lesssim$" will be used in usual manner, namely, if $\Phi, \Psi : \mathcal{A} \to \mathbb{R}$ are given functions, where $\mathcal{A}$ is some abstract domain (it can be either a subset of Euclidean space, as well as a set of functions), we will write that $\Phi \lesssim \Psi$ if there is a constant $C > 0$ such that $\Phi(a) \le C\Psi(a)$, for every $a \in \mathcal{A}$. When $n \in \mathbf{N}$, we denote: $Q' = [0,1]^{n-1}$, $Q = [0,1]^n = Q' \times (0,1)$. By $Lip(\Omega)$ we denote space of Lipschitz functions defined on the set $\Omega \subseteq \mathbb{R}^n$, while by $Lip_0(\Omega)$ we denote those elements of $Lip(\Omega)$ which have compact support in Ω. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then x' will stand for $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. By $\ln x$ we denote the natural logarithm of a positive number x.

2.2 Orlicz, Orlicz–Sobolev and Orlicz–Slobodetskii spaces

2.2.1 Orlicz space

When $\Psi : [0, \infty) \to [0, \infty)$ is a nondecreasing convex continuous function such that $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = +\infty$, the space

$$L^{\Psi}(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \int_{\Omega} \Psi(s|f(x)|)\, dx < \infty \text{ for some } s > 0 \right\}$$

is called *Orlicz space* (see e.g. [49]). It is a Banach space equipped with the *Luxemburg norm*:

$$\|f\|_{L^{\Psi}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \Psi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\}.$$

As is well known, when $\Psi(\lambda) = \lambda^p$ and $p \ge 1$, then $L^{\Psi}(\Omega) = L^p(\Omega)$ is the usual Lebesgue space. The same notation will be used for vector functions, $u : \Omega \to \mathbb{R}^m$, with the formal difference that instead of $|u(x)|$ we shall work with the Euclidean norm of the vector $u(x)$.

We shall write that $\Psi \in \Delta_2$ if it satisfies the Δ_2-condition: $\Psi(2\lambda) \le C\Psi(\lambda)$, for every $\lambda > 0$, with a constant C independent of λ. Symbol $\Psi \in \Delta_2^c$ will mean that the *Legendre conjugate* of Ψ, that is, $\Psi^*(s) := \sup_{t>0}\{st - \Psi(t)\}$, satisfies the Δ_2-condition.

We will be using the following statement (see e.g. [9, Proposition 2]).

Proposition 2.1 *Let M be a Young function and (X, μ) be the measurable space equipped with the measure μ. Then the expression*

$$\|f\|_{L^{\Psi}(X,\mu),\alpha} := \inf \left\{ \lambda > 0 : \int_{X} \Psi\left(\frac{|f(x)|}{\lambda} \right) \mu(dx) \le \alpha \right\}.$$

defines a complete norm on

$$L^{\Psi}(X, \mu) := \left\{ f \in L^1_{loc}(X) : \int_{\Omega} \Psi(s|f(x)|)\, \mu(dx) < \infty \textit{ for some } s > 0 \right\}$$

for each $\alpha \in (0, \infty)$. Moreover, all norms $\|\cdot\|_{L^{\Psi}(X,\mu),\alpha}$, $\alpha \in (0, \infty)$ are equivalent.

2.2.2 Orlicz–Sobolev space

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded domain, $k \in \mathbf{N}$, and $\Psi : [0, \infty) \to [0, \infty)$ be a nondecreasing convex continuous function such that $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = +\infty$. The *Orlicz–Sobolev space* $W^{k,p}(\Omega)$ is the linear set

$$\{u \in L^1_{loc}(\Omega) : D^{\alpha}u \in L^{\Psi}(\Omega) \text{ for every } \alpha : |\alpha| \le k\} \tag{2.2}$$

equipped with the norm:

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{\alpha:|\alpha|\le k} \|D^\alpha u\|_{L^\Psi(\Omega)}.$$

Here $D^\alpha u$ means the distributional derivative of u.

We define the space $W^{k,\Phi}_\infty(\Omega)$, (respectively $W^{k,\Phi}_L(\Omega)$) as the completion of $C^\infty(\bar\Omega)$ (respectively $Lip(\Omega)$) in the norm of the space $W^{k,\Phi}(\Omega)$.

2.2.3 Orlicz–Slobodetskii space $Y^{\Psi,\Phi}$

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded domain, $\Psi, \Phi : [0,\infty) \to [0,\infty)$ be nondecreasing convex continuous functions such that $\Psi(0) = \Phi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \lim_{t\to\infty} \Phi(t) = +\infty$. By $Y^{\Psi,\Phi}(\Omega)$ we denote the space of all $u \in L^\Psi(\Omega)$, for which the seminorm

$$I^\Phi(u,\Omega) := \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{dxdy}{|x-y|^{n-1}} \tag{2.3}$$

is finite. We equip it with the norm:

$$\|u\|_{Y^{\Psi,\Phi}(\Omega)} := \|u\|_{L^\Psi(\Omega)} + J^\Phi(u,\Omega),$$

where

$$J^\Phi(u,\Omega) := \inf\left\{\lambda > 0 : I^\Phi\left(\frac{u}{\lambda},\Omega\right) \le 1\right\}$$

is the Luxemburg-type seminorm.

Analogously one can define $Y^{\Psi,\Phi}(u,M)$, $I^\Phi(u,M)$, and $J^\Phi(u,M)$, where $M \subseteq \mathbb{R}^k$ is an arbitrary n-dimensional rectifiable set ($n \le k$) and instead of the Lebesgue measure we consider the n-dimensional Hausdorff measure $\mathcal{H}^n$ defined on M.

By $Y^{\Psi,\Phi}_\infty(\Omega)$ (respectively $Y^{\Psi,\Phi}_L(\Omega)$) we will mean the completion of set

$$C^\infty(\bar\Omega) \cap Y^{\Psi,\Phi}(\Omega) \text{ (respectively } Lip(\Omega) \cap Y^{\Psi,\Phi}(\Omega))$$

with respect to the norm $\|u\|_{Y^{\Psi,\Phi}(\Omega)}$. Analogously, if $M \subseteq \mathbb{R}^k$ is an arbitrary n-dimensional rectifiable set ($n \le k$), by $Y^{\Psi,\Phi}_\infty(M)$ (resp. $Y^{\Psi,\Phi}_L(M)$) we mean the completion of set

$$C^\infty(M) \cap Y^{\Psi,\Phi}(M) \text{ (respectively } Lip(M) \cap Y^{\Psi,\Phi}(M))$$

in the norm $Y^{\Psi,\Phi}(M)$, where by $C^\infty(M)$ we mean the set of functions which are defined on certain neighborhood of M in $\mathbb{R}^k$ and are smooth as functions on $\mathbb{R}^k$.

Remark 2.1 Let $\Omega \subseteq \mathbb{R}^n$ be an open domain with locally Lipschitz boundary, $\Psi(\lambda) = \Phi(\lambda) = |\lambda|^p$, $1 < p < \infty$. Then

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \sim \|u\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{p+n-2}}\,dx\,dy\right)^{1/p},$$

which is the norm of u in the Slobodetskii space $W^{1-\frac{1}{p},p}(\partial\Omega)$, see e.g. [33].

2.3 Basic assumptions

In the sequel we will be dealing with the following assumptions comming from papers by Kita [27,28].

Definition 2.1 *(Assumption A (Kita pair))* We assume that $a, b : [0,\infty) \to [0,\infty)$ are strictly positive continuous functions such that

(a) $\int_0^1 a(s)/s\,ds < \infty$, $\int_1^\infty \frac{a(s)}{s}\,ds = +\infty$;
(b) $b(\cdot)$ is non-decreasing, $\lim_{s\to\infty} b(s) = +\infty$.
(c) there exist constants $c_1 > 0$, $s_0 \geq 0$ such that

$$\int_0^s \frac{a(t)}{t}\,dt \leq c_1 b(c_1 s) \quad \text{for all} \quad s > s_0, \tag{2.4}$$

and in the case $s_0 > 0$ mapping $s \mapsto \frac{a(s)}{s}$ is bounded when $s \neq 0$ is near to 0.

We define

$$\Phi(t) := \int_0^t a(s)\,ds \quad \text{and} \quad \Psi(t) := \int_0^t b(s)\,ds, \text{ where } t \geq 0. \tag{2.5}$$

Remark 2.2 We always have $\Phi(t) \leq \widetilde{C}\Psi(C_1 t)$, with universal constants, so that $\Phi \prec \Psi$. Moreover, the following three statements (a), (b), (c) are equivalent (see e.g. Proposition 5.1 in [25]):

(a) $\Psi \in \Delta_2^c$
(b) Φ and Ψ are equivalent Orlicz functions,
(c) $\Phi \in \Delta_2^c$.

Example 2.1 Simple computation shows that pair $(\Phi(\lambda), \Psi(\lambda)) = (\lambda(\log(2+\lambda))^\alpha, \lambda(\log(2+\lambda))^{\alpha+1})$ where $\alpha > 0$ is a Kita pair.

2.4 Trace operator

Let us briefly recall basic claims from [23, Theorems 3.10 and 3.13]. The original formulation holds with $u \in C^\infty(\bar{\Omega})$ however, the proof follows by the same arguments with no difference for $u \in Lip(\Omega)$ as well.

Theorem 2.1 ([23], embedding theorem, inequalities in modulars) *Let Assumption A be satisfied (see Definition 2.1) and let Ω be a bounded domain with locally Lipschitz boundary. Then we have*

(i) *If* (2.4) *holds with* $s_0 = 0$*, then for every* $u \in Lip(\Omega)$*,*

$$\int_{\partial\Omega} \Psi(|u(x)|)\mathcal{H}^{n-1}(dx) + I^{\Phi}(u, \partial\Omega) \leq C\left(\int_{\Omega} \Psi(C_1|u(x)|)\,dx + \int_{\Omega} \Psi(C_2|\nabla u(x)|)\,dx\right), \tag{2.6}$$

with constants C, C_1, C_2 *independent of* u*.*

(ii) *If* (2.4) *holds with some* $s_0 > 0$*, then for every* $u \in Lip(\Omega)$*,*

$$\int_{\partial\Omega} \Psi(|u(x)|)\mathcal{H}^{n-1}(dx) + I^{\Phi}(u, \partial\Omega) \leq C\left(\int_{\Omega} \Psi(C_1|u(x)|)\,dx + \int_{\Omega} \Psi(C_2|\nabla u(x)|)\,dx + \int_{\Omega} |\nabla u(x)|\,dx\right), \tag{2.7}$$

with constants C, C_1, C_2 *independent of* u*.*

Theorem 2.2 ([23], embedding theorem, inequalities in norms) *Let Assumption A be satisfied (see Definition* 2.1*) and let* Ω *be a bounded domain with a locally Lipschitz boundary. Then there exists a constant* D *such that for every* $u \in Lip(\Omega)$ *we have*

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \leq D\|u\|_{W^{1,\Psi}(\Omega)}. \tag{2.8}$$

2.4.1 Trace operator

Let the assumptions in Theorems 2.1 and 2.2 be satisfied and let $u \in W^{1,\Psi}_{Lip}(\Omega)$. Consider any sequence $u_m \in L(\bar{\Omega})$ convergent to u in the norm of the space $W^{1,\Psi}(\Omega)$. Then $\{u_m\}$ is a Cauchy sequence in $Y^{\Psi,\Phi}(\partial\Omega)$ (convergence in the norm), so that it converges some element $\bar{u} \in Y^{\Psi,\Phi}_L(\partial\Omega)$. It is easy to observe that $\bar{u}$ is independent of the choice of the sequence $\{u_m\} \subseteq L(\bar{\Omega})$, converging to u. It allows to extend the standard definition of the trace operator:

$$\text{Tr}\ u := \lim_{m\to\infty} u_m = \bar{u} \in Y^{\Psi,\Phi}_L(\partial\Omega), \tag{2.9}$$

where the convergence holds in the norm of $Y^{\Psi,\Phi}(\partial\Omega)$.

As a consequence we obtain the following result.

Theorem 2.3 ([23], embedding theorem) *Let Assumption A be satisfied (see Definition* 2.1*) and let* Ω *be a bounded domain with a locally Lipschitz boundary. Then trace operator* $\text{Tr} : W^{1,\Psi}_L(\Omega) \mapsto Y^{\Psi,\Phi}_L(\partial\Omega)$ *is well defined by expression* (2.9) *and there exists a constant* D *such that for every* $u \in W^{1,\Psi}_L(\Omega)$ *we have*

$$\|\,\text{Tr}\ u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \leq D\|u\|_{W^{1,\Psi}(\Omega)}. \tag{2.10}$$

3 Heat kernel estimates

Let $u \in Lip(\partial\Omega)$. We will define function $\widetilde{u} \in Lip(\Omega)$ such that $\mathrm{Tr}\,\widetilde{u} = u$ using the Gaussian kernel. Let

$$\widetilde{E}(s,t) := \frac{1}{2^{n-1}(\pi t)^{(n-1)/2}} \exp(-s^2/4t),\ s \geq 0, t > 0$$
$$\text{and } E(x,t) := \widetilde{E}(|x|,t),\ x \in \mathbb{R}^{n-1},\ t > 0,$$

be the heat kernel. Then E obeys the following properties

1.
$$\int_{\mathbb{R}^{n-1}} E(x,t)\,dx = 1 \quad \text{for every } t > 0,$$

2.
$$\lim_{t\to 0} \int_{\mathbb{R}^{n-1}} \phi(x)E(x,t)\,dx = \phi(0) \quad \text{for every } \phi \in Lip_0(\mathbb{R}^{n-1}),$$

3.
$$E_t(x,t) = \Delta_x E(x,t).$$

We will start our construction with the case when $\Omega = Q$, and assuming that $u \in Lip_0(Q')$.

We define

$$\widetilde{u}(x,t) = (e^{-t\Delta}u)(x) := \begin{cases} \int_{\mathbb{R}^{n-1}} u(y)E(x-y,t)\,dy = (E(\cdot,t) * u)(x) & \text{when } t > 0, \\ u(x) & \text{when } t = 0, \end{cases} \tag{3.1}$$

where $g * u$ is the usual convolution. We have the following observation.

Lemma 3.1 *For any $u \in Lip_0(Q')$ we have $\widetilde{u} \in Lip(\bar{Q})$. In particular* $\mathrm{Tr}\,\widetilde{u} = u$.

The remaining estimates and our final result will be established in several steps presented in in the sequel.

4 Estimates of function $\widetilde{u}$

4.1 Presentation of main results

We start with the following result.

Proposition 4.1 ([23]) *Let $u \in Lip(\mathbb{R}^{n-1})$ be supported in Q', and $\widetilde{u}$ be defined by* (3.1). *Then for any $\lambda > 0$ and any convex function $R : [0,\infty) \to [0,\infty)$ and for any $T > 0$, we have*

$$\int_0^T \int_{Q'} R\left(\frac{|\widetilde{u}(x',t)|}{\lambda}\right) dx'\, dt \le T \int_{\mathbb{R}^{n-1}} R\left(\frac{|u(x')|}{\lambda}\right) dx'. \tag{4.1}$$

In particular, if R is a N-function, we have $\|\widetilde{u}\|_{L^R(Q)} \le \|u\|_{L^R(Q')}$.

Our goal is to show that in some cases the function $\widetilde{u}$ can have better integrability properties than u. As main result of this section we obtain the following lemma.

Lemma 4.1 *Let $\widetilde{u}$ be given by (3.1), where $u \in Lip_0(Q')$. Moreover, let (Φ, Ψ) be as in Assumption A (see Definition 2.1) and suppose that Φ satisfies the following estimate:*

$$\Phi(xy) \le D(1 + G(x) + G(x)\Phi(y)), \quad \textit{with } D \textit{ independent of } x, y, \tag{4.2}$$

and continuous, nondecreasing function G such that $\int_0^1 G(c|\ln t|)\, dt < \infty$ for any constant $c > 0$. Then

$$\int_{(x,t)\in Q} \Psi\left(\frac{|\widetilde{u}(x,t)|}{\lambda}\right) dxdt \lesssim 1 + \int_{Q'} \Phi\left(\frac{C|u(x)|}{\lambda}\right) dx, \textit{ for every } \lambda > 0, \tag{4.3}$$

$$\|\widetilde{u}\|_{L^\Psi(Q'\times(0,1))} \lesssim \|u\|_{L^\Phi(Q')}. \tag{4.4}$$

where constant $C > 0$ above is independent of u.

Remark 4.1 1. The condition (4.2) implies that Φ satisfies Δ_2-condition near infinity, i.e. *there exists $N > 0$ and $C > 0$ such that $\Phi(2y) \le C\Phi(y)$ whenever $y > N$.*
2. Assume that $\Phi(s) = s^\alpha h(|\ln s|)$ where h is continuous, positive for positive arguments, nondecreasing and satisfies Δ_2-condition near ∞ (in the sense of definition given above). Then

$$\Phi(xy) \lesssim 1 + G(x) + G(x)\Phi(y), \text{ where } G(x) = \max\{x^\alpha, \Phi(x))\}.$$

Indeed, this follows from the following estimates:

$$\begin{aligned}\Phi(xy) &= x^\alpha y^\alpha h(|\ln x| + |\ln y|) \le x^\alpha y^\alpha\, (h(2|\ln x|) + h(2|\ln y|))\\ &\lesssim x^\alpha y^\alpha (1 + h(|\ln x|) + h(|\ln y|))\\ &= y^\alpha \Phi(x) + x^\alpha \Phi(y) + x^\alpha y^\alpha = y^\alpha(\Phi(x) + x^\alpha) + x^\alpha \Phi(y)\\ &\lesssim G(x) y^\alpha + G(x)\Phi(y)\\ &\lesssim G(x) + G(x)\Phi(y).\end{aligned}$$

3. In the original formulations of Lemmas 4.1 and 4.1 we assume $u \in C^\infty$ instead of u being Lipschitz. The statements we formulate now hold true without any changes in their proofs.

4.2 Proof of Lemma 4.1

The proof will be proceeded by the known results of Kita [28, Theorems 2.1 and 2.7] and sequence of lemmas where we derive certain pointwise estimates.

Theorem 4.1 ([28]) *Let $a(s), b(s), \Phi(t), \Psi(t)$ be as in Assumption A (see Definition 2.1), with $s_0 \geq 0$. Then there exist constants $C_1, C_2, C_3 > 0$ such that*

$$\int_{\mathbb{R}^n} \Psi(C_3|f(x)|)\,dx \leq C_1 s_0 \int_{\mathbb{R}^n} |f(x)|\,dx + C_2 \int_{\mathbb{R}^n} \Phi(Mf(x))\,dx,$$

for all $f \in L^1(\mathbb{R}^n)$, where $Mf(x)$ is the Hardy–Littlewood maximal function of $f(x)$.

Lemma 4.2 *Let $\beta > -2$ and*

$$F_\beta(x) = x \int_x^\infty q^\beta e^{-q}\,dq. \tag{4.5}$$

Then there exist constants C_0, C_1 (depending on β) such that for every $y \in \mathbb{R}_+$ we have

$$\sup_{\{x \leq y\}} F_\beta(x) \leq C_0 \chi_{y > C_1} + F_\beta(y) \chi_{y \leq C_1}.$$

Proof Function F_β is increasing up to certain x_β and it is bounded. Therefore inequality follows with $C_0 = \sup F < \int_0^\infty q^{\beta+1} e^{-q}\,dq$, $C_1 = x_\beta$. □

Lemma 4.3 *Let $s \in \mathbb{R}_+$, $t > 0, \alpha > 0$ and*

$$\widetilde{E}_\alpha(s,t) := \frac{1}{t^\alpha} \exp(-s^2/4t),$$
$$B_\alpha(s, t_0, r) := \frac{1}{2r} \int_{(t_0 - r, t_0 + r)} \widetilde{E}_\alpha(s,t) \chi_{t>0}\,dt.$$

Then for any $s, t_0, r > 0$ we have

$$\sup_{r>0} B_\alpha(s, t_0, r) \lesssim \widetilde{E}_\alpha\left(s, \frac{3}{2} t_0\right) + \frac{1}{s^{2\alpha}} F_{\alpha-2}\left(\frac{s^2}{8t_0}\right) + \frac{1}{s^{2\alpha}} \chi_{t_0 < \frac{s^2}{8C_1}},$$

where $F_\beta(x)$ is given by (4.5), C_1 is the same as in Lemma 4.2.

Proof We start with the case $0 < r < \frac{t_0}{2}$. Let us find the negative integer $k \leq -2$ such that $t_0 2^k \leq r \leq t_0 2^{k+1}$, so that for $B = B_\alpha(s, t_0, r)$ we have

$$\begin{aligned} B &\lesssim \frac{1}{2r} \int_{(t_0 - r, t_0 + r)} \frac{1}{t^\alpha} e^{-\frac{s^2}{4t}}\,dt \\ &\lesssim \frac{1}{2^k t_0} \cdot e^{-\frac{s^2}{4(\frac{3}{2} t_0)}} \cdot t_0^{-\alpha} |(t_0 - r, t_0 + r)| \lesssim \widetilde{E}_\alpha\left(s, \frac{3}{2} t_0\right), \end{aligned}$$

because when $t \in (t_0 - r, t_0 + r)$ we have $\frac{t_0}{2} \le t \le \frac{3t_0}{2}$ and $|(t_0 - r, t_0 + r)| = 2r \le 2^{k+2} t_0$.

In case $\frac{t_0}{2} \le r < t_0$, we have (we change variables, substituting $q = \frac{s^2}{4t}$)

$$B \lesssim \left(\frac{1}{2t_0} \int_0^{2t_0} \frac{1}{t^\alpha} e^{-\frac{s^2}{4t}} \, dt \right)$$
$$\lesssim \frac{1}{s^{2\alpha}} \left(\frac{s^2}{8t_0} \right) \int_{\frac{s^2}{8t_0}} q^{\alpha-2} e^{-q} \, dq = \frac{1}{s^{2\alpha}} F_{\alpha-2} \left(\frac{s^2}{8t_0} \right).$$

Finally, if $r \ge t_0$, we proceed similarly as above, to get

$$B \lesssim \frac{1}{2r} \int_0^{2r} \frac{1}{t^\alpha} e^{-\frac{s^2}{4t}} \, dt$$
$$\lesssim \frac{1}{s^{2\alpha}} F_{\alpha-2} \left(\frac{s^2}{8r} \right),$$

then we use Lemma 4.2 (note that $\frac{s^2}{8r} \le \frac{s^2}{8t_0} =: y$) to get in this case

$$B \lesssim \frac{1}{s^{2\alpha}} \chi_{\frac{s^2}{8C_1} > t_0} + \frac{1}{s^{2\alpha}} F_{\alpha-2} \left(\frac{s^2}{8t_0} \right) \chi_{\frac{s^2}{8C_1} \le t_0}.$$

Therefore the estimate holds in all considered cases. □

Lemma 4.4 *Let $u \in L^1(Q')$, $Q' = [0,1]^{n-1}$, and $\widetilde{u}$ be given by (3.1), M^2 is as in (2.1). Then there exists constant $C_1 > 0$ such that for any $t_0 > 0$ we have*

$$M^2(\widetilde{u}\chi_{t>0})(x, t_0) \lesssim (\mathcal{T}_0 u)\left(x, \frac{3}{2} t_0\right) + (\mathcal{T}_1 u)(x, 2t_0) + (\mathcal{T}_2 u)(x, 2C_1 t_0),$$

where

$$(\mathcal{T}_0 u)(x,t) := \int_{\mathbb{R}^{n-1}} \tilde{E}(|x-y|, t) |u(y)| \, dy,$$
$$(\mathcal{T}_1 u)(x,t) := \int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^{n-1}} F_{\frac{n-5}{2}} \left(\frac{|x-y|^2}{4t} \right) |u(y)| \, dy,$$
$$(\mathcal{T}_2 u)(x,t) := \int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^{n-1}} \chi_{t < \frac{|x-y|^2}{4}} |u(y)| \, dy.$$

The estimate holds with some constant which is independent of u, x, t.

Proof Applying Fubini's theorem we obtain

$$
\begin{aligned}
&\frac{1}{2r}\int_{(t_0-t,t_0+r)} |\widetilde{u}(x,t)|\chi_{t>0}\,dt \\
&\quad\lesssim \frac{1}{2r}\int_{(t_0-r,t_0+r)}\left(\int_{\mathbb{R}^{n-1}} \widetilde{E}_{\frac{n-1}{2}}(|x-y|,t)|u(y)|\,dy\right)\chi_{t>0}\,dt \\
&\quad= \int_{\mathbb{R}^{n-1}}\left(\frac{1}{2r}\int_{(t_0-r,t_0+r)} \widetilde{E}_{\frac{n-1}{2}}(|x-y|,t)\chi_{t>0}\,dt\right)|u(y)|\,dy.
\end{aligned}
$$

Now it suffices to apply Lemma 4.3. □

Remark 4.2 We observe that under the notation in Lemma 4.4 we have

1. when u is nonnegative then $(\mathcal{T}_0u)(x,t) = \widetilde{u}(x,t)$, in general $(\mathcal{T}_0u)(x,t) = \widetilde{|u|}(x,t)$ (see 3.1);
2. C_1 is as in Lemma 4.2 with $\beta = \frac{n-5}{2}$.

Lemma 4.5 *Let $u \in L^1(\mathbb{R}^{n-1})$ and let $(\mathcal{T}_1u)(x,t_0)$ be as in Lemma 4.4. Then for any convex function R and every $t_0 > 0$ we have*

$$
\int_{\mathbb{R}^{n-1}} R(\mathcal{T}_1u)(x,t_0)\,dx \le \int_{\mathbb{R}^{n-1}} R(\widetilde{C}|u(y)|)\,dy,
$$

where $\widetilde{C} = \int_{\mathbb{R}^{n-1}} |z|^{-(n-1)}F_{\frac{n-5}{2}}(|z|^2)dz$.

Proof We have

$$
R((\mathcal{T}_1u)(x,t_0)) = R\left(\int_{\mathbb{R}^{n-1}} \frac{\frac{1}{|x-y|^{n-1}}F_{\frac{n-5}{2}}\left(\frac{|x-y|^2}{4t_0}\right)}{\widetilde{C}}\cdot(\widetilde{C}|u(y)|)\,dy\right)
$$

and $\widetilde{C} = \int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^{n-1}}F_{\frac{n-5}{2}}(\frac{|x-y|^2}{4t_0})\,dy = \int_{\mathbb{R}^{n-1}} |z|^{-(n-1)}F_{\frac{n-5}{2}}(|z|^2)dz$. Therefore by Jensen's inequality

$$
R((\mathcal{T}_1u)(x,t_0)) \le \int_{\mathbb{R}^{n-1}} \frac{\frac{1}{|x-y|^{n-1}}F_{\frac{n-5}{2}}\left(\frac{|x-y|^2}{4t_0}\right)}{\widetilde{C}}\cdot R(\widetilde{C}|u(y)|)\,dy.
$$

Integrating the above expression over $\mathbb{R}^{n-1}$, then applying Fubini's theorem, we get

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}} R((\mathcal{T}_1u)(x,t_0))\,dx &\le \int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^{n-1}} \frac{\frac{1}{|x-y|^{n-1}}F_{\frac{n-5}{2}}\left(\frac{|x-y|^2}{4t_0}\right)}{\widetilde{C}}\,dx\right) R(\widetilde{C}|u(y)|)\,dy \\
&= \int_{\mathbb{R}^{n-1}} R(\widetilde{C}|u(y)|)\,dy.
\end{aligned}
$$

This finishes the proof of the lemma. □

Lemma 4.6 *Let $\mathcal{T}_2$ be as in Lemma 4.4, R be an arbitrary convex function and $Q' = [0,1]^{n-1}$. Then for any $t > 0$ and measurable function u supported in $Q^{'}$ we have*

$$\int_{Q'} R((\mathcal{T}_2\widetilde{u})(x,t))\,dx \prec \int_{Q'} R(C(1+|\ln t|)|u(x)|)\,dx,$$

with constant $C > 0$ independent of u.

Proof If $x \in Q'$ and $y \in Q'$, then $|x-y| \le \sqrt{n-1}$. We have

$$\mathcal{L} := \int_{Q'} R((\mathcal{T}_2 u)(x,t))\,dx = \int_{Q'} R\left(\int_{Q'} \frac{1}{|x-y|^{n-1}} \chi_{\{t<\frac{|x-y|^2}{4}<\frac{n-1}{4}\}} u(y)\,dy\right) dx.$$

It suffices to show that

$$\mathcal{L} \precsim \begin{cases} \int_{\mathbb{R}^{n-1}} R(C_0|\ln t||u(y)|)dy & \text{when } t < \frac{1}{64} \\ \int_{\mathbb{R}^{n-1}} R(C_0|u(y)|)dy & \text{when } t \ge \frac{1}{64} \end{cases} \tag{4.6}$$

where C_0 is independent in u. We start with the case $t < \frac{1}{64}$. For this, let

$$C(x,t) := \int_{Q'} \frac{1}{|x-y|^{n-1}} \chi_{\{t<\frac{|x-y|^2}{4}<\frac{n-1}{4}\}} dy,$$
$$A(x,t) := \left\{ y \in Q^{'} : t < \frac{|x-y|^2}{4} \right\}.$$

We have

$$|A(x,t)| = \int_{A(x,t)} 1dy \overset{z:=x-y}{=} \int_{z\in\{x-Q^{'}\}\cap\{|z|\ge 2\sqrt{t}\}} dz = |\{x-Q^{'}\} \cap \{|z| \ge 2\sqrt{t}\}|.$$

Let us fix $x^{'} = (x_2,\ldots,x_{n-1}) \in Q^{'}$ and consider the mapping

$$(0,1) \ni x_1 \mapsto |\{(x_1, x_2 \ldots, x_{n-1}) - Q^{'}\} \cap \{|z| \ge 2\sqrt{t}\}| =: f_1(x_1).$$

We will show that $f_1(x_1)$ takes its minimal value at $x_1 = 1/2$. Indeed, let $x = (x_1, x^{'}) \in (0,1) \times (0,1)^{n-2}$. We have

$$f_1(x_1) = \int_{z_1 \in \{x_1-(0,1)\}} \left\{ \int_{z^{'} \in \{x^{'}-(0,1)^{n-2}\}} \chi_{\{|z^{'}|^2 \ge 4t - |z_1|^2\}} dz^{'} \right\} dz_1.$$

Define for $s \in (0,1)$:

$$h(s) := \int_{z^{'} \in \{x^{'}-(0,1)^{n-2}\}} \chi_{\{|z^{'}|^2 \ge 4t - s^2\}} dz^{'},$$
$$\mathcal{C}(s) := \{z^{'} \in x^{'} - (0,1)^{n-2} : |z^{'}|^2 \ge 4t - s^2\}.$$

Note that as for $s_1 \le s_2$ we have $\mathcal{C}(s_1) \subseteq \mathcal{C}(s_2)$, therefore function $s \mapsto h(s)$ is nondecreasing. Moreover, let

$$A(x_1) := [-1 + x_1, x_1] \cap \left[-\frac{1}{2}, \frac{1}{2}\right] \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right],$$
$$B(x_1) := [-1 + x_1, x_1] \Big\backslash \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Then $|A(x_1)| + |B(x_1)| = 1$, sets $A(x_1)$, $B(x_1)$ are disjoint and $\inf\{h(|z_1|^2) : z_1 \in B(x_1)\} \ge h(\frac{1}{4}) \ge \sup\{h(|z_1|^2) : z_1 \in [-\frac{1}{2}, \frac{1}{2}]\}$. As we have

$$\begin{aligned} f_1(x_1) &= \int_{z_1 \in A(x_1)} h(|z_1|^2) dz_1 + \int_{z_1 \in B(x_1)} h(|z_1|^2) dz_1 \\ &= \int_{z_1 \in [-\frac{1}{2}, \frac{1}{2}]} h(|z_1|^2) dz_1 + \left\{ \int_{z_1 \in B(x_1)} h(|z_1|^2) dz_1 - \int_{z_1 \in [-\frac{1}{2}, \frac{1}{2}] \backslash A(x_1)} h(|z_1|^2) dz_1 \right\}, \end{aligned}$$

sets $B(x_1)$ and $[-\frac{1}{2}, \frac{1}{2}] \backslash A(x_1)$ have the same measure, therefore the expression in brackets $\{\dots\}$ is nonnegative. Therefore $f_1(x_1)$ achieves its minimal value when $\{\dots\} = 0$, in particular when $A(x_1) = [-\frac{1}{2}, \frac{1}{2}]$, $B(x_1) = \emptyset$, i. e. in case when $x_1 = \frac{1}{2}$.

By an obvious modification of the above argument we have

$$\begin{aligned} |A(x, t)| &\ge |\{(1/2, \dots, 1/2) - Q'\} \cap \{|z| \ge 2\sqrt{t}\}| \\ &= |(1/2, \dots, 1/2)^{n-1} \cap \{|z| \ge 2\sqrt{t}\}|, \end{aligned}$$

and right hand side is nonzero provided that $2\sqrt{t} < |(1/2, \dots, 1/2)| = \sqrt{n-1}/2$. When $t < 1/16$ this is always satisfied and in that case $C(x, t) > 0$. Moreover, by simple observation we have

$$\begin{aligned} C(x, t) &= \int_{A(x,t)} \frac{1}{|x - y|^{n-1}} dy = \int_{\{\frac{x}{2\sqrt{t}} - \frac{1}{2\sqrt{t}} Q'\} \cap \{|z| \ge 1\}} \frac{1}{|z|^{n-1}} dz \\ &\le \int_{\{1 \le |z| \le \frac{\sqrt{n-1}}{2\sqrt{t}}\}} \frac{1}{|z|^{n-1}} dz = \omega_{n-2} \ln r \Big|_1^{\frac{\sqrt{n-1}}{2\sqrt{t}}} = \omega_{n-2} \ln \frac{\sqrt{n-1}}{2\sqrt{t}} \\ &\le \frac{\omega_{n-2}}{2} \ln(n-1) + \frac{\omega_{n-2}}{2} |\ln t| =: a_1 + a_2 |\ln t|. \end{aligned}$$

where ω_{n-2} is measure of unit sphere in $\mathbb{R}^{n-2}$ is the case $n > 2$ and $\omega_0 = 2$. On the other hand

$$\begin{aligned} C(x, t) &= \int_{\{\frac{x}{2\sqrt{t}} - \frac{1}{2\sqrt{t}} Q'\} \cap \{|z| \ge 1\}} \frac{1}{|z|^{n-1}} dz \\ &\ge \int_{\{\frac{x}{2\sqrt{t}} - \frac{1}{2\sqrt{t}} Q'\} \cap \{|z| \ge 1\} \cap \frac{1}{2\sqrt{t}} (-\frac{1}{2}, \frac{1}{2})^{n-1}} \frac{1}{|z|^{n-1}} dz \ge \int_{\{\{|z| \ge 1\} \cap \frac{1}{2\sqrt{t}} (0, \frac{1}{2})^{n-1}} \frac{1}{|z|^{n-1}} dz \end{aligned}$$

$$\geq \frac{1}{2^{n-1}} \int_{1<|z|<\frac{1}{4\sqrt{t}}} \frac{1}{|z|^{n-1}} dz = 2^{-(n-2)} \omega_{n-2} \ln r|_1^{\frac{1}{4\sqrt{t}}}$$
$$= 2^{-(n-2)} \omega_{n-2} \ln \left(\frac{1}{4\sqrt{t}}\right) \geq 6-(n-2)\omega_{n-2} \ln 2.$$

This easily implies

$$\sup_{t\in[0,\frac{1}{64}]} \frac{\sup_{y\in Q'} C(y,t)}{\inf_{x\in Q'} C(x,t)} \prec 1. \tag{4.7}$$

By Jensen's inequality, Fubini's theorem and the above estimates we can estimate further:

$$\mathcal{L} \leq \int_{Q'} \int_{Q'} \frac{\frac{1}{|x-y|^{n-1}} \chi_{\{t<\frac{|x-y|^2}{4}<\frac{n-1}{4}\}}}{C(x,t)} R\left(C(x,t)|u(y)|\right) dydx$$
$$\leq \int_{Q'} \left(\int_{Q'} \frac{\frac{1}{|x-y|^{n-1}} \chi_{\{t<\frac{|x-y|^2}{4}<\frac{n-1}{4}\}}}{C(x,t)} dx \right) R(C(1+|\ln t|)|u(y)|)\, dy$$
$$\overset{(4.7)}{\prec} \int_{Q'} R(C(1+|\ln t|)|u(y)|)\, dy,$$

where C is independent on u.

When $t > \frac{1}{64}$ the computations become simpler as then we have

$$\frac{1}{|x-y|^{n-1}} \chi_{t<\frac{|x-y|^2}{4}} \leq \frac{1}{|x-y|^{n-1}} \chi_{\frac{1}{4}<|x-y|} \leq 4^{n-1},$$

which implies

$$R\left(\int_{Q'} \frac{1}{|x-y|^{n-1}} \chi_{\{t<\frac{|x-y|^2}{4}\}} u(y)\, dy \right) \leq R\left(\int_{Q'} 4^{n-1} u(y)\, dy \right)$$
$$\leq \int_{Q'} R(4^{n-1} u(y))\, dy.$$

□

We are now in position to prove Lemma 4.1.

Proof of Lemma 4.1 Let $\lambda > 0$ and denote

$$\mathcal{L} := \int_{(0,1)} \int_{Q'} \Psi \left(\frac{|\widetilde{u}(x,t)|}{\lambda} \right) dxdt$$
$$= \int_{Q'} \left(\int_{\mathbb{R}} \Psi \left(\frac{|\widetilde{u}(x,t)|}{\lambda} \chi_{t\in(0,1)} \right) dt \right) dx.$$

We apply Theorem 4.1 to the internal integral, then Fubini Theorem, to get

$$\mathcal{L} \lesssim \int_{\mathbb{R}} \int_{Q'} \Phi\left(\frac{M^2\left(\widetilde{u}(x,t)\chi_{t\in(0,1)}\right)}{\lambda}\right) dxdt + s_0 \int_{\mathbb{R}} \int_{Q'} \frac{|\widetilde{u}(x,t)|\chi_{t\in(0,1)}}{\lambda}\, dxdt.$$

This is further estimated with the help of Lemma 4.4:

$$\begin{aligned}\mathcal{L} \lesssim & \int_{Q'} \int_{(0,1)} \Phi\left(\frac{C\left(|\mathcal{T}_0 u(x, \frac{3}{2}t)|\right)}{\lambda}\right) dxdt + \int_{Q'} \int_{(0,1)} \Phi\left(\frac{C\,(\mathcal{T}_1 u(x,2t))}{\lambda}\right) dxdt \\ & + \int_{Q'} \int_{(0,1)} \Phi\left(\frac{C\,((\mathcal{T}_2 u)(x, 2C_1 t))}{\lambda}\right) dxdt + s_0 \int_{\mathbb{R}} \int_{Q'} \frac{|\widetilde{u}(x,t)|}{\lambda}\, dxdt \\ & =: \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}.\end{aligned}$$

Applying Proposition 4.1 and Lemma 4.5 with $R = \Phi$, we obtain

$$\mathcal{A} + \mathcal{B} \lesssim \int_{Q'} \Phi\left(\frac{C|u(x)|}{\lambda}\right) dx.$$

Moreover, according to Lemma 4.6, we have

$$\mathcal{C} \lesssim \int_{(0,1)} \int_{Q'} \Phi(C_2(1+|\ln 2C_1 t|)|u(x)|)\, dx \lesssim \int_{(0,1)} \int_{Q'} \Phi(C_3(1+|\ln t|)|u(x)|)\, dx,$$

with some general constants $C_1, C_2, C_3 > 0$. This together with (4.2) gives $\mathcal{C} \lesssim 1 + \int_{Q'} \Phi(\frac{|u(x)|}{\lambda})\, dx$. To get (4.3) it suffices to note that if $s_0 > 0$, then

$$\begin{aligned}\frac{\mathcal{D}}{s_0} &= \int_Q \frac{|\widetilde{u}(x,t)|}{\lambda}\, dxdt \le \int_Q \Phi^*(1)\, dxdt + \int_Q \Phi\left(\frac{|\widetilde{u}(x,t)|}{\lambda}\right) dxdt \\ &\lesssim 1 + \int_{Q'} \Phi\left(\frac{|u(x)|}{\lambda}\right) dx.\end{aligned}$$

The last estimate follows from Proposition 4.1. This implies (4.3). Inequality (4.4) follows directly from (4.3) and Proposition 2.1 after the substitution $\lambda = \|u\|_{L^{\Phi}(Q')}$. □

5 Estimates of function $\nabla_x \widetilde{u}$

5.1 Presentation of main results

Our approach will be based on the following results.

Proposition 5.1 (Lemma 3.3, [23]) *For any* $k \in \{1, \dots, n-1\}$, *any* $0 < \varepsilon < \frac{1}{2}$, *and any* $u \in Lip_0(Q')$ *such that* $\mathrm{dist}(\mathrm{supp}\, u, \partial Q') \geq \varepsilon$, *we have*

$$\left|\frac{\partial \widetilde{u}}{\partial x_k}(x,t)\right| \leq \int_{Q'} P(|x-y|,t)\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dy + \widetilde{C}|u(x)|, \text{ where}$$
$$P(s,t) = \frac{1}{2^n(\sqrt{2\pi})^{n-1}} \frac{1}{t^{(n-1)/2}} \frac{s^2}{t} e^{-\frac{s^2}{4t}}, \text{when } s \in \mathbb{R}, \text{ in particular},$$
$$P(|z|,t) = 2E(z,t)\frac{|z|^2}{4t}, \text{ when } z \in \mathbb{R}^{n-1}, \widetilde{C} = \frac{1}{\sqrt{2\pi e} \cdot \varepsilon}.$$

Proposition 5.2 ([23]) *For any convex function* R, *any* $0 < \varepsilon < \frac{1}{2}$ *and any function* $u \in Lip_0(Q')$ *such that* $\mathrm{dist}(\mathrm{supp}\, u, \partial Q') \geq \varepsilon$ *we have*

$$\int_{Q'}\int_{(0,1)} R(|\nabla_x \widetilde{u}|)\, dxdt \lesssim \int_{Q'}\int_{Q'} R\left(\frac{C_1|u(y)-u(x)|}{|x-y|}\right)\frac{1}{|x-y|^{n-3}}\, dxdy + \int_{Q'} R(C_2|u(x)|)\, dx, \tag{5.1}$$

with $C_1 = 2\sqrt{n-1}\int_{\mathbb{R}^{n-1}} |w|^2 e^{-|w|^2}\, dw$, $C_2 = \frac{2\sqrt{n-1}}{\sqrt{2\pi e}\cdot\varepsilon}$.

In particular, when $\nabla_x u$ *denotes the spatial gradient of* u, *we have*

$$\int_{Q'}\int_{(0,1)} R(|\nabla_x \widetilde{u}|)\, dxdt \lesssim \int_{Q'}\int_{Q'} R\left(\frac{C_1|u(y)-u(x)|}{|x-y|}\right)\frac{1}{|x-y|^{n-2}}\, dxdy + \int_{Q'} R(C_2|u(x)|)\, dx = J^R(C_1 u, Q') + \int_{Q'} R(C_2|u(x)|)\, dx.$$

Proof We use argument preceding (3.5) in the proof of Lemma 3.2 in [23], to get the slightly more precise statement. ◻

Corollary 5.1 ([23]) *If* $u \in Y_L^{\Phi,\Phi}(Q')$ *is supported in the interior of* Q' *then the spatial derivative of* $\widetilde{u}$ *satisfies* $|\nabla_x \widetilde{u}| \in L^\Phi(Q)$.

Remark 5.1 As before we remark that the original formulations of Lemmas 5.1 and 5.2 and Corollary 5.1 deal with $u \in C_0^\infty(Q')$ instead of $u \in Lip_0(Q')$ but their statements hold with Lipschitz u by the same arguments.

Remark 5.2 Estimate (5.1) shows that

$$\int_{Q'}\int_{(0,1)} R(|\nabla_x \widetilde{u}|)\, dxdt \lesssim \int_{Q'}\int_{Q'} R\left(\frac{C_1|u(y)-u(x)|}{|x-y|}\right)\frac{\omega(x,y)}{|x-y|^{n-2}}\, dxdy + \int_{Q'} R(C_2|u(x)|)\, dx,$$

where $\omega(x,y)=\frac{1}{|x-y|}$. This implies the estimates in the norms

$$\|\nabla_x \widetilde{u}\|_{L^R(Q)} \precsim \|u\|_{Y_\omega^{R,R}(Q')}$$

where $Y_\omega^{R,R}(Q^{'})$ is weighted Orlicz–Slobodetskii space introduced in [11,12]. In particular more precise statement than that in Corollary 5.1 holds.

Corollary 5.2 *Let $Y_{\omega,L}^{\Phi,\Phi}(Q')$ denote the completion of Lipschitz functions in $Y_\omega^{\Phi,\Phi}(Q')$. If $u \in Y_{\omega,L}^{\Phi,\Phi}(Q')$ is supported in the interior of Q' then the spatial derivative of $\widetilde{u}$ satisfies $|\nabla_x \widetilde{u}| \in L^\Phi(Q)$.*

Note that we have $Y^{\Phi,\Phi}(Q') \subseteq Y_\omega^{\Phi,\Phi}(Q')$.

Our main goal is to show that in general stronger property follows. Namely, that under an assumption $u \in Y_L^{\Psi,\Phi}(\partial\Omega)$, where u is supported in the interior of Q' and (Ψ,Φ) is as in Assumption A (see Definition 2.1), we have $|\nabla_x \widetilde{u}| \in L^\Psi(Q)$. Let us recall that in general we have $\Phi \prec \Psi$ (see Remark 2.2) and it may happen that $\Phi \not\approx \Psi$.

The main result of this subsection reads as follows.

Lemma 5.1 *Let $0<\varepsilon<1/2$ and $\widetilde{u}$ be given by (3.1), where $u \in Lip_0(Q')$ and $\operatorname{dist}(\operatorname{supp} u, \partial Q') \ge \varepsilon$. Moreover, let (Φ,Ψ) be as in Assumption A (see Definition 2.1) and Φ satisfies the following estimation:*

$$\Phi(xy) \le C(1+G(x)+G(x)\Phi(y)), \text{ with } C \text{ independent of } x, y, \tag{5.2}$$

where G is continuous, nondecreasing, locally bounded and such that

$$\sup_{s<\frac{1}{2}} \frac{1}{s}\int_0^{s^2} \frac{G(c|\ln t|)}{|\ln t|}\,dt < \infty$$

for any $c>0$. Then for every $\lambda>0$

$$\begin{aligned}
&\int_{(0,1)}\int_{Q'} \Psi\left(\frac{|\nabla_x \widetilde{u}(x,t)|}{\lambda}\right)dxdt \\
&\quad \precsim 1+\int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}}\Phi\left(\frac{1}{\lambda}\frac{|u(x)-u(y)|}{|x-y|}\right)dxdy+\int_{Q'}\Phi\left(\frac{|u(x)|}{\lambda}\right)dx, \quad \text{and}
\end{aligned} \tag{5.3}$$

$$\|\nabla_x \widetilde{u}(x,t)\|_{L^\Psi(Q)} \precsim \|u\|_{Y^{\Phi,\Phi}(Q')}. \tag{5.4}$$

Remark 5.3 Constant bounds in the above estimations depend on $\epsilon>0$.

5.2 Proof of Lemma 5.1

First we prove several auxiliary claims. We shall proceed quickly when the proofs are similar to that of the previous subsection.

Lemma 5.2 *Let $s \in \mathbb{R}_+$, $t > 0$, $\alpha > 0$, $\gamma \geq 0$ and*

$$\widetilde{E}_{\alpha,\gamma}(s,t) := \frac{1}{t^\alpha}\left(\frac{s^2}{4t}\right)^\gamma \exp(-s^2/4t),$$
$$B_{\alpha,\gamma}(s,t_0,r) := \frac{1}{2r}\int_{(t_0-r,t_0+r)} \widetilde{E}_{\alpha,\gamma}(s,t)\chi_{t>0}\,dt.$$

Then for any $s, t_0, r > 0$ we have

$$\sup_{r>0} B_{\alpha,\gamma}(s,t_0,r) \lesssim \widetilde{E}_{\alpha,\gamma}\left(s, \frac{3}{2}t_0\right) + \frac{1}{s^{2\alpha}} F_{\alpha+\gamma-2}\left(\frac{s^2}{8t_0}\right)\chi_{t_0 \geq \frac{s^2}{8C_2}} + \frac{1}{s^{2\alpha}}\chi_{t_0 < \frac{s^2}{4C_2}},$$

where $F_\beta(x)$ is given by (4.5), $C_2 = C_2(\alpha,\gamma) > 0$ is a given constant.

Proof The proof is the little modification of Lemma 4.3 with the difference that instead of $\widetilde{E}_\alpha(s,t)$ we now deal with $\widetilde{E}_{\alpha,\gamma}(s,t) = \widetilde{E}_\alpha(s,t)(\frac{s^2}{4t})^\gamma$. Repeating the proof of Lemma 4.3 we observe that

$$B_{\alpha,\gamma}(s,t_0,r) \lesssim \widetilde{E}_{\alpha,\gamma}\left(s, \frac{3}{2}t_0\right)$$

in case $0 < r < \frac{t_0}{2}$. In case $\frac{t_0}{2} \leq r < t_0$ we have the estimation

$$\begin{aligned}\widetilde{E}_{\alpha,\gamma}(s,t) &\lesssim \frac{1}{2t_0}\int_0^{2t_0} \frac{1}{t^\alpha}\left(\frac{s^2}{4t}\right)^\gamma e^{-\frac{s^2}{4t}}\,dt\\ &\lesssim \frac{1}{s^{2\alpha}}\frac{s^2}{8t_0}\int_{\frac{s^2}{8t_0}}^\infty q^{\alpha+\gamma-2}e^{-q}\,dq = \frac{1}{s^{2\alpha}}F_{\alpha+\gamma-2}\left(\frac{s^2}{8t_0}\right),\end{aligned}$$

while in case $r \geq t_0$ we compute that

$$B_{\alpha,\gamma}(s,t_0,r) \lesssim \frac{1}{s^{2\alpha}}F_{\alpha+\gamma-2}\left(\frac{s^2}{4r}\right).$$

Using Lemma 4.2 dealing with $y = \frac{s^2}{8t_0}$ and $\beta = \alpha+\gamma-2 > -2$, we obtain that when $\frac{t_0}{2} \leq r$

$$\begin{aligned}\widetilde{E}_{\alpha,\gamma}(s,t) \lesssim \frac{1}{s^{2\alpha}}F_{\alpha+\gamma-2}\left(\frac{s^2}{8t_0}\right)\chi_{\frac{s^2}{8C_2}\leq t_0}\\ + \frac{1}{s^{2\alpha}}\chi_{\frac{s^2}{8C_2}>t_0},\end{aligned}$$

with some positive constant $C_2 > 0$ depending on $\alpha+\gamma$. This completes the proof. □

Lemma 5.3 *Let $u \in Lip_0(Q')$ and $\widetilde{u}$ be given by (3.1). Then for any $t_0 > 0$ we have*

$$\begin{aligned} M^2(\nabla_x \widetilde{u} \chi_{t>0})(x,t_0) \lesssim & \int_{Q'} P(|x-y|, \frac{3}{2}t) \left(\frac{|u(x)-u(y)|}{|x-y|} \right) dy \\ & + \int_{Q' \cap \{y: t \geq \frac{|x-y|^2}{8C_2}\}} \frac{1}{|x-y|^{n-1}} F_{\frac{n-3}{2}} \left(\frac{|x-y|^2}{8t} \right) \\ & \left(\frac{|u(x)-u(y)|}{|x-y|} \right) dy \\ & + \int_{Q' \cap \{y: t < \frac{|x-y|^2}{8C_2}\}} \frac{1}{|x-y|^{n-1}} \left(\frac{|u(x)-u(y)|}{|x-y|} \right) dy + |u(x)|, \end{aligned}$$

where constant $C_2 > 0$ does not depend on u.

Proof According to Proposition 5.1 we have

$$\begin{aligned} \frac{1}{2r} \int_{(t_0-r, t_0+r)} |\nabla_x \widetilde{u}| \chi_{t>0} \, dt \lesssim & \int_{Q'} \left(\frac{1}{2r} \int_{(t_0-r,t_0+r)} P(|x-y|, t) \chi_{t>0} \, dt \right) \\ & \times \left(\frac{|u(x)-u(y)|}{|x-y|} \right) dy + |u(x)|. \end{aligned}$$

Now it suffices to estimate $\frac{1}{2r} \int_{(t_0-r,t_0+r)} P(|x-y|, t)\chi_{t>0} \, dt$ with help of Lemma 5.2 with $\alpha = \frac{n-1}{2}$, $\gamma = 1$. □

Lemma 5.4 *Let $i \in \{1, 2, 3\}$ and $(\mathcal{P}_i u)(x,t)$ be defined by:*

$$\begin{aligned} (\mathcal{P}_0 u)(x,t) &:= \int_{Q'} P(|x-y|, t) \left(\frac{|u(x)-u(y)|}{|x-y|} \right) dy \\ (\mathcal{P}_1 u)(x,t) &:= \int_{Q' \cap \{y: t \geq \frac{|x-y|^2}{4C_2}\}} \frac{1}{|x-y|^{n-1}} F_{\frac{n-3}{2}} \left(\frac{|x-y|^2}{4t} \right) \left(\frac{|u(x)-u(y)|}{|x-y|} \right) dy \\ (\mathcal{P}_2 u)(x,t) &:= \int_{Q' \cap \{y: t < \frac{|x-y|^2}{4C_2}\}} \frac{1}{|x-y|^{n-1}} \left(\frac{|u(x)-u(y)|}{|x-y|} \right) dy. \end{aligned}$$

where $C_2 > 0$ is the same as in Lemma 5.3 (it does not depend on u). Let $0 < \epsilon < 1/2$ be given and $u \in Lip_0(Q')$ be such that $\mathrm{dist}(\mathrm{supp} u, \partial Q') \geq \epsilon$. *Then we have for any $t_0 > 0$*

$$M^2(\nabla_x \widetilde{u} \chi_{t>0})(x,t_0) \lesssim (\mathcal{P}_0 u)\left(x, \frac{3}{2}t_0\right) + (\mathcal{P}_1 u)(x, 2t_0) + (\mathcal{P}_2 u)(x, 2t_0) + |u(x)|.$$

Moreover, let R be such that

$$R(xy) \lesssim (1 + G(x) + G(x)R(y)), \text{ where,} \tag{5.5}$$

$$\sup_{s<\frac{1}{2}} \frac{1}{s} \int_0^{s^2} \frac{G(|\ln t|)}{|\ln t|}\, dt < \infty. \tag{5.6}$$

and G is nonincreasing and locally bounded. Then for any convex function R we have for any $T > 0$

$$\int_{(0,T)} \int_{Q'} R\left((\mathcal{P}_i u)(x,t)\right) dxdt \lesssim 1 + \int_{Q'} \int_{Q'} \frac{1}{|x-y|^{n-2}} R\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy,$$

where $i \in \{0, 1, 2\}$*, with some constant* $C > 0$ *which is independent of* u.

Proof In this proof constant $C > 0$ will denote some general constant independent of u. It can be different even in the same line. We start with the estimate of $\mathcal{P}_0$. For this, we apply arguments from the proof of Lemma 3.2 in [23], to get

$$\int_{(0,T)} \int_{Q'} R((\mathcal{P}_0 u)(x,t))\, dxdt \lesssim \int_{Q'} \int_{Q'} \frac{1}{|x-y|^{n-3}} R\left(C \frac{|u(x)-u(y)|}{|x-y|}\right) dxdy,$$

whenever $T > 0$. For reader's convenience we submit them. As $C_1 := \int_{\mathbb{R}^{n-1}} P(|x-y|, t)dy > 0$ does not depend on t, we have from Jensen's Inequality:

$$\begin{aligned} R((\mathcal{P}_0 u)(x,t)) &= R\left(\left(\int_{Q'} \left(\frac{P(|x-y|,t)}{C_1}\right)\left(\frac{C_1|u(y)-u(x)|}{|x-y|}\right) dy\right)\right) \\ &\le \int_{Q'} \left(\frac{P(|x-y|,t)}{C_1}\right) R\left(\frac{C_1|u(y)-u(x)|}{|x-y|}\right) dy \end{aligned}$$

Simple computation shows that for any $s > 0$

$$\int_{(0,T)} P(s,t)\, dt \lesssim \left(\frac{s^2}{4}\right)^{-(n-3)/2} \int_{\frac{s^2}{4}}^{\infty} q^{(n-3)/2} e^{-q}\, dq \lesssim s^{-(n-3)} \lesssim s^{-(n-2)},$$

and we apply the estimate (5.5) on R involving $\bar{x} = C_1$, $\bar{y} = \frac{|u(y)-u(x)|}{|x-y|}$.

The proof of the estimate of $\mathcal{P}_1$ goes more or less along the lines of that of Lemma 4.5. Namely, we observe that for $x \in Q'$

$$R\left((\mathcal{P}_1 u)(x,t)\right) \le R\left(\int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^{n-1}} F_{\frac{n-3}{2}} \chi_{\{t \ge \frac{|x-y|^2}{4C_2}\}} \left(\frac{|x-y|^2}{4t}\right)\left(\frac{|u(x)-u(y)|}{|x-y|} \chi_{x,y\in Q'}\right) dy\right)$$

$$\leq \int_{\mathbb{R}^{n-1}} \frac{\frac{1}{|x-y|^{n-1}} F_{\frac{n-3}{2}} \chi_{\{t \geq \frac{|x-y|^2}{4C_2}\}} \left(\frac{|x-y|^2}{4t}\right)}{\widetilde{C}} R\left(\frac{\widetilde{C}|u(x)-u(y)|}{|x-y|} \chi_{x,y \in Q'}\right) dy,$$

where we choose $\widetilde{C} = \int_{\mathbb{R}^{n-1}} \frac{1}{|x-y|^{n-1}} F_{\frac{n-3}{2}} (\frac{|x-y|^2}{4t})\, dy = \int_{\mathbb{R}^{n-1}} \frac{1}{|z|^{n-1}} F_{\frac{n-3}{2}} (|z|^2) dz$. Therefore

$$\int_{Q'} \int_{(0,T)} R\left((\mathcal{P}_1 u)(x,t)\right)\, dt dx \leq$$
$$\int_{Q'} \int_{Q'} \frac{1}{|x-y|^{n-1}} \{A(x,y)\}\, R\left(C \frac{|u(x)-u(y)|}{|x-y|}\right) dx dy, \text{ where}$$
$$A(x,y) = \int_{(0,T)} F_{\frac{n-3}{2}} \left(\frac{|x-y|^2}{4t}\right) \chi_{\{t \geq \frac{|x-y|^2}{4C_2}\}}\, dt.$$

Now it suffices to verify that for small s,

$$\int_0^T F_{\frac{n-3}{2}} \left(\frac{s^2}{4t}\right) \chi_{\frac{s^2}{4t} \leq C_2}\, dt \lesssim \int_{\frac{s^2}{4C_2}}^T \frac{s^2}{t}\, dt \lesssim s^2 |\ln s| \lesssim s.$$

Now let us estimate $\mathcal{P}_2$. We have (with $C = \frac{1}{4C_2}$),

$$R((\mathcal{P}_2 u)(x,t)) = \begin{cases} R\left(\int_{Q' \cap \{y: t < |x-y|^2\}} \frac{\frac{1}{|x-y|^{n-1}} \chi_{t < C|x-y|^2}}{C(x,t)} \left(C(x,t) \frac{|u(x)-u(y)|}{|x-y|}\right) dy\right) := \mathcal{A}(C) & \text{if } |Q' \cap \{y : t < C|x-y|^2\}| > 0 \\ 0 & \text{if } |Q' \cap \{y : t < C|x-y|^2\}| = 0. \end{cases}$$

Where we chose at first the positive constant

$$C(x,t) = \int_{Q'} \frac{1}{|x-y|^{n-1}} \chi_{\{t < C|x-y|^2\}}\, dy = \int_{\frac{x-Q'}{\sqrt{t}}} \frac{1}{|z|^{n-1}} \chi_{\{|z| > \frac{1}{\sqrt{C}}\}}\, dy.$$

We can assume that C in the condition $t < C|x-y|^2$ is bigger than one and that $C(x,t)$ is positive. This is because if $C_1 < C_2$ we have $\mathcal{A}(C_1) \leq \mathcal{A}(C_2)$, therefore if we enlarge C, then we get $R((\mathcal{P}_3 u)) \leq \mathcal{A}(C)$, which is sufficient for our analysis.

Observe that

$$C(x,t) \geq \int_{\frac{1}{\sqrt{t}} \cdot [0,1/2]^{n-1}} \frac{1}{|w|^{n-1}} \chi_{|w| > \frac{1}{\sqrt{C}}}\, dw = \frac{1}{2^{n-1}} \int_{\frac{1}{\sqrt{t}} \cdot [-1/2,1/2]^{n-1}} \frac{1}{|w|^{n-1}} \chi_{|w| > \frac{1}{\sqrt{C}}}\, dw$$
$$\geq \int_{\frac{1}{\sqrt{t}} \cdot B(0,\frac{1}{2})} \frac{1}{|w|^{n-1}} \chi_{|w| > \frac{1}{\sqrt{C}}}\, dw \gtrsim (1 + |\ln t|),$$

when t is sufficiently small. On the other hand we always have $C(x,t) \lesssim (1 + |\ln t|)$, for every $t \in (0,1)$, therefore

$$(1 + |\ln t|) \lesssim C(x,t) \lesssim (1 + |\ln t|), \text{ when } t < C_0, \text{ and } x \in Q',$$

where C_0 does is some general positive constant. Combining this with Jensen's inequality we obtain the following inequality for small t:

$$R((\mathcal{P}_2u)(x,t)) \le R\left(\int_{Q'} \frac{\frac{1}{|x-y|^{n-1}}\chi_{t<C|x-y|^2}}{C(x,t)}\left(C(x,t)\frac{|u(x)-u(y)|}{|x-y|}\right)dy\right)$$
$$\le \int_{Q'} \frac{\frac{1}{|x-y|^{n-1}}\chi_{t<C|x-y|^2}}{C(x,t)} R\left(C(x,t)\frac{|u(x)-u(y)|}{|x-y|}\right)dy$$
$$\precsim \int_{Q'} \frac{\frac{1}{|x-y|^{n-1}}\chi_{t<C|x-y|^2}}{1+|\ln t|} R\left(C(1+|\ln t|)\frac{|u(x)-u(y)|}{|x-y|}\right)dy.$$

Let $\bar{x} := C(1+|\ln t|)$, $\bar{y} := \frac{|u(x)-u(y)|}{|x-y|}$. Using the condition (5.5) on $R(\bar{x}\bar{y})$ and estimating further, we get

$$\int_{(0,C_0)}\int_{Q'} R((\mathcal{P}_2u)(x,t))\,dx\,dt$$
$$\precsim \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-1}}\left\{\int_0^{\min\{T,C|x-y|^2\}} \frac{1}{1+|\ln t|}\,dt\right\}dx\,dy$$
$$+\int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}}\left\{\frac{1}{|x-y|}\int_0^{\min\{T,C|x-y|^2\}} \frac{G(C(1+|\ln t|))}{1+|\ln t|}\,dt\right\}dx\,dy$$
$$+\int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}}\left\{\frac{1}{|x-y|}\int_0^{\min\{T,C|x-y|^2\}} \frac{G(C(1+|\ln t|))}{1+|\ln t|}\,dt\right\}$$
$$\times R\left(\frac{|u(x)-u(y)|}{|x-y|}\right)dx\,dy$$
$$\precsim 1+\int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} R\left(\frac{|u(x)-u(y)|}{|x-y|}\right)dx\,dy$$

When $T \ge t \ge C_0$, the estimates become simples as then we have

$$R((\mathcal{P}_2u)(x,t)) \le R\left(\int_{Q'} \frac{\frac{1}{|x-y|^{n-1}}\chi_{C_0<t<C|x-y|^2}}{b(x)}\left(b(x)\frac{|u(x)-u(y)|}{|x-y|}\right)dy\right)$$
$$\le R\left(\int_{Q'} \frac{\sqrt{C/C_0}\frac{1}{|x-y|^{n-2}}}{b(x)}\left(b(x)\frac{|u(x)-u(y)|}{|x-y|}\right)dy\right)$$

where we chose $b(x) = \int_{Q'} \sqrt{C/C_0}\frac{1}{|x-y|^{n-2}}\,dy \sim 1$ (as $x \in Q'$). Therefore and by Jensen's inequality

$$R((\mathcal{P}_2u)(x,t)) \precsim \int_{Q'} \frac{1}{|x-y|^{n-2}} R\left(C\frac{|u(x)-u(y)|}{|x-y|}\right)dy. \tag{5.7}$$

After integrating it over $(C_0, T) \times Q^{'}$ and using (5.5) again, we obtain

$$\int_{(C_0,T)} \int_{Q'} R((\mathcal{P}_2 u)(x,t))\, dx\, dt \precsim 1 + \int_{Q'} \int_{Q'} \frac{1}{|x-y|^{n-2}} R\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dx\, dy,$$

which finishes the proof. □

Proof of Lemma 5.1 We note that for an arbitrary $\lambda > 0$ we have by Theorem 4.1,

$$\begin{aligned} I &:= \int_{(0,1)} \int_{Q'} \Psi\left(\frac{|\nabla_x \widetilde{u}(x,t)|}{\lambda}\right) dxdt = \int_{Q'} \left\{\int_{\mathbb{R}} \Psi\left(\frac{|\nabla_x \widetilde{u}(x,t)|\chi_{t\in(0,1)}}{\lambda}\right) dt\right\} dx \\ &\lesssim s_0 \int_{Q'} \int_{(0,1)} \frac{|\nabla_x \widetilde{u}(x,t)|}{\lambda}\, dtdx + \int_{Q'} \left\{\int_{\mathbb{R}} \Phi\left(\frac{M^2|\nabla_x \widetilde{u}(x,t)|\chi_{t\in(0,1)}}{\lambda}\right) dt\right\} dx \\ &=: \mathcal{L}_1 + \mathcal{L}_2. \end{aligned}$$

Moreover, $|f(x,t)| \le M^2 f(x,t)$ and $\int_Q |h(x,t)|\, dxdt \le \Phi^*(1) + \int_Q \Phi(|h(x,t)|)\, dxdt$, hence $\mathcal{L}_1 \prec 1 + \mathcal{L}_2$ and it remains to estimate the expression $\mathcal{L}_2$.

According to Lemma 5.3 we have (with the same notation)

$$M^2(|\nabla_x \widetilde{u}|\chi_{t>0})(x,t_0) \lesssim \sum_{i=0}^{2} (\mathcal{P}_i u)(x,t) + |u(x)|.$$

Applying Lemma 5.4 with $R = \Phi$ and observing that $\Phi(\sum_{j=1}^4 a_j) \le \frac{1}{4}\sum_{j=1}^4 \Phi(4a_j) \lesssim 1 + \sum_{j=1}^4 \Phi(a_j)$, we arrive at

$$\begin{aligned} \mathcal{L}_2 &\lesssim 1 + \sum_{j=0}^{2} \int_{Q'} \int_{(0,1)} \Phi\left(\frac{(4\mathcal{P}_i u)(x,t)}{\lambda}\right) dxdt + \int_{Q'} \Phi\left(\frac{4|u(x)|}{\lambda}\right) dx \\ &\lesssim 1 + \int_{Q'} \int_{Q'} \frac{1}{|x-y|^{n-2}} \Phi\left(\frac{1}{\lambda} \frac{|u(x)-u(y)|}{|x-y|}\right) dxdy + \int_{Q'} \Phi\left(\frac{|u(x)|}{\lambda}\right) dx. \end{aligned}$$

This gives (5.3). The choice of $\lambda_0 = I^{\Phi}(u, Q') + \|u\|_{L^{\Phi}(Q')}$ implies

$$\int_Q \Psi\left(\frac{|\nabla_x \widetilde{u}(x,t)|}{\lambda_0}\right) \le C,$$

which together with Proposition 2.1 gives (5.4). This ends the proof of the lemma. □

6 Estimates of function $\partial_t \widetilde{u}$

6.1 Presentation of results and discussion

Our goal here is to continue our estimates of heat extension operator, dealing now with the time derivative of $\widetilde{u}$.

We start with the following lemma. The Lemma was originally proven for $u \in C_0^\infty(Q^{'})$. We remark that assumption $u \in Lip_0(Q')$ does not change final conclusion and the proof under such assumption follows by the same arguments.

Lemma 6.1 ([23]) *Let* $u \in Lip_0(Q')$, $\widetilde{u}$ *be given by* (3.1), $0 < \varepsilon < \frac{1}{2}$ *and let* $\mathrm{dist}(\mathrm{supp}\, u, \partial Q') \geq \varepsilon$. *Then we have*

$$\left|\frac{\partial \widetilde{u}}{\partial t}(x,t)\right| \leq \widetilde{C}_1 \int_{Q'} S(|x-y|,t)\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dy + \widetilde{C}_2 \frac{1}{\varepsilon^2}|u(x)|, \text{ where}$$

$$S(s,t) := \frac{1}{2^n(\sqrt{2\pi})^{n-1}}\left\{\frac{s}{2\sqrt{t}} + \left(\frac{s}{2\sqrt{t}}\right)^3\right\}\frac{1}{t^{n/2}}e^{-\frac{s^2}{4t}}, \text{ in particular}$$

$$S(|z|,t) = \left\{\frac{|z|}{2\sqrt{t}} + \left(\frac{|z|}{2\sqrt{t}}\right)^3\right\}\widetilde{E}(|z|,t)\cdot\frac{1}{2\sqrt{t}}, \text{ when } z \in \mathbb{R}^{n-1}, \tag{6.1}$$

with $\widetilde{C}_1 = 2n$, $\widetilde{C}_2 = \frac{1}{\sqrt{\pi}}(\sqrt{3})^3 e^{-\frac{9}{4}} \cdot (n-1)$.

A wishful thinking would expect the inequality of type

$$\int_{Q'}\int_{(0,1)} R(|\partial_t \widetilde{u}(x,t)|)\, dxdt \lesssim \int_{Q'}\int_{Q'} R\left(\frac{|u(y)-u(x)|}{|x-y|}\right)\frac{1}{|x-y|^{n-2}}\, dxdy + \int_{Q'} R\left(|u(x)|\right)\, dx,$$

dealing with an arbitrary convex function R. It seems quite difficult to prove such an inequality. However, we have obtained such result under the special assumption stated below.

Definition 6.1 *(Assumption B)*

1. $R(\lambda) = \lambda P(\lambda)$, where $P(ab) \prec 1 + P(a) + P(b)$,
2. P is nondecreasing,
3. Function $\frac{P(\lambda)}{\lambda}$ is nonincreasing for large arguments.

Remark 6.1 ([23])

1. Condition **1.**, **2.**, and **3.** above imply $P(x^n) \leq C_n P(x)$ for big arguments. It gives inequality $P(y) \leq C_\kappa y^\kappa$ for large arguments, with any $\kappa > 0$.
2. Let us consider $P(\lambda) = (\ln(2+\lambda))^\alpha$, with any $\alpha > 0$. Then P satisfies **1**, **2**, and **3**. Moreover, $P(\lambda) \prec 1 + \lambda^\beta$, with any $\beta > 0$.
3. We have that $\int_0^a P(\frac{1}{s})\, ds < \infty$ for every $a > 0$.

We obtained the following result (formulated originally for $u \in C_0^\infty(Q')$).

Proposition 6.1 ([23]) *If R satisfies an Assumption B (see Definition 6.1), then for any function $u \in Lip_0(Q')$ we have*

$$\int_{Q'}\int_{(0,1)} R\left(|\partial_t \widetilde{u}(x,t)|\right) dxdt \lesssim 1 + \int_{Q'} R\left(|u(x)|\right) dx + \int_{Q'}\int_{Q'} R\left(\frac{|u(y)-u(x)|}{|x-y|}\right)\frac{1}{|x-y|^{n-2}}\, dxdy.$$

Remark 6.2 It follows from the proof of Proposition 6.1 presented in [23] that when

$$\mathcal{B}u(x,t) := \int_{Q'} S(|x-y|,t)\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dy,$$

then for any u supported in Q' and any R satisfying Condition **B**, we have

$$\int_{Q'}\int_{(0,T)} R(\mathcal{B}u)(x,t))\, dxdt \lesssim 1 + \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} R\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy,$$

for every $0 < T < \infty$.

It is clear from Proposition 6.1 that under certain assumptions on function R, condition $u \in Y_L^{R,R}(Q')$ implies $\partial_t \widetilde{u} \in L^R(Q)$. Following our previous schema, we would like to prove that the condition $u \in Y^{\Phi,\Phi}(Q')$ implies $\partial_t \widetilde{u} \in L^\Psi(Q)$, where (Φ, Ψ) is as in Assumption A. We do not know if this is true in general. However, we have the following result.

Lemma 6.2 *Let $0 < \epsilon < 1/2$, $u \in Lip_0(Q')$ is such that* $\mathrm{dist}(\mathrm{supp} u, \partial Q') \geq \epsilon$ *and $\widetilde{u}$ is given by* (3.1). *Moreover, let (Φ, Ψ) be as in Assumption A (see Definition 2.1) and Φ satisfies Assumption B (see Definition 6.1). Then we have for every $\lambda > 0$:*

$$\int_{(0,1)}\int_{Q'} \Psi\left(\frac{|\partial_t \widetilde{u}(x,t)|}{\lambda}\right) dxdt \lesssim 1 + \int_{Q'} \Phi\left(\frac{|u(x)|}{\lambda}\right) dx + \int_{Q'}\int_{Q'} \frac{|\ln|x-y||}{|x-y|^{n-2}} \Phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|}\right) dxdy, \tag{6.2}$$

$$\int_{(0,1)}\int_{Q'} \Psi\left(\frac{|\partial_t \widetilde{u}(x,t)|}{\lambda}\right) dxdt \lesssim 1 + \int_{Q'} \Phi\left(\frac{|u(x)|}{\lambda}\right) dx + \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} \ln\left(\frac{|u(x)-u(y)|}{\lambda|x-y|}\right) \Phi\left(\frac{|u(x)-u(y)|}{\lambda|x-y|}\right) dxdy. \tag{6.3}$$

Remark 6.3 Note that under the assumptions of Lemma 6.2 we have

$$\int_{(0,1)}\int_{Q'} \Psi\left(\frac{|\partial_t \widetilde{u}(x,t)|}{\lambda}\right) dxdt$$
$$\lesssim 1 + \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2+\delta}} \Phi\left(\frac{1}{\lambda}\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy$$
$$+ \int_{Q'} \Phi\left(\frac{|u(x)|}{\lambda}\right) dx. \tag{6.4}$$

and parameter $\delta > 0$ above can be taken arbitrarily. Moreover, the above inequality is very close to the inequality:

$$\int_{(0,1)}\int_{Q'} \Psi\left(\frac{|\partial_t \widetilde{u}(x,t)|}{\lambda}\right) dxdt$$
$$\lesssim 1 + \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} \Phi\left(\frac{1}{\lambda}\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy$$
$$+ \int_{Q'} \Phi\left(\frac{|u(x)|}{\lambda}\right) dx.$$

We do not know if (6.4) holds with $\delta = 0$.

We are now in position to prove the presented result.

6.2 Proof of Lemma 6.2

We start with the following result which extends Lemma 5.2.

Lemma 6.3 *Let $s \in \mathbb{R}_+$, $t > 0, \alpha > 0$, $v(x) = x + x^3$ and*

$$\widetilde{E}_{\alpha,v}(s,t) := \frac{1}{t^\alpha} v\left(\frac{s}{2\sqrt{t}}\right) \exp(-s^2/4t),$$
$$B_{\alpha,v}(s,t_0,r) := \frac{1}{2r}\int_{(t_0-r,t_0+r)} \widetilde{E}_{\alpha,v}(s,t)\chi_{t>0}\, dt,$$

Then for any $s, t_0, r > 0$ we have

$$\sup_{r>0} B_{\alpha,v}(s,t_0,r) \lesssim \widetilde{E}_{\alpha,v}\left(s, \frac{3}{2}t_0\right) + \frac{1}{s^{2\alpha}}\frac{s^2}{4t_0}\chi_{t_0 \geq \frac{s^2}{4C_3}} + \frac{1}{s^{2\alpha}}\chi_{t_0 < \frac{s^2}{4C_3}},$$

with some constant $C_3 > 0$.

Proof We note that under an assumption of Lemma 5.2 we have $\widetilde{E}_{\alpha,w}(s,t) = \widetilde{E}_{\alpha,\frac{1}{2}}(s,t) + \widetilde{E}_{\alpha,\frac{3}{2}}(s,t)$. The estimate follows from Lemma 5.2 by summing up the estimates for $\widetilde{E}_{\alpha,\frac{1}{2}}(s,t)$ and $\widetilde{E}_{\alpha,\frac{3}{2}}(s,t)$, after we note that $F_\beta(x) \lesssim 1$ for large arguments, so that $F_\beta(\frac{s^2}{4t_0})$ can be estimated by constant when $t_0 < \frac{s^2}{4C_3}$. On the other hand $F_\beta(x) \sim x$ for small x. □

Lemma 6.4 *Let $u \in Lip_0(Q')$ and $\widetilde{u}$ be as in Lemma 6.2 and let $S(s,t)$ be as in Lemma 6.1. Then for any $t > 0$,*

$$
\begin{aligned}
M^2(\partial_t \widetilde{u}\chi_{t>0})(x,t) &\lesssim \int_{Q'} S\left(|x-y|, \frac{3}{2}t\right)\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dy \\
&\quad + \int_{Q'\cap\{y: t \ge \frac{|x-y|^2}{4C_3}\}} \frac{1}{|x-y|^n}\frac{|x-y|^2}{4t}\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dy \\
&\quad + \int_{Q'\cap\{y: t < \frac{|x-y|^2}{4C_3}\}} \frac{1}{|x-y|^n}\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dy + |u(x)| \\
&=: (\mathcal{C}_1 u)(x,t) + (\mathcal{C}_2 u)(x,t) + (\mathcal{C}_3 u)(x,t) + |u(x)|,
\end{aligned}
$$

with some constant C_3, which is not dependent on u.

Proof According to Lemma 6.1, we have

$$
\begin{aligned}
&\frac{1}{2r}\int_{(t_0-r,t_0+r)} |\partial_t \widetilde{u}|\chi_{t>0}\, dt \lesssim |u(x)| + \\
&\int_{Q'}\left(\frac{1}{2r}\int_{(t_0-r,t_0+r)} S(|x-y|,t)\chi_{t>0}\, dt\right)\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dy.
\end{aligned}
$$

Using the notation of Lemmas 6.1 and 6.3 we have $S(s,t) = \widetilde{E}_{\frac{n}{2},v}(s,t)$. Therefore the result follows from Lemma 6.3 applied with $\alpha = \frac{n}{2}$. □

We are now to establish our crucial estimates for $\partial_t \widetilde{u}$. We have the following result.

Lemma 6.5 *Let $u \in Lip_0(Q')$ and $\widetilde{u}$ be as in Lemma 6.2, $i \in \{1,2,3\}$ and $(\mathcal{C}_i u)(x,t)$ be the same as in Lemma 6.4 and R satisfies Assumption B (see Definition 6.1). Then we have*

$$
\begin{aligned}
&\int_{(0,1)}\int_{Q'} R\left((\mathcal{C}_1 u)(x,t)\right) dxdt \\
&\lesssim 1 + \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} R\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy, \qquad (6.5)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{(0,1)}\int_{Q'} R\left((\mathcal{C}_2 u)(x,t)\right) dxdt \\
&\lesssim 1 + \int_{Q'}\int_{Q'} \frac{|\ln|x-y||}{|x-y|^{n-2}} R\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy, \qquad (6.6)
\end{aligned}
$$

$$\int_{(0,1)} \int_{Q'} R\left((\mathcal{C}_2 u)(x,t)\right)\, dxdt$$
$$\lesssim 1 + \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} \ln\left(\frac{|u(x)-u(y)|}{|x-y|}\right) R\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy, \quad (6.7)$$
$$\int_{(0,1)} \int_{Q'} R\left((\mathcal{C}_3 u)(x,t)\right)\, dxdt$$
$$\lesssim \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} R\left(\frac{|u(x)-u(y)|}{|x-y|}\right) dxdy, \quad (6.8)$$

Proof Denote for simplicity $h(x,y) := \frac{|u(x)-u(y)|}{|x-y|}$. The proof will be divided into several steps.

Step 1 (proof of 6.5). The estimate for $i = 1$ is a consequence of Remark 6.2 as $(\mathcal{C}_1 u)(x,t) = Bu(x, \frac{3}{2}t)$.

Step 2 (reduction argument). We note that for $\theta := 2\alpha - 1$, $\alpha \in (1/2, 1)$:

$$\begin{aligned}(\mathcal{C}_2 u)(x,t) &= \int_{Q'\cap\{y: t\ge \frac{|x-y|^2}{4C_3}, h(x,y)\le \left(\frac{1}{\sqrt{t}}\right)^{\theta}\}} \frac{1}{|x-y|^n}\frac{|x-y|^2}{4t} h(x,y)\, dy, \text{ where}\\ &\quad + \int_{Q'\cap\{y: t\ge \frac{|x-y|^2}{4C_3}, \left(\frac{1}{\sqrt{t}}\right)^{\theta} < h < \frac{1}{\sqrt{t}}\}} \frac{1}{|x-y|^n}\frac{|x-y|^2}{4t} h(x,y)\, dy\\ &\quad + \int_{Q'\cap\{y: t\ge \frac{|x-y|^2}{4C_3}, h > \frac{1}{\sqrt{t}}\}} \frac{1}{|x-y|^n}\frac{|x-y|^2}{4t} h(x,y)\, dy\\ &:= \mathcal{B}_1 u(x,t) + \mathcal{B}_2 u(x,t) + \mathcal{B}_3 u(x,t).\end{aligned}$$

By the convexity argument $R(\sum_{i=1}^3 \mathcal{B}_i u) \le \frac{1}{3}\sum_{i=1}^3 R(3\mathcal{B}_i u)$ and $R(Ca) \lesssim 1 + R(a)$, therefore it suffices to prove (6.5) with $\mathcal{B}_i u$ instead of $\mathcal{C}_2 u$.

Step 3 (proof of (6.6) and (6.7) with $\mathcal{B}_1 u$ instead of $\mathcal{C}_2 u$).

For this purpose we note that

$$\mathcal{B}_1 u \lesssim \int_{Q'\cap\{y: t\ge \frac{|x-y|^2}{4C_3}, h(x,y)\le (\frac{1}{\sqrt{t}})^{\theta}\}} \frac{1}{|x-y|^{n-2}} t^{-1-\theta/2}\, dy \lesssim t^{-\frac{1}{2}-\frac{\theta}{2}} =: t^{-\alpha}.$$

Hence $R(\mathcal{B}_1 u) \lesssim 1 + R(t^{-\alpha})$, consequently

$$\int_{Q'}\int_0^1 R(\mathcal{B}_1 u) \lesssim 1 + \int_0^1 R(t^{-\alpha})\, dt \lesssim 1.$$

Step 4 (proof of (6.6) and (6.7) with $\mathcal{B}_2 u$ instead of $\mathcal{C}_2 u$).

As $\int_{Q'\cap\{y:t\geq\frac{|x-y|^2}{4C_3}\}}\frac{1}{|x-y|^{n-2}}t^{-1/2}\,dy\sim 1$, we can apply Jensen's inequality to get

$$R(\mathcal{B}_2u)\lesssim\int_{Q'\cap\{y:t\geq\frac{|x-y|^2}{4C_3},\left(\frac{1}{\sqrt{t}}\right)^{\theta}<h(x,y)\leq\frac{1}{\sqrt{t}}\}}\frac{1}{|x-y|^{n-2}}t^{-1/2}R\left(\frac{ch(x,y)}{\sqrt{t}}\right)dy, \tag{6.9}$$

with some constant $c>0$. We have from Assumption B $P(\frac{ch}{\sqrt{t}})\lesssim 1+P(h)+P(\frac{1}{\sqrt{t}})\lesssim P(\frac{1}{\sqrt{t}})$ on the considered set of integration. Therefore $R(\frac{ch}{\sqrt{t}})\lesssim\frac{1}{\sqrt{t}}hP(\frac{ch}{\sqrt{t}})\lesssim\frac{1}{\sqrt{t}}hP(\frac{1}{\sqrt{t}})$ on this set. This implies

$$\begin{aligned}I&:=\int_{Q'}\int_0^1R(\mathcal{B}_2u)(x,t)\,dxdt\lesssim\int\int\int_A\frac{1}{|x-y|^{n-2}}\frac{1}{t}P\left(\frac{1}{\sqrt{t}}\right)h(x,y)\,dydxdt\\&\lesssim\int_{Q'}\int_{Q'}\frac{1}{|x-y|^{n-2}}\left\{\int_{(\frac{1}{h})^{\frac{2}{\theta}}}^{\frac{1}{h^2}}\frac{1}{t}P\left(\frac{1}{\sqrt{t}}\right)dt\right\}h(x,y)\chi_{h\leq\frac{2\sqrt{C_3}}{|x-y|}}\,dydxdt,\end{aligned}$$

where $A:=\{Q'\times Q'\times(0,1):t\geq\frac{|x-y|^2}{4C_3},(\frac{1}{\sqrt{t}})^{\theta}<h<\frac{1}{\sqrt{t}}\}$. Now we estimate the internal integral in brackets $\{\dots\}$ denoting it by X. Note that when $\frac{1}{\sqrt{t}}\in(h,h^{1/\theta})$, we get $P(h)\lesssim P(\frac{1}{\sqrt{t}})\lesssim 1+P(h)$ (see Remark 6.1, part 1). This (and the conditions $h<\frac{1}{\sqrt{t}}$, $h>1$) imply that $X\lesssim P(h)\int_{(\frac{1}{h})^{\frac{2}{\theta}}}^{\frac{1}{h^2}}\frac{1}{t}\,dt\sim P(h)|\ln h|$. On set of integration we have the condition $h\leq\frac{C}{|x-y|}$ with some constant $C>0$. Therefore both inequalities hold:

$$\begin{aligned}I&\lesssim\int_{Q'}\int_{Q'}\frac{|\ln|x-y||}{|x-y|^{n-2}}R(h(x,y))\,dxdy\quad\text{and}\\I&\lesssim\int_{Q'}\int_{Q'}\frac{|\ln h(x,y)|}{|x-y|^{n-2}}R(h(x,y))\,dxdy,\end{aligned}$$

which implies our assertion.

Step 5 (proof of (6.6) and (6.7) with $\mathcal{B}_3u$ instead of $\mathcal{C}_2u$).

For this purpose we apply inequality

$$R(\mathcal{B}_3u)\lesssim\int_{Q'\cap\{y:t\geq\frac{|x-y|^2}{4C_3},h(x,y)>\frac{1}{\sqrt{t}}\}}\frac{1}{|x-y|^{n-2}}t^{-1/2}R\left(\frac{ch(x,y)}{\sqrt{t}}\right)dy,$$

which is obtained by similar arguments as the ones to get (6.9). Observing that $P(\frac{ch}{\sqrt{t}})\lesssim P(h)$, we get, by similar computations as in previous step that

$$J := \int_{Q'} \int_0^1 R(\mathcal{B}_3 u)(x,t)\, dxdt \lesssim \int\int\int_B \frac{1}{|x-y|^{n-2}} \frac{1}{t} R(h(x,y))\, dydxdt$$
$$\lesssim \int_{Q'} \int_{Q'} \frac{1}{|x-y|^{n-2}} \left\{ \int_{\max\{(\frac{1}{h})^2, \frac{|x-y|^2}{4C_3}\}}^1 \frac{1}{t}\, dt \right\} R(h(x,y))\, dydxdt,$$

where $B := \{Q' \times Q' \times (0,1) : t \geq \frac{|x-y|^2}{4C_3}, h \geq \frac{1}{\sqrt{t}}\}$. This easily gives the desired estimates.

Step 6 (proof of (6.8)).

To this goal we again use the fact that when t is sufficiently small, i.e. $t < t_0$ for some $t_0 \in (0,1)$, we have $\int_{Q' \cap \{y: t < \frac{|x-y|^2}{4C_3}\}} \frac{\sqrt{t}}{|x-y|^n}\, dy \sim 1$ and then we apply Jensen's inequality to get

$$\begin{aligned} R(\mathcal{C}_3 u) &\lesssim \int_{Q' \cap \{y: t < \frac{|x-y|^2}{4C_3}\}} \frac{\sqrt{t}}{|x-y|^n} R\left(\frac{h}{\sqrt{t}}\right) dy \\ &= \int_{Q' \cap \{y: t < \frac{|x-y|^2}{4C_3}\}} \frac{1}{|x-y|^n} h(x,y) P\left(\frac{h}{\sqrt{t}}\right) dy \\ &= \int_{Q' \cap \{y: t < \frac{|x-y|^2}{4C_3}, h(x,y) \leq (\frac{1}{\sqrt{t}})^\theta\}} \frac{1}{|x-y|^n} h(x,y) P\left(\frac{h}{\sqrt{t}}\right) dy \\ &\quad + \int_{Q' \cap \{y: t < \frac{|x-y|^2}{4C_3}, h(x,y) > (\frac{1}{\sqrt{t}})^\theta\}} \frac{1}{|x-y|^n} h(x,y) P\left(\frac{h}{\sqrt{t}}\right) dy \\ &=: A(x,t) + B(x,t), \end{aligned}$$

recalling that $\theta = 2\alpha - 1$. Now we estimate $A(x,t)$ and $B(x,t)$ separately. To deal with A, we note that (as $h \leq (\frac{1}{\sqrt{t}})^\theta$), we have $P(\frac{h}{\sqrt{t}}) \lesssim 1 + P(h) + P(\frac{1}{\sqrt{t}}) \lesssim P(\frac{1}{\sqrt{t}})$. Consequently

$$\begin{aligned} I_1 &:= \int_{Q'} \int_0^1 A(x,t)\, dxdt \\ &\lesssim \int_{Q'} \int_{Q'} \frac{1}{|x-y|^{n-1}} \left\{ \int_0^{\frac{|x-y|^2}{4C_3}} \frac{1}{|x-y|} \left(\frac{1}{\sqrt{t}}\right)^\theta P\left(\frac{1}{\sqrt{t}}\right) dt \right\} dydx. \end{aligned}$$

Now we estimate the integral in brackets $\{\cdot\}$ denoted by Y. Note that on set of integration we have $\frac{1}{|x-y|} \lesssim \frac{1}{\sqrt{t}}$, moreover, $P(\lambda) \lesssim \lambda^\varepsilon$ for arbitrary $\varepsilon > 0$ (see Remark 6.1). Taking this into account we get:

$$Y \lesssim \int_0^{\frac{|x-y|^2}{4C_3}} t^{-(\alpha+\varepsilon)}\, dt \sim |x-y|^\beta,$$

where $\beta = 2(1-\alpha-\varepsilon) > 0$, (it is enough to take sufficiently small ε). This implies $I_1 \lesssim 1$.

To estimate the term with B we note that on set of integration we have $P(\frac{h}{\sqrt{t}}) \lesssim P(h)$, so that

$$I_2 := \int_{Q'}\int_0^1 B(x,t)\,dxdt \lesssim \int_{Q'}\int_{Q'} \frac{1}{|x-y|^n}\left\{\int_0^{\frac{|x-y|^2}{4C_3}} 1dt\right\} R(h(x,y))\,dydx$$
$$\lesssim \int_{Q'}\int_{Q'} \frac{1}{|x-y|^{n-2}} R(h(x,y))\,dydx.$$

The estimates when $t > t_0$ become simpler as on set if integration we have $\sqrt{4C_3t_0} < |x-y|$, therefore we omit them. Lemma is proved. □

Proof of Lemma 6.2 The proof if obvious modification of the proof of Lemma 5.1 and is based on Lemmas 6.4 and 6.5. □

7 Final results

We are now to present our main results. For this purpose we introduce the new space of functions. Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz boundary domain. By $Y^{\Psi,\Phi}_{log}(\partial\Omega)$ we will mean the modification of the space $Y^{\Psi,\Phi}(\partial\Omega)$, where the seminorm:

$$I^{\Phi}(u,\partial\Omega) = \int_{\partial\Omega}\int_{\partial\Omega} \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)\frac{1}{|x-y|^{n-2}} d\sigma(x)d\sigma(y),$$

is substituted by

$$I^{\Phi}_{\log}(u,\partial\Omega) := \int_{\partial\Omega}\int_{\partial\Omega} \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)\frac{|\ln|x-y||}{|x-y|^{n-2}} d\sigma(x)d\sigma(y),$$

where σ is the $n-1$-dimensional Hausdorff measure on $\partial\Omega$. By $Y^{\Psi,\Phi}_{L,log}(\partial\Omega)$ we will mean the completion of set $\{u \in Lip(\partial\Omega) \cap Y^{\Psi,\Phi}_{log}(\partial\Omega)\}$ in the norm of $Y^{\Psi,\Phi}_{log}(\partial\Omega)$.

Our first final result reads as follows.

Theorem 7.1 (Theorem about extension) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz boundary domain, $u \in Y^{\Phi,\Phi}_{L,\log}(\partial\Omega)$, where (Ψ,Φ) is as in Assumption A (see Definition 2.1) and let Φ satisfy Assumption B (see Definition 6.1). Then there exists function $\widetilde{u} \in W^{1,\Psi}_L(\Omega)$ such that $\mathrm{Tr}\,\widetilde{u} = u$, moreover, we have*

$$\int_\Omega \Psi(|\widetilde{u}|)\,dx + \int_\Omega \Psi(|\nabla\widetilde{u}|)\,dx$$
$$\lesssim 1 + \int_{\partial\Omega} \Phi(|u|)d\sigma(x)$$
$$+ \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\ln|x-y||}{|x-y|^{n-2}}\Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)\,dxdy.$$

Consequently

$$\|\widetilde{u}\|_{W^{1,\Psi}(\Omega)} \leq C\|u\|_{Y^{\Phi,\Phi}_{\log}(\partial\Omega)},$$

with constant C independent of u.

Remark 7.1 As a consequence of Theorem 7.1 we obtain inequality

$$\int_\Omega \Psi(|\widetilde{u}|)\,dx + \int_\Omega \Psi(|\nabla\widetilde{u}|)\,dx \\ \leq C_\delta \left\{1+\int_{\partial\Omega} \Phi(|u|)d\sigma(x) + \int_{\partial\Omega}\int_{\partial\Omega} \frac{1}{|x-y|^{n-2+\delta}} \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)\right\} dxdy,$$

with an arbitrary $\delta > 0$ independent on u. Note that $\int_{\partial\Omega} \Phi(|u|)d\sigma(x) \prec 1 + \int_{\partial\Omega} \Psi(|u|)d\sigma(x)$, so this inequality is very close to the following one:

$$\int_\Omega \Psi(|\widetilde{u}|)\,dx + \int_\Omega \Psi(|\nabla\widetilde{u}|)\,dx \\ \prec \left\{1+\int_{\partial\Omega} \Psi(|u|)d\sigma(x) + \int_{\partial\Omega}\int_{\partial\Omega} \frac{1}{|x-y|^{n-2}} \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)\right\} dxdy, \tag{7.1}$$

and the above implies norm inequality $\|\widetilde{u}\|_{W^{1,\Psi}(\Omega)} \leq C\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)}$. If one could find extension operator $u \mapsto \tilde{u}$ from u defined on $\partial\Omega$ to $\tilde{u}$ defined on Ω for which inequality (7.1) holds, it would imply that trace operator from Theorem 2.3, acting from $W^{1,\Psi}(\Omega)$ to $Y^{\Psi,\Phi}(\partial\Omega)$, is a surjection. However we have not proven such property dealing with heat extension operator, our result seems to support that conjecture.

Proof of Theorem 7.1 Using standard covering arguments (see e.g. the book [33], or [34]), suitable partition of the unity on $\partial\Omega$ and the biLipschitz equivalence of sets $B(x_0, r)\cap\Omega$, where $x_0 \in \partial\Omega$, r is sufficiently small, with the cube $Q = Q' \times (0, 1) = (0, 1)^n$, we observe that the proof reduces to the case when we deal with the heat extension operator from Q' to Q. Then we use Lemmas 4.1, 5.1 and 6.2. This requires to verify the condition (4.2) and (5.2).

We start with the verification of (4.2). We have

$$\Phi(xy) = xyP(xy) \lesssim xy(1 + P(x) + P(y)) = xy + y\Phi(x) + x\Phi(y) =: \mathcal{L}.$$

Let $G(x, y) := \max\{x, \Phi(x)\}$. When $y > 1$ we have $y \prec \Phi(y)$. Consequently $\mathcal{L} \precsim G(x)$ when $y < 1$ and $\mathcal{L} \precsim G(x)\Phi(y)$ when $y > 1$. This implies

$$\Phi(xy) \precsim G(x) + G(x)\Phi(y).$$

The verification of the condition $\int_0^1 G(|\ln t|)dt < \infty$ follows from the following two estimates:

$$\int_0^1 |\ln t| dt \overset{w=\ln\frac{1}{t}}{=} \int_0^\infty we^{-w} dw < \infty,$$

$$\int_0^1 |\ln t| P(|\ln t|) dt \overset{w=\ln\frac{1}{t}}{=} \int_0^\infty wP(w)e^{-w} dw$$
$$\prec 1 + \int_1^\infty w^2 \left(\frac{P(w)}{w}\right) e^{-w} dw \prec 1 + \int_1^\infty w^2 e^{-w} dw < \infty.$$

This completes the proof of (4.2). To verify (5.2) we have to check that

$$\sup_{s<\frac{1}{2}} \frac{1}{s} \int_0^{s^2} \frac{G(c|\ln t|)}{|\ln t|} dt < \infty \quad \text{for any } c > 0.$$

This follows from chain of inequalities where $s < 1/2$:

$$\frac{1}{s} \int_0^{s^2} \frac{G(c|\ln t|)}{|\ln t|} dt = \frac{1}{s} \int_0^{s^2} \frac{c|\ln t| + c|\ln t| P(|\ln t|)}{|\ln t|} dt$$
$$\prec 1 + \frac{1}{s} \int_0^{s^2} P(|\ln t|) dt,$$
$$\int_0^{s^2} P(|\ln t|) dt \overset{w=\ln\frac{1}{t}}{=} \int_{-2\ln s}^\infty P(w)e^{-w} dw = \int_{-2\ln s}^\infty w \left(\frac{P(w)}{w}\right) e^{-w} dw$$
$$\overset{w>\ln 4}{\prec} \int_{-2\ln s}^\infty we^{-w} dw \prec \int_{-2\ln s}^\infty e^{-w/2} dw = \int_{-\ln s}^\infty e^{-w} dw = s.$$

□

Our final result establishes regularity properties of solutions to heat equation with the initial condition in weighted Orlicz Slobodetskii space.

Theorem 7.2 *Let* $u \in Y^{\Phi,\Phi}_{L,\log}(\mathbb{R}^{n-1} \times \{0\})$*, (regularity property)* (Ψ, Φ) *be as in Assumption A (see Definition 2.1),* Φ *satisfy Assumption B (see Definition 6.1),* $T > 0$. *Moreover, let* $\tilde{u} \in W^{1,1}_{loc}(\mathbb{R}^{n-1} \times (0, T))$ *be the solution to heat equation*

$$\tilde{u}(x) := \begin{cases} \tilde{u}_t(x,t) = \Delta u(x,t), & \text{in } \mathbb{R}^{n-1} \times (0,T) \\ \tilde{u}(x,0) = u(x) & \text{on } \mathbb{R}^{n-1} \times \{0\} \end{cases} \tag{7.2}$$

Then $\widetilde{u} \in W^{1,\Psi}_L(\mathbb{R}^{n-1} \times (0, T))$ *and we have*

$$\int_{\mathbb{R}^{n-1}\times(0,T)} \Psi(|\widetilde{u}|)\, dxdt + \int_{\mathbb{R}^{n-1}\times(0,T)} \Psi(|\nabla\widetilde{u}|)\, dxdt$$
$$\lesssim 1 + \int_{\mathbb{R}^{n-1}} \Phi(|u|) dx$$

$$+\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|\ln|x-y||}{|x-y|^{n-2}}\Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)dxdy;$$
$$\|\widetilde{u}\|_{W^{1,\Psi}((\mathbb{R}^{n-1}\times(0,T)))} \lesssim \|u\|_{Y^{\Phi,\Phi}_{\log}(\mathbb{R}^{n-1})},$$

and constants in the above estimates are independent of u.

Proof The proof is based on the choice of suitable Lipschitz resolution of unity on $\mathbb{R}^{n-1}$: $\{\phi_i\}_{i\in\mathbb{N}}$ with the control of Lipschitz constants and supports of the ϕ_i's where supp$\phi_i \subseteq Q_i^{'}$ and $Q_i^{'}$'s are unit cubes. The we provide the estimates for $u_i = u\phi_i$. The details are left to the reader. □

Example 7.1 The pair $(\Phi(\lambda), \Psi(\lambda)) = (\lambda(\log(2+\lambda))^{\alpha}, \lambda(\log(2+\lambda))^{\alpha+1})$ where $\alpha > 0$ obeys assumptions of Theorems 7.1 and 7.2.

Remark 7.2 [Open questions]

1. We do not know what is the optimal space for the initial data u to have the solution of (7.2) in Orlicz–Sobolev space $W^{1,\Psi}(\mathbb{R}^{n-1}\times(0,T))$.
2. It would be interesting to know under what conditions one has:$u \in Y^{\Phi,\Phi}_{\omega_1}(\Omega) \Rightarrow \tilde{u} \in W^{1,\Psi}_{\omega_2}(\Omega\times(0,T))$ where Ω is the given domain and ω_1, ω_2 are given measures defined on Ω and $\Omega\times(0,T)$, respectively.

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