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Littlewood–Paley Theory for Triangle Buildings

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Abstract For the natural two-parameter filtration $(\mathcal{F}_\lambda : \lambda \in P)$ on the boundary of a triangle building, we define a maximal function and a square function and show their boundedness on $L^p(\Omega_0)$ for $p \in (1, \infty)$. At the end, we consider $L^p(\Omega_0)$ boundedness of martingale transforms. If the building is of $\mathrm{GL}(3, \mathbb{Q}_p)$, then Ω_0 can be identified with p-adic Heisenberg group.

Keywords Affine building · Littlewood–Paley theory · Square function · Maximal function · Multi-index filtration · Heisenberg group · p-adic numbers

Mathematics Subject Classification Primary 22E35 · 51E24 · 60G42

1 Introduction

Let $(\Omega, \mathcal{F}, \pi)$ be a σ-finite measure space. A sequence of σ-algebras $(\mathcal{F}_n : n \in \mathbb{Z})$ is a filtration if $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Given f a locally integrable function on Ω by $\mathbb{E}[f|\mathcal{F}_n]$, we denote its conditional expectation value with respect to $\mathcal{F}_n$. Let M^* and S denote, respectively, the maximal function and the square function defined by

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$$M^* f = \sup_{n \in \mathbb{Z}} |f_n|,$$

and

$$Sf = \Big(\sum_{n \in \mathbb{Z}} |d_n f|^2 \Big)^{1/2}, \tag{1.1}$$

where $d_n f = f_n - f_{n-1}$. The Hardy and Littlewood maximal estimate (see [8]) implies that

$$\pi\big(\{M^* f > \lambda\}\big) \leq \lambda^{-1} \int_{M^* f > \lambda} |f| \, \mathrm{d}\pi,$$

from where it is easy to deduce that for $p \in (1, \infty]$

$$\big\| M^* f \big\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}.$$

For the square function, if $p \in (1, \infty)$, then there is $C_p > 1$ such that

$$C_p^{-1} \|f\|_{L^p} \leq \|Sf\|_{L^p} \leq C_p \|f\|_{L^p}. \tag{1.2}$$

The inequality (1.2) goes back to Paley [12], and has been reproved in many ways, for example, [2–4, 7, 10]. Its main application is in proving the L^p-boundedness of martingale transforms (see [2]), that is, for operators of the form

$$Tf = \sum_{n \in \mathbb{Z}} a_n d_n f$$

where $(a_n : n \in \mathbb{Z})$ is a sequence of uniformly bounded functions such that a_{n+1} is $\mathcal{F}_n$-measurable.

In 1975, Cairoli and Walsh (see [5]) have started to generalize the theory of martingales to two-parameter cases. Let us recall that a sequence of σ-fields $(\mathcal{F}_{n,m} : n, m \in \mathbb{Z})$ is a two-parameter filtration if

$$\mathcal{F}_{n+1,m} \subset \mathcal{F}_{n,m}, \qquad \text{and} \qquad \mathcal{F}_{n,m+1} \subset \mathcal{F}_{n,m}. \tag{1.3}$$

Then $(f_{n,m} : n, m \in \mathbb{Z})$ is a two-parameter martingale if

$$\mathbb{E}[f_{n+1,m} | \mathcal{F}_{n,m}] = f_{n,m}, \qquad \text{and} \qquad \mathbb{E}[f_{n,m+1} | \mathcal{F}_{n,m}] = f_{n,m}. \tag{1.4}$$

Observe that conditions (1.3) and (1.4) impose a structure only for comparable indices. In that generality, it is hard, if not impossible, to build the Littlewood–Paley theory.

This lead to the introduction of other (smaller) classes of martingales (see [19,20]). In particular, in [5], Cairoli and Walsh introduced the following condition

$$\mathbb{E}[f|\mathcal{F}_{n,\infty}|\mathcal{F}_{\infty,m}] = \mathbb{E}[f|\mathcal{F}_{\infty,m}|\mathcal{F}_{n,\infty}] = f_{n,m} \tag{F_4}$$

where

$$\mathcal{F}_{n,\infty} = \sigma\Big(\bigcup_{m\in\mathbb{Z}} \mathcal{F}_{n,m}\Big), \quad \text{and} \quad \mathcal{F}_{\infty,m} = \sigma\Big(\bigcup_{n\in\mathbb{Z}} \mathcal{F}_{n,m}\Big).$$

Under (F_4), the result obtained by Jensen, Marcinkiewicz, and Zygmund in [9] implies that the maximal function

$$M^* f = \sup_{n,m\in\mathbb{Z}} |f_{n,m}| \tag{1.5}$$

is bounded on $L^p(\Omega)$ for $p \in (1, \infty]$. In this context, the square function is defined by

$$Sf = \Big(\sum_{n,m\in\mathbb{Z}} |d_{n,m} f|^2\Big)^{1/2} \tag{1.6}$$

where $d_{n,m}$ denote the double difference operator, i.e.

$$d_{n,m} f = f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}.$$

In [11], it was observed by Metraux that the boundedness of S on $L^p(\Omega)$ for $p \in (1, \infty)$ is implied by the one parameter Littlewood–Paley theory. Also the concept of a martingale transform has a natural generalization, that is,

$$Tf = \sum_{n,m\in\mathbb{Z}} a_{n,m} d_{n,m} f$$

where $(a_{n,m} : n, m \in \mathbb{Z})$ is a sequence of uniformly bounded functions such that $a_{n+1,m+1}$ is $\mathcal{F}_{n,m}$-measurable.

In this article, we are interested in a case when the condition (F_4) is not satisfied. The simplest example may be obtained by considering the Heisenberg group together with the non-isotropic two parameter dilations

$$\delta_{s,t}(x, y, z) = (sx, ty, stz).$$

Since in this setup the dyadic cubes do not posses the same properties as the Euclidean cubes, it is more convenient to work on the p-adic version of the Heisenberg group. We observe that this group can be identified with Ω_0, a subset of a boundary of the building of $\mathrm{GL}(3, \mathbb{Q}_p)$ consisting of the points opposite to a given ω_0. The set Ω_0 has a natural two-parameter filtration $(\mathcal{F}_{n,m} : n, m \in \mathbb{Z})$ (see Sect. 2 for details). The maximal function and the square function are defined by (1.5) and (1.6), respectively. The results we obtain are summarized in the following three theorems.

Theorem A *For each $p \in (1, \infty]$, there is $C_p > 0$ such that for all $f \in L^p(\Omega_0)$*

$$\left\| M^* f \right\|_{L^p} \leq C_p \| f \|_{L^p}.$$

Theorem B *For each $p \in (1, \infty)$, there is $C_p > 1$ such that for all $f \in L^p(\Omega_0)$*

$$C_p^{-1} \| f \|_{L^p} \leq \| S f \|_{L^p} \leq C_p \| f \|_{L^p}.$$

Theorem C *If $(a_{n,m} : n, m \in \mathbb{Z})$ is a sequence of uniformly bounded functions such that $a_{n+1,m+1}$ is $\mathcal{F}_{n,m}$-measurable, then the martingale transform*

$$T f = \sum_{n,m \in \mathbb{Z}} a_{n,m} d_{n,m} f$$

is bounded on $L^p(\Omega_0)$, for all $p \in (1, \infty)$.

Let us briefly describe methods we use. First, we observe that instead of (F_4) the stochastic basis satisfies the remarkable identity (2.2). Based on it, we show that the following pointwise estimate holds

$$M^*(|f|) \leq C\big(L^* R^* L^* R^*(|f|) + R^* L^* R^* L^*(|f|)\big) \tag{1.7}$$

proving the maximal theorem. Thanks to the two-parameter Khintchine's inequality, to bound the square function S, it is enough to show Theorem C. To do so, we define a new square function $\mathcal{S}$ which has a nature similar to the square function used in the presence of (F_4). Then, we adapt the technique developed by Duoandikoetxea and Rubio de Francia in [6] (see Theorem 3). This implies L^p-boundedness of $\mathcal{S}$. Since S does not preserve the L^2 norm, the lower bound requires an extra argument. Namely, we view the square function S as an operator with values in $L^p(\ell^2)$ and take its dual. As a consequence of Theorem 3 and the identity (4.7), the latter is bounded on L^p.

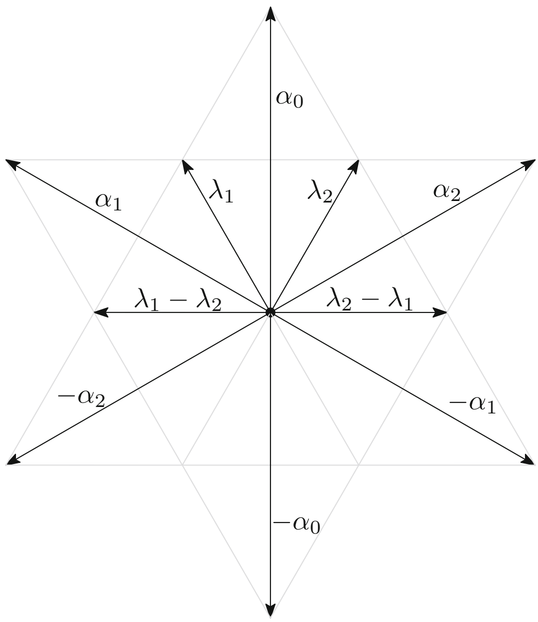
Finally, let us comment on the behavior of the maximal function M^* close to L^1. Based on the pointwise estimate (1.7), in view of [8], we conclude that M^* is of weak-type for functions in the Orlicz space $L(\log L)^3$. To better understand the maximal function M^*, we investigate exact behavior close to L^1. This together with weighted estimates is the subject of the forthcoming paper. It is also interesting how to extend Theorems A, B and C to higher rank and other types of affine buildings.

1.1 Notation

For two quantities $A > 0$ and $B > 0$, we say that $A \lesssim B$ ($A \gtrsim B$) if there exists an absolute constant $C > 0$ such that $A \leq CB$ ($A \geq CB$).

If $\lambda \in P$ we set $|\lambda| = \max\{|\lambda_1|, |\lambda_2|\}$.

Fig. 1 A_2 root system



2 Triangle Buildings

2.1 Coxeter Complex

We recall basic facts about the A_2 root system and the $\tilde{A}_2$ Coxeter group. A general reference is [1]. Let $\mathfrak{a}$ be the hyperplane in $\mathbb{R}^3$ defined as

$$\mathfrak{a} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}.$$

We denote by $\{e_1, e_2, e_3\}$ the canonical orthonormal basis of $\mathbb{R}^3$ with respect to the standard scalar product $\langle \cdot, \cdot \rangle$. We set $\alpha_1 = e_2 - e_1$, $\alpha_2 = e_3 - e_2$, $\alpha_0 = e_3 - e_1$ and $I = \{0, 1, 2\}$. The A_2 root system is defined by

$$\Phi = \{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2\}.$$

We choose the base $\{\alpha_1, \alpha_2\}$ of Φ. The corresponding positive roots are $\Phi^+ = \{\alpha_0, \alpha_1, \alpha_2\}$. Denote by $\{\lambda_1, \lambda_2\}$ the basis dual to $\{\alpha_1, \alpha_2\}$; its elements are called the *fundamental co-weights*. Their integer combinations, form the *co-weight lattice* P. As in Fig. 1, we always draw λ_1 pointing up and to the left and λ_2 up and to the right. Likewise $\lambda_1 - \lambda_2$ is drawn pointing directly left, while $\lambda_2 - \lambda_1$ points directly right. Because $\langle \lambda_1, \alpha_0 \rangle = \langle \lambda_2, \alpha_0 \rangle = 1$, we see that for any $\lambda \in P$ the expression $\langle \lambda, \alpha_0 \rangle$ represents the vertical level of λ. For $\lambda = i\lambda_1 + j\lambda_2$, that level is $i + j$.

Let $\mathcal{H}$ be the family of affine hyperplanes, called *walls*,

$$H_{j;k} = \{x \in \mathfrak{a} : \langle x, \alpha_j \rangle = k\}$$

where $j \in I, k \in \mathbb{Z}$. To each wall $H_{j;k}$, we associate $r_{j;k}$ the orthogonal reflection in $\mathfrak{a}$, i.e.

$$r_{j;k}(x) = x - \big(\langle x, \alpha_j \rangle - k\big)\alpha_j.$$

Set $r_1 = r_{1;0}$, $r_2 = r_{2;0}$ and $r_0 = r_{0;1}$. The *finite Weyl group* W_0 is the subgroup of $\mathrm{GL}(\mathfrak{a})$ generated by r_1 and r_2. The *affine Weyl group* W is the subgroup of $\mathrm{Aff}(\mathfrak{a})$ generated by r_0, r_1 and r_2.

Let $\mathcal{C}$ be the family of open connected components of $\mathfrak{a} \setminus \bigcup_{H \in \mathcal{H}} H$. The elements of $\mathcal{C}$ are called *chambers*. By C_0, we denote the fundamental chamber, i.e.

$$C_0 = \{x \in \mathfrak{a} : \langle x, \alpha_1 \rangle > 0, \langle x, \alpha_2 \rangle > 0, \langle x, \alpha_0 \rangle < 1\}.$$

The group W acts simply transitively on $\mathcal{C}$. Moreover, $\overline{C_0}$ is a fundamental domain for the action of W on $\mathfrak{a}$ (see e.g. [1, VI, §1-3]). The vertices of C_0 are $\{0, \lambda_1, \lambda_2\}$. The set of all vertices of all $C \in \mathcal{C}$ is denoted by $V(\Sigma)$. Under the action of W, $V(\Sigma)$ is made up of three orbits, $W(0)$, $W(\lambda_1)$, and $W(\lambda_2)$. Vertices in the same orbit are said to have the same *type*. Any chamber $C \in \mathcal{C}$ has one vertex in each orbit or in other words one vertex of each of the three types.

The family $\mathcal{C}$ may be regarded as a simplicial complex Σ by taking as the simplexes all non-empty subsets of vertices of C, for all $C \in \mathcal{C}$. Two chambers C and C' are *i-adjacent* for $i \in I$ if $C = C'$ or if there is $w \in W$ such that $C = wC_0$ and $C' = wr_iC_0$. Since $r_i^2 = 1$ this defines an equivalence relation.

The *fundamental sector* is defined by

$$\mathcal{S}_0 = \{x \in \mathfrak{a} : \langle x, \alpha_1 \rangle > 0, \langle x, \alpha_2 \rangle > 0\}.$$

Given $\lambda \in P$ and $w \in W_0$ the set $\lambda + w\mathcal{S}_0$ is called a *sector* in Σ with *base vertex* λ. The angle spanned by a sector at its base vertex is $\pi/3$.

2.2 The Definition of Triangle Buildings

For the theory of affine buildings, we refer the reader to [13]. See also the first author's expository paper [14], for an elementary introduction to the p-adics, and to precisely the sort of the buildings which this paper deals with.

A simplicial complex $\mathscr{X}$ is an *$\tilde{A}_2$ building*, or as we like to call it, a *triangle building*, if each of its vertices is assigned one of the three types, and if it contains a family of subcomplexes called *apartments* such that

1. Each apartment is type-isomorphic to Σ,
2. Any two simplexes of $\mathscr{X}$ lie in a common apartment,
3. For any two apartments, $\mathscr{A}$ and $\mathscr{A}'$, having a chamber in common, there is a type-preserving isomorphism $\psi : \mathscr{A} \to \mathscr{A}'$ fixing $\mathscr{A} \cap \mathscr{A}'$ pointwise.

We assume also that the system of apartments is *complete*, meaning that any subcomplex of $\mathscr{X}$ type-isomorphic to Σ is an apartment. A simplex C is a *chamber* in $\mathscr{X}$

if it is a chamber for some apartment. Two chambers of $\mathscr{X}$ are *i-adjacent* if they are i-adjacent in some apartment. For $i \in I$ and for a chamber C of $\mathscr{X}$, let $q_i(C)$ be equal to

$$q_i(C) = |\{C' \in \mathscr{X} : C' \sim_i C\}| - 1.$$

It may be proved that $q_i(C)$ is independent of C and of i. Denote the common value by q, and assume local finiteness: $q < \infty$. Any *edge* of $\mathscr{X}$, i.e., any 1-simplex, is contained in precisely $q + 1$ chambers.

It follows from the axioms that the ball of radius one about any vertex x of $\mathscr{X}$ is made up of x itself, which is of one type, $q^2 + q + 1$ vertices of a second type, and a further $q^2 + q + 1$ vertices of the third type. Moreover, adjacency between vertices of the second and third types makes them into, respectively, the points and the lines of a finite projective plane.

A subcomplex $\mathscr{S}$ is called a *sector* of $\mathscr{X}$ if it is a sector in some apartment. Two sectors are called *equivalent* if they contain a common subsector. Let Ω denote the set of equivalence classes of sectors. If x is a vertex of $\mathscr{X}$ and $\omega \in \Omega$, there is a unique sector denoted $[x, \omega]$ which has base vertex x and represents ω.

Given any two points ω and $\omega' \in \Omega$, one can find two sectors representing them which lie in a common apartment. If that apartment is unique, we say that ω and ω' are *opposite*, and denote the unique apartment by $[\omega, \omega']$. In fact, ω and ω' are opposite precisely when the two sectors in the common apartment point in opposite directions in the Euclidean sense.

2.3 Filtrations

We fix once and for all an origin vertex $O \in \mathscr{X}$ and a point $\omega_0 \in \Omega$. Choose O so that it has the same type as the origin of Σ. Let $\mathscr{S}_0 = [O, \omega_0]$ be the sector representing ω_0 with base vertex O. By Ω_0, we denote the subset of Ω consisting of ω's opposite to ω_0. For purposes of motivation only, we recall that if $\mathscr{X}$ is the building of $\mathrm{GL}(3, \mathbb{Q}_p)$, then Ω_0 can be identified with the p-adic Heisenberg group (see Appendix 1 for details).

Let $\mathscr{A}_0$ be any apartment containing $\mathscr{S}_0$. By ψ, we denote the type-preserving isomorphism between $\mathscr{A}_0$ and Σ such that $\psi(\mathscr{S}_0) = -S_0$. We set $\rho = \psi \circ \rho_0$ where ρ_0 is the retraction from $\mathscr{X}$ to $\mathscr{A}_0$. With these definitions, $\rho : \mathscr{X} \to \Sigma$ is a type-preserving simplicial map, and for any $\omega \in \Omega_0$ the apartment $[\omega, \omega_0]$ maps bijectively to Σ with ω_0 mapping to the bottom (of Fig. 1) and ω mapping to the top.

For any vertex x of $\mathscr{X}$, define the subset $E_x \subset \Omega_0$ to consist of all ω's such that x belongs to $[\omega, \omega_0]$; an equivalent condition is that $[x, \omega_0] \subseteq [\omega, \omega_0]$. Fix $\lambda \in P$. By $\mathcal{F}_\lambda$, we denote the σ-field generated by sets E_x for $x \in \mathscr{X}$ with $\rho(x) = \lambda$. There are countably many such x, and the corresponding sets E_x are mutually disjoint, and hence, $\mathcal{F}_\lambda$ is a countably generated atomic σ-field.

Let $\preceq$ denote the partial order on P where $\lambda \preceq \mu$ if and only if $\langle \lambda - \mu, \alpha_1 \rangle \leq 0$ and $\langle \lambda - \mu, \alpha_2 \rangle \leq 0$. If we draw and orient Σ as in Fig. 1, then $\lambda \preceq \mu$ exactly when μ lies in the sector pointing upward from λ.

Proposition 2.1 *If $\lambda \preceq \mu$, then $\mathcal{F}_\lambda \subset \mathcal{F}_\mu$.*

Proof Choose any vertex x so that $\rho(x) = \mu$. Because $\lambda \preceq \mu$, there is a unique vertex y in the sector $[x, \omega_0]$ so that $\rho(y) = \lambda$. For any $\omega \in E_x$, the apartment $[\omega, \omega_0]$ contains x, and hence, it contains $[x, \omega_0]$, which hence contains y. This establishes that $E_x \subseteq E_y$. In other words, each atom of $\mathcal{F}_\mu$ is a subset of some atom of $\mathcal{F}_\lambda$. Hence, each atom of $\mathcal{F}_\lambda$ is a disjoint union of atoms of $\mathcal{F}_\mu$. □

In fact, Proposition 2.1 says that $(\mathcal{F}_\lambda : \lambda \in P) = \big(\mathcal{F}_{i\lambda_1 + j\lambda_2} : i, j \in \mathbb{Z}\big)$ is a two parameter filtration. Let

$$\mathcal{F} = \sigma\Big(\bigcup_{\lambda \in P} \mathcal{F}_\lambda\Big).$$

Let π denote the unique σ-additive measure on $(\Omega_0, \mathcal{F})$ such that for $E_x \in \mathcal{F}_\lambda$

$$\pi(E_x) = q^{-2\langle\lambda, \alpha_0\rangle}.$$

All σ-fields in this paper should be extended so as to include π-null sets.

A function $f(\omega)$ on Ω_0 is $\mathcal{F}_\lambda$-measurable if it depends only on that part of the apartment $[\omega, \omega_0]$ which retracts under ρ to the sector pointing downward from λ. For $i, j \in \mathbb{Z}$ set

$$\mathcal{F}_{i,\infty} = \sigma\Big(\bigcup_{j' \in \mathbb{Z}} \mathcal{F}_{i\lambda_1 + j'\lambda_2}\Big), \qquad \mathcal{F}_{\infty,j} = \sigma\Big(\bigcup_{i' \in \mathbb{Z}} \mathcal{F}_{i'\lambda_1 + j\lambda_2}\Big).$$

A function $f(\omega)$ on Ω_0 is $\mathcal{F}_{i,\infty}$-measurable (respectively $\mathcal{F}_{\infty,j}$-measurable) if it depends only on that part of the apartment which retracts to a certain "lower" half-plane with boundary parallel to λ_2 (respectively λ_1).

If $\mathcal{F}'$ is σ-subfield of $\mathcal{F}$, we denote by $\mathbb{E}[f|\mathcal{F}']$ the Radon–Nikodym derivative with respect to $\mathcal{F}'$. If $\mathcal{F}''$ is another σ-subfield of $\mathcal{F}$, we write

$$\mathbb{E}[f|\mathcal{F}'|\mathcal{F}''] = \mathbb{E}\big[\mathbb{E}[f|\mathcal{F}']\big|\mathcal{F}''\big].$$

The σ-field generated by $\mathcal{F}' \cup \mathcal{F}''$ is denoted by $\mathcal{F}' \vee \mathcal{F}''$. We write $f_\lambda = \mathbb{E}_\lambda f = \mathbb{E}[f|\mathcal{F}_\lambda]$ for $\lambda \in P$. If $\lambda \preceq \mu$, then it follows from Proposition 2.1 that $\mathbb{E}_\mu \mathbb{E}_\lambda = \mathbb{E}_\lambda \mathbb{E}_\mu = \mathbb{E}_\lambda$.

We note that the Cairoli–Walsh condition (F_4) introduced in [5] is not satisfied, i.e.

$$\mathbb{E}_{\lambda+\lambda_1}\mathbb{E}_{\lambda+\lambda_2} \neq \mathbb{E}_\lambda.$$

Instead of (F_4), we have

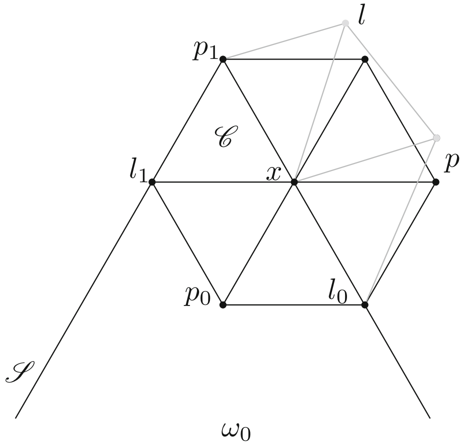
Lemma 2.2 *For a locally integrable function f on Ω_0*

$$\mathbb{E}[f_{\lambda+\lambda_1}|\mathcal{F}_{\lambda+\lambda_2}|\mathcal{F}_{\lambda+\lambda_1}] = q^{-1} f_{\lambda+\lambda_1} - q^{-1}\mathbb{E}[f_{\lambda+\lambda_1}|\mathcal{F}_{\lambda+\lambda_1-\lambda_2} \vee \mathcal{F}_\lambda] + f_\lambda, \quad (2.1)$$

$$\big(\mathbb{E}_{\lambda+\lambda_2}\mathbb{E}_{\lambda+\lambda_1}\big)^2 = q^{-1}\mathbb{E}_{\lambda+\lambda_2}\mathbb{E}_{\lambda+\lambda_1} + (1 - q^{-1})\mathbb{E}_\lambda, \quad (2.2)$$

and likewise if we exchange λ_1 and λ_2.

Fig. 2 Residue of x



Proof For the proof of (2.1) it is enough to consider $f = \mathbf{1}_{E_{p_1}}$ where p_1 is a vertex in $\mathscr{X}$ such that $\rho(p_1) = \lambda + \lambda_1$. Let $\mathscr{S}$ be the sector $[p_1, \omega_0]$ and let x be the unique vertex of $\mathscr{S}$ with $\rho(x) = \lambda$. The ball in $\mathscr{X}$ of radius 1 around x has the structure of a finite projective plane. In Fig. 2, the spot marked x is for vertices of $\mathscr{X}$ which retract via ρ to λ. Recall that E_x is an atom of the σ-field $\mathcal{F}_\lambda$. The spot marked p_1 is for vertices retracting to $\lambda + \lambda_1$; the spot marked l is for vertices retracting to $\lambda + \lambda_2$; the spot marked l_1 is for vertices retracting to $\lambda + \lambda_1 - \lambda_2$; etc. In the ball of radius 1 around x, only x itself retracts to the spot marked x. The line-type vertex known as l_0 is the only vertex in the ball retracting to its spot; q line-type vertices retract to the same spot as l_1; the remaining q^2 line-type vertices retract to the spot marked l. Likewise, p_0 is the unique point-type vertex of the ball retracting to its spot; q point-type vertices retract to the spot marked p; q^2 retract to the same spot as p_1. It follows that

$$\mathbb{E}[\mathbf{1}_{E_{p_1}} | \mathcal{F}_\lambda] = q^{-2}\mathbf{1}_{E_x} = q^{-2} \sum_{p' \nsim l_0} \mathbf{1}_{E_{p'}} = q^{-2} \sum_{l \nsim p_0} \mathbf{1}_{E_l}$$

and

$$\mathbb{E}[\mathbf{1}_{E_{p_1}} | \mathcal{F}_{\lambda+\lambda_1-\lambda_2} \vee \mathcal{F}_\lambda] = q^{-1}\mathbf{1}_{E_x \cap E_{l_1}} = q^{-1} \sum_{\substack{p' \sim l_1 \\ p' \nsim l_0}} \mathbf{1}_{E_{p'}}$$

where p' runs through the point-type vertices of the ball, l runs through the line-type vertices of the ball, and $\sim$ stands for the incidence relation. We have

$$\mathbb{E}[\mathbf{1}_{E_{p_1}} | \mathcal{F}_{\lambda+\lambda_2}] = q^{-1} \sum_{\substack{l \sim p_1 \\ l \nsim p_0}} \mathbf{1}_{E_l}. \tag{2.3}$$

Therefore, we obtain

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{E_{p_1}}|\mathcal{F}_{\lambda+\lambda_2}|\mathcal{F}_{\lambda+\lambda_1}] &= q^{-2}\sum_{\substack{l\sim p_1\\ l\nsim p_0}}\sum_{\substack{p'\sim l\\ p'\nsim l_0}}\mathbf{1}_{E_{p'}} = q^{-1}\mathbf{1}_{E_{p_1}} + q^{-2}\sum_{\substack{p'\nsim l_0\\ p'\nsim l_1}}\mathbf{1}_{E_{p'}}\\ &= q^{-1}\mathbf{1}_{E_{p_1}} + q^{-2}\sum_{p'\nsim l_0}\mathbf{1}_{E_{p'}} - q^{-2}\sum_{\substack{p'\sim l_1\\ p'\nsim l_0}}\mathbf{1}_{E_{p'}},\end{aligned} \tag{2.4}$$

which finishes the proof of (2.1). Applying one more average to the next to the last expression of (2.4), we get

$$\mathbb{E}[\mathbf{1}_{E_{p_1}}|\mathcal{F}_{\lambda+\lambda_2}|\mathcal{F}_{\lambda+\lambda_1}|\mathcal{F}_{\lambda+\lambda_2}] = q^{-2}\sum_{\substack{l\sim p_1\\ l\nsim p_0}}\mathbf{1}_{E_l} + q^{-3}\sum_{\substack{p'\nsim l_0\\ p'\nsim l_1}}\sum_{\substack{l\sim p'\\ l\nsim p_0}}\mathbf{1}_{E_l}.$$

For any line $l \nsim p_0$, there are q points p' such that $p' \sim l$ and $p' \nsim l_0$ and among them there is exactly one incident to l_1. Hence, in the last sum, each line $l \nsim p_0$ appears $q-1$ times. Thus, we can write

$$q^{-3}\sum_{\substack{p'\nsim l_0\\ p'\nsim l_1}}\sum_{\substack{l\sim p'\\ l\nsim p_0}} = q^{-3}(q-1)\sum_{l\nsim p_0}\mathbf{1}_{E_l} = (1-q^{-1})\mathbb{E}[\mathbf{1}_{E_{p_1}}|\mathcal{F}_\lambda]$$

proving (2.2). □

The following lemma describes the composition of projections on the same level.

Lemma 2.3 *If $k, j \in \mathbb{Z}$ are such that $k \geq j \geq 0$ or $k \leq j \leq 0$ then*

$$\mathbb{E}_{\lambda+k(\lambda_2-\lambda_1)}\mathbb{E}_\lambda = \mathbb{E}_{\lambda+k(\lambda_2-\lambda_1)}\mathbb{E}_{\lambda+j(\lambda_2-\lambda_1)}\mathbb{E}_\lambda. \tag{2.5}$$

Proof We carry out the proof for $k \geq j \geq 0$. For any $\omega \in \Omega_0$, there is a connected chain of vertices $(x_i : 0 \leq i \leq k) \subseteq [\omega, \omega_0]$ with $\rho(x_i) = \lambda + k(\lambda_2 - \lambda_1)$. Suppose, conversely, that $(x_i : 0 \leq i \leq k)$ is a connected chain of vertices and that $\rho(x_i) = \lambda + k(\lambda_2 - \lambda_1)$. Construct a subcomplex $\mathscr{B} \subset \mathscr{X}$ by putting together $([x_i, \omega_0] : 0 \leq i \leq k)$, the edges between the x_i's and the triangles pointing downward from those edges to ω_0. Referring to Fig. 3, the extra triangle pointing downward from the first edge has vertices x_0, x_1, and y_0. Note that $[x_0, \omega_0] \cap [x_1, \omega_0] = [y_0, \omega_0]$. Proceeding one step at a time, one may verify that the restriction of ρ to $\mathscr{B}$ is an injection and that $\mathscr{B}$ and $\rho(\mathscr{B})$ are isomorphic complexes.

By basic properties of affine buildings, one knows it is possible to extend $\mathscr{B}$ to an apartment. Any such apartment will retract bijectively to Σ, and will be of the form form $[\omega, \omega_0]$ where ω is the equivalence class represented by the upward pointing sectors of the apartment. Moreover, using the definition of π one may calculate that

$$\pi(\{\omega \in \Omega_0 : \mathscr{B} \subseteq [\omega, \omega_0]\}) = q^{-2\langle\lambda,\alpha_0\rangle-k}.$$

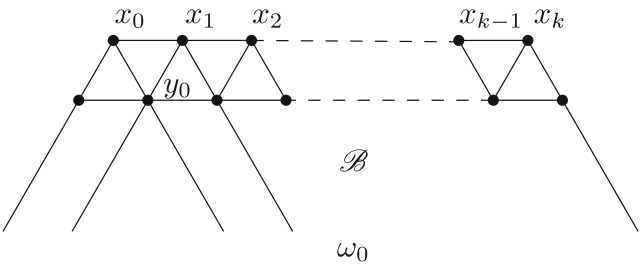


Fig. 3 The complex $\mathscr{B}$

The important point is that the measure of the set depends only on the level of λ and the length of the chain.

Basic properties of affine buildings imply that any apartment containing x_0 and x_k contains the entire chain. Hence,

$$\pi(E_{x_0} \cap E_{x_k}) = \pi(\{\omega \in \Omega_0 : \mathscr{B} \subseteq [\omega, \omega_0]\}) = q^{-2\langle \lambda, \alpha_0 \rangle - k}.$$

Fix x_0. Proceeding one step at a time, one sees there are q^k connected chains $(x_i : 0 \leq i \leq k)$ with $\rho(x_i) = \lambda + k(\lambda_2 - \lambda_1)$. Consequently

$$\mathbb{E}_{\lambda+k(\lambda_2-\lambda_1)} \mathbf{1}_{x_0} = q^{-k} \sum_{(x_i : 0 \leq i \leq k)} \mathbf{1}_{x_k}.$$

Likewise

$$\begin{aligned} \mathbb{E}_{\lambda+k(\lambda_2-\lambda_1)} \mathbb{E}_{\lambda+j(\lambda_2-\lambda_1)} \mathbf{1}_{x_0} &= q^{-j} \mathbb{E}_{\lambda+k(\lambda_2-\lambda_1)} \sum_{(x_i : 0 \leq i \leq j)} \mathbf{1}_{x_j} \\ &= q^{-j} q^{-(k-j)} \sum_{(x_i : 0 \leq i \leq j)} \sum_{(x_i : j \leq i \leq k)} \mathbf{1}_{x_k}, \end{aligned}$$

which is the same thing. □

Consider $\mathbb{E}_\lambda \mathbb{E}_\mu$. If $\lambda \preceq \mu$ then the product is equal to $\mathbb{E}_\lambda$; similarly if $\mu \preceq \lambda$. If λ and μ are incomparable, the following lemma allows us to reduce to the case where λ and μ are on the same level.

Lemma 2.4 *Suppose $\lambda \in P$ and*

$$\lambda' = \lambda - i\lambda_1, \qquad \mu = \lambda' + k(\lambda_2 - \lambda_1), \qquad \tilde{\mu} = \mu + (\lambda_2 - \lambda_1)$$

for $i, k \in \mathbb{N}$. Then for any locally integrable function f on Ω_0

$$\mathbb{E}[f | \mathcal{F}_\lambda | \mathcal{F}_\mu] = \mathbb{E}[f | \mathcal{F}_{\lambda'} | \mathcal{F}_\mu], \tag{2.6}$$

$$\mathbb{E}[f | \mathcal{F}_\mu | \mathcal{F}_\lambda] = \mathbb{E}[f | \mathcal{F}_\mu | \mathcal{F}_{\lambda'}], \tag{2.7}$$

$$\mathbb{E}[f | \mathcal{F}_\lambda | \mathcal{F}_\mu \vee \mathcal{F}_{\tilde{\mu}}] = \mathbb{E}[f | \mathcal{F}_{\lambda'} | \mathcal{F}_\mu] \tag{2.8}$$

$$\mathbb{E}[f | \mathcal{F}_\mu \vee \mathcal{F}_{\tilde{\mu}} | \mathcal{F}_\lambda] = \mathbb{E}[f | \mathcal{F}_\mu | \mathcal{F}_{\lambda'}] \tag{2.9}$$

and likewise if we exchange λ_1 and λ_2.

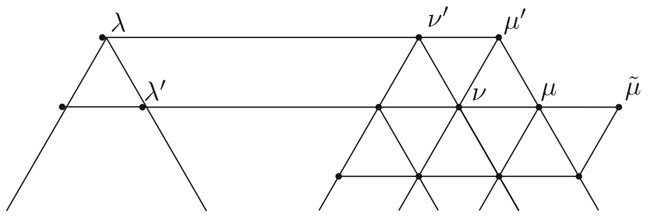


Fig. 4 Notation used in Lemma 2.4

Proof We first prove (2.6) for $i = 1$ and $k = 1$. Because $\mathbb{E}[f|\mathcal{F}_{\lambda'}] = \mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_{\lambda'}]$, it is sufficient to consider $f = \mathbf{1}_{E_{p_1}}$ where $\rho(p_1) = \lambda$. Use Fig. 2 to fix the notation, and note that if p_1 retracts to λ, then x retracts to λ' and p to μ. One calculates:

$$\mathbb{E}[\mathbf{1}_{E_{p_1}}|\mathcal{F}_\lambda|\mathcal{F}_\mu] = \mathbb{E}[\mathbf{1}_{E_{p_1}}|\mathcal{F}_\mu] = q^{-3} \sum_{\substack{p\sim l_0 \\ p\neq p_0}} \mathbf{1}_{E_p} = q^{-2}\mathbb{E}[\mathbf{1}_{E_x}|\mathcal{F}_\mu]$$
$$= \mathbb{E}[\mathbf{1}_{E_{p_1}}|\mathcal{F}_{\lambda'}|\mathcal{F}_\mu].$$

Next consider the case $i = 1, k > 1$. Set $\mu' = \mu+\lambda_1$, $\nu = \mu+\lambda_1-\lambda_2$ and $\nu' = \nu+\lambda_1$ (see Fig. 4). Since $\mathcal{F}_\mu$ is a subfield of $\mathcal{F}_{\mu'}$, we have

$$\mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_{\mu'}|\mathcal{F}_\mu].$$

Thus, applying Lemma 2.3, we obtain

$$\mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_{\mu'}|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_{\nu'}|\mathcal{F}_{\mu'}|\mathcal{F}_\mu]$$
$$= \mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_{\nu'}|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_\nu|\mathcal{F}_\mu]$$

where in the last step we have used the case $k = 1$. Now apply induction on k and Lemma 2.3 again to get

$$\mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_\nu|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_{\lambda'}|\mathcal{F}_\nu|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_{\lambda'}|\mathcal{F}_\mu].$$

To extend to the case $i > 1$, use induction on i and observe that

$$\mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_\lambda|\mathcal{F}_{\mu'}|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_{\lambda'+\lambda_1}|\mathcal{F}_{\mu'}|\mathcal{F}_\mu]$$
$$= \mathbb{E}[f|\mathcal{F}_{\lambda'+\lambda_1}|\mathcal{F}_\mu] = \mathbb{E}[f|\mathcal{F}_{\lambda'}|\mathcal{F}_\mu].$$

The proof of (2.8) is analogous, starting with the case $i = 1, k = 0$. Identity that (2.6) can be read as $\mathbb{E}_\mu\mathbb{E}_\lambda = \mathbb{E}_\mu\mathbb{E}_{\lambda'}$. The expectation operators are orthogonal projections with respect to the usual inner product, and taking adjoints gives $\mathbb{E}_\lambda\mathbb{E}_\mu = \mathbb{E}_{\lambda'}\mathbb{E}_\mu$ which is (2.7). To be more precise, one takes the inner product of either side of (2.7) with some nice test function, applies self-adjointness, and reduces to (2.6). Likewise, (2.9) follows from (2.8). □

Lemma 2.5 *Suppose* $\lambda = i\lambda_1 + j\lambda_2$, $\mu = \lambda + k(\lambda_1 - \lambda_2)$. *Then for any locally integrable function* f *on* Ω_0

$$\mathbb{E}[f|\mathcal{F}_\mu|\mathcal{F}_\lambda] = \begin{cases} \mathbb{E}[f|\mathcal{F}_\mu|\mathcal{F}_{i,\infty}] & \textit{if } k \geq 0, \\ \mathbb{E}[f|\mathcal{F}_\mu|\mathcal{F}_{\infty,j}] & \textit{if } k \leq 0. \end{cases}$$

Proof Suppose $k \geq 0$. By Lemma 2.4 for any $j' \geq 0$, we have

$$\mathbb{E}_\mu \mathbb{E}_{\lambda + j'\lambda_2} = \mathbb{E}_\mu \mathbb{E}_\lambda.$$

So if g is $\mathcal{F}_{\lambda+j'\lambda_2}$-measurable and compactly supported, then

$$\begin{aligned} \langle g, \mathbb{E}_{i,\infty}\mathbb{E}_\mu f\rangle = \langle \mathbb{E}_\mu \mathbb{E}_{i,\infty} g, f\rangle &= \langle \mathbb{E}_\mu g, f\rangle \\ &= \langle \mathbb{E}_\mu \mathbb{E}_{\lambda+j'\lambda_2} g, f\rangle \\ &= \langle \mathbb{E}_\mu \mathbb{E}_\lambda g, f\rangle = \langle g, \mathbb{E}_\lambda \mathbb{E}_\mu f\rangle. \end{aligned}$$

The test functions g which we use are sufficient to distinguish between one $\mathcal{F}_{i,\infty}$-measurable function and another. Since $\mathbb{E}_{i,\infty}\mathbb{E}_\mu f$ and $\mathbb{E}_\lambda \mathbb{E}_\mu f$ are both $\mathcal{F}_{i,\infty}$-measurable, the proof is done.

3 Littlewood-Paley Theory

3.1 Maximal Functions

The natural maximal function M^* for a locally integrable function f on Ω_0 is defined by

$$M^* f = \max_{\lambda \in P} |f_\lambda|.$$

In addition, we define two auxiliary single-parameter maximal functions

$$L^* f = \max_{i \in \mathbb{Z}} \mathbb{E}[\,|f|\,|\,\mathcal{F}_{i,\infty}], \qquad R^* f = \max_{j \in \mathbb{Z}} \mathbb{E}[\,|f|\,|\,\mathcal{F}_{\infty,j}].$$

Lemma 3.1 *Let* $\lambda \in P$ *and* $k \in \mathbb{N}$. *For any non-negative locally integrable function* f *on* Ω_0

$$\left(\mathbb{E}_{\lambda+k\lambda_2}\mathbb{E}_{\lambda+k\lambda_1}\right)^2 f \geq (1 - q^{-1})\mathbb{E}_\lambda f.$$

Proof We may assume $\lambda = 0$. Let us define (see Fig. 5)

$$\begin{aligned} &\mu = k\lambda_1, && \mu' = \lambda_1 + (k-1)\lambda_2, && \mu'' = k\lambda_2, \\ &\nu = (k-1)\lambda_1, && \nu' = \lambda_1 + (k-2)\lambda_2, && \nu'' = (k-1)\lambda_2. \end{aligned}$$

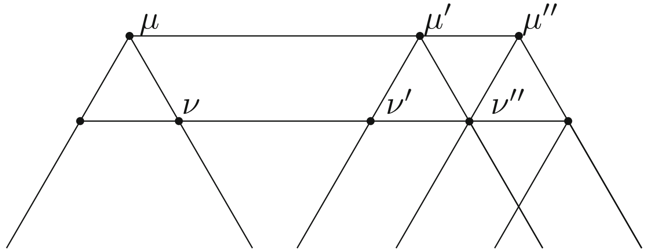


Fig. 5 Notation used in Lemma 3.1

We show

$$\mathbb{E}_{\mu''}\mathbb{E}_{\mu}\mathbb{E}_{\mu''}\mathbb{E}_{\mu} - q^{-1}\mathbb{E}_{\mu''}\mathbb{E}_{\mu}\mathbb{E}_{\mu'}\mathbb{E}_{\mu} = \mathbb{E}_{\nu''}\mathbb{E}_{\nu}\mathbb{E}_{\nu''}\mathbb{E}_{\nu} - q^{-1}\mathbb{E}_{\nu''}\mathbb{E}_{\nu}\mathbb{E}_{\nu'}\mathbb{E}_{\nu}. \quad (3.1)$$

Let $g = \mathbb{E}[f|\mathcal{F}_{\mu}]$. By two applications of Lemma 2.3, we can write

$$\mathbb{E}[g|\mathcal{F}_{\mu''}|\mathcal{F}_{\mu}] = \mathbb{E}[g|\mathcal{F}_{\mu'}|\mathcal{F}_{\mu''}|\mathcal{F}_{\mu'}|\mathcal{F}_{\mu}]$$

and by Lemma 2.2

$$\begin{aligned}\mathbb{E}[g|\mathcal{F}_{\mu'}|\mathcal{F}_{\mu''}|\mathcal{F}_{\mu'}] = q^{-1}\mathbb{E}[g|\mathcal{F}_{\mu'}] + \mathbb{E}[g|\mathcal{F}_{\nu''}] \\ -q^{-1}\mathbb{E}[g|\mathcal{F}_{\mu'}|\mathcal{F}_{\nu'} \vee \mathcal{F}_{\nu''}].\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}[g|\mathcal{F}_{\mu''}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}] - q^{-1}\mathbb{E}[g|\mathcal{F}_{\mu'}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}] \\ &\quad = \mathbb{E}[g|\mathcal{F}_{\nu''}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}] - q^{-1}\mathbb{E}[g|\mathcal{F}_{\mu'}|\mathcal{F}_{\nu'} \vee \mathcal{F}_{\nu''}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}].\end{aligned}$$

By repeated application of Lemma 2.4, we have

$$\mathbb{E}[g|\mathcal{F}_{\nu''}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}] = \mathbb{E}[f|\mathcal{F}_{\mu}|\mathcal{F}_{\nu''}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}] = \mathbb{E}[f|\mathcal{F}_{\nu}|\mathcal{F}_{\nu''}|\mathcal{F}_{\nu}|\mathcal{F}_{\nu''}]$$

and

$$\begin{aligned}\mathbb{E}[g|\mathcal{F}_{\mu'}|\mathcal{F}_{\nu'} \vee \mathcal{F}_{\nu''}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}] &= \mathbb{E}[f|\mathcal{F}_{\mu}|\mathcal{F}_{\mu'}|\mathcal{F}_{\nu'} \vee \mathcal{F}_{\nu''}|\mathcal{F}_{\mu}|\mathcal{F}_{\mu''}] \\ &= \mathbb{E}[f|\mathcal{F}_{\nu}|\mathcal{F}_{\nu'}|\mathcal{F}_{\nu}|\mathcal{F}_{\nu''}]\end{aligned}$$

which finishes the proof of (3.1). By iteration of (3.1), we obtain

$$\begin{aligned}&\mathbb{E}_{\mu''}\mathbb{E}_{\mu}\mathbb{E}_{\mu''}\mathbb{E}_{\mu} - q^{-1}\mathbb{E}_{\mu''}\mathbb{E}_{\mu}\mathbb{E}_{\mu'}\mathbb{E}_{\mu} \\ &\quad = \mathbb{E}_{\lambda_2}\mathbb{E}_{\lambda_1}\mathbb{E}_{\lambda_2}\mathbb{E}_{\lambda_1} - q^{-1}\mathbb{E}_{\lambda_2}\mathbb{E}_{\lambda_1}\mathbb{E}_{\lambda_1}\mathbb{E}_{\lambda_1}\end{aligned}$$

which together with Lemma 2.2 implies

$$\mathbb{E}_{\mu''}\mathbb{E}_{\mu}\mathbb{E}_{\mu''}\mathbb{E}_{\mu} = q^{-1}\mathbb{E}_{\mu''}\mathbb{E}_{\mu}\mathbb{E}_{\mu'}\mathbb{E}_{\mu} + (1-q^{-1})\mathbb{E}_0.$$

□

Theorem 1 *For each $p \in (1, \infty]$ there is $C_p > 0$ such that*

$$\left\|L^* f\right\|_{L^p} \le C_p \|f\|_{L^p}, \qquad \left\|R^* f\right\|_{L^p} \le C_p \|f\|_{L^p}, \tag{3.2}$$

$$\left\|M^* f\right\|_{L^p} \le C_p \|f\|_{L^p}. \tag{3.3}$$

Proof Inequalities (3.2) are two instances of Doob's well-known maximal inequality for single parameter martingales (see e.g. [15]). To show (3.3), consider a non-negative $f \in L^p(\Omega_0, \mathcal{F}_\mu)$. Fix $\lambda \in P$. Since $f \in L^p(\Omega_0, \mathcal{F}_{\mu'})$ for any $\mu' \succeq \mu$ we may assume $\mu \succeq \lambda$. Let

$$\nu = \lambda + \langle \mu - \lambda, \alpha_0 \rangle \lambda_1, \qquad \nu'' = \lambda + \langle \mu - \lambda, \alpha_0 \rangle \lambda_2.$$

By Lemma 3.1,

$$(1-q^{-1})\mathbb{E}_\lambda f \le \mathbb{E}_{\nu''}\mathbb{E}_{\nu}\mathbb{E}_{\nu''}\mathbb{E}_{\nu} f.$$

If $\lambda = i\lambda_1 + j\lambda_2$, then repeated application of Lemma 2.5 gives

$$\begin{aligned}\mathbb{E}_{\nu''}\mathbb{E}_{\nu}\mathbb{E}_{\nu''}\mathbb{E}_{\nu} f = \mathbb{E}_{\nu''}\mathbb{E}_{\nu}\mathbb{E}_{\nu''}\mathbb{E}_{\nu}\mathbb{E}_{\mu} f &= \mathbb{E}[f|\mathcal{F}_{\infty,j}|\mathcal{F}_{i,\infty}|\mathcal{F}_{\infty,j}|\mathcal{F}_{i,\infty}] \\ &\le L^* R^* L^* R^* f.\end{aligned}$$

By taking the supremum over $\lambda \in P$, we get

$$(1-q^{-1})M^* f \le L^* R^* L^* R^* f.$$

Hence, by (3.2), we obtain (3.3) for $f \in L^p(\Omega_0, \mathcal{F}_\mu)$. Finally, a standard Fatou's lemma argument establishes the theorem for arbitrary $f \in L^p(\Omega_0)$.

3.2 Square Function

Let f be a locally integrable function on Ω_0. Given $i, j \in \mathbb{Z}$, we define projections

$$L_i f = \mathbb{E}[f|\mathcal{F}_{i,\infty}] - \mathbb{E}[f|\mathcal{F}_{i-1,\infty}], \quad R_j f = \mathbb{E}[f|\mathcal{F}_{\infty,j}] - \mathbb{E}[f|\mathcal{F}_{\infty,j-1}].$$

Note that L_i (respectively R_j) is the martingale difference operator for the filtration $\left(\mathcal{F}_{i,\infty} : i \in \mathbb{Z}\right)$ (respectively $\left(\mathcal{F}_{\infty,j} : j \in \mathbb{Z}\right)$). For $\lambda = i\lambda_1 + j\lambda_2$, we set

$$D_\lambda f = L_i R_j f, \qquad D_\lambda^\star f = R_j L_i f.$$

The following development is inspired by that of Stein and Street in [17]. We start by defining the corresponding square function.

$$\mathcal{S} f = \Big(\sum_{\lambda \in P} |D_\lambda f|^2 \Big)^{1/2}.$$

We will also need its dual counterpart

$$\mathcal{S}^\star f = \Big(\sum_{\lambda \in P} |D_\lambda^\star f|^2 \Big)^{1/2}.$$

Theorem 2 *For every $p \in (1, \infty)$ there is $C_p > 1$ such that*

$$C_p^{-1} \|f\|_{L^p} \leq \|\mathcal{S} f\|_{L^p} \leq C_p \|f\|_{L^p}, \quad C_p^{-1} \|f\|_{L^p} \leq \left\|\mathcal{S}^\star f\right\|_{L^p} \leq C_p \|f\|_{L^p}.$$

Moreover, on $L^2(\Omega_0)$ square functions $\mathcal{S}$ and $\mathcal{S}^\star$ preserve the norm.

Proof Since

$$S_L(f) = \Big(\sum_{i \in \mathbb{Z}} |L_i f|^2 \Big)^{1/2} \quad \text{and} \quad S_R(f) = \Big(\sum_{j \in \mathbb{Z}} |R_j f|^2 \Big)^{1/2}$$

preserve the norm on $L^2(\Omega_0)$, we have

$$\begin{aligned} \int \sum_{i,j \in \mathbb{Z}} |L_i R_j f|^2 \, \mathrm{d}\pi &= \sum_{j \in \mathbb{Z}} \int \sum_{i \in \mathbb{Z}} |L_i R_j f|^2 \, \mathrm{d}\pi \\ &= \sum_{j \in \mathbb{Z}} \int |R_j f|^2 d\pi = \int |f|^2 \, \mathrm{d}\pi. \end{aligned} \tag{3.4}$$

Hence, $\mathcal{S}$ preserves the norm.

For $p \neq 2$, we use the two-parameter Khintchine inequality (see [12]) and bounds on single parameter martingale transforms (see [2,15,18]). Let $(\epsilon_i : i \in \mathbb{Z})$ and $(\epsilon'_j : j \in \mathbb{Z})$ be sequences of real numbers, with absolute values bounded above by 1. For $N \in \mathbb{N}$, we consider the operator

$$T_N = \sum_{|i|, |j| \leq N} \epsilon_i \epsilon'_j D_{i\lambda_1 + j\lambda_2}$$

which may be written as a composition $\mathcal{L}_N \mathcal{R}_N$ where

$$\mathcal{L}_N = \sum_{|i| \leq N} \epsilon_i L_i, \quad \mathcal{R}_N = \sum_{|j| \leq N} \epsilon'_j R_j.$$

Since by Burkholder's inequality (see [2,15]) the operators $\mathcal{R}_N$ and $\mathcal{L}_N$ are bounded on $L^p(\Omega_0)$ with bounds uniform in N, we have

$$\|T_N f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Setting r_k to be the Rademacher function, by Khintchine's inequality, we get

$$\begin{aligned}&\int \Big(\sum_{|i|,|j|\leq N} |D_{i\lambda_1+j\lambda_2} f|^2 \Big)^{p/2} \,\mathrm{d}\pi \\ &\quad\lesssim \int \int_0^1 \int_0^1 \Big| \sum_{|i|,|j|\leq N} r_i(s) r_j(t) D_{i\lambda_1+j\lambda_2} f \Big|^p \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\pi,\end{aligned}$$

which is bounded by $\|f\|_{L^p}^p$. Finally, let N approach infinity and use the monotone convergence theorem to get

$$\|\mathcal{S} f\|_{L^p} \lesssim \|f\|_{L^p}.$$

For the opposite inequality, we take $f \in L^p(\Omega_0) \cap L^2(\Omega_0)$ and $g \in L^{p'}(\Omega_0) \cap L^2(\Omega_0)$ where $1/p' + 1/p = 1$. By polarization of (3.4) and the Cauchy–Schwarz and Hölder inequalities, we obtain

$$\langle f, g\rangle = \int \sum_{\lambda \in P} D_\lambda f \overline{D_\lambda g} \,\mathrm{d}\pi \leq \langle \mathcal{S} f, \mathcal{S} g\rangle \leq \|\mathcal{S} f\|_{L^p} \|\mathcal{S} g\|_{L^{p'}} \lesssim \|\mathcal{S} f\|_{L^p} \|g\|_{L^{p'}}.$$

□

Given a set $\{v_\lambda : \lambda \in P\}$ of vectors in a Banach space, we say that $\sum_{\lambda \in P} v_\lambda$ converges *unconditionally* if, whenever we choose a bijection $\phi : \mathbb{N} \to P$,

$$\sum_{n=1}^{\infty} v_{\phi(n)} \qquad \text{exists, and is independent of } \phi.$$

Equivalently, we may ask that for any increasing, exhaustive sequence $(F_N : N \in \mathbb{N})$ of finite subsets of P, the limit

$$\lim_{N\to\infty} \sum_{\lambda \in F_N} v_\lambda \qquad \text{exists.}$$

The following proposition provides a Calderón reproducing formula.

Proposition 3.2 *For each $p \in (1, \infty)$ and any $f \in L^p(\Omega_0)$,*

$$f = \sum_{\lambda \in P} D_\lambda D_\lambda^\star f$$

where the sum converges in $L^p(\Omega_0)$ unconditionally.

Proof Fix an increasing and exhaustive sequence $(F_N : N \in \mathbb{N})$ of finite subsets of P. Let

$$I_N(f) = \sum_{\lambda \in F_N} D_\lambda D_\lambda^\star f.$$

For $f \in L^p(\Omega_0)$ and $g \in L^{p'}(\Omega_0)$, where $1/p + 1/p' = 1$, we have

$$\begin{aligned} |\langle I_N(f) - I_M(f), g\rangle| &= \Big| \sum_{\lambda \in F_N \setminus F_M} \langle D_\lambda^\star f, D_\lambda^\star g\rangle \Big| \\ &\le \Big\| \Big(\sum_{\lambda \in F_N \setminus F_M} (D_\lambda^\star f)^2 \Big)^{1/2} \Big\|_{L^p} \big\| \mathcal{S}^\star(g) \big\|_{L^{p'}}. \end{aligned} \tag{3.5}$$

In particular,

$$|\langle I_N(f), g\rangle| \le \big\| \mathcal{S}^\star(f) \big\|_{L^p} \big\| \mathcal{S}^\star(g) \big\|_{L^{p'}},$$

whence $\|I_N(f)\|_{L^p} \lesssim \|f\|_{L^p}$ uniformly in N. Consequently, it is enough to prove convergence for $f \in L^p(\Omega_0) \cap L^2(\Omega_0)$. From (3.5) and the bounded convergence theorem, it follows that for any positive ϵ, $\|I_N(f) - I_M(f)\|_{L^p} \le \epsilon$ whenever M and N are large enough. This shows that the limit exists. Finally, for $g \in L^{p'}(\Omega_0) \cap L^2(\Omega_0)$, the polarized version of (3.4) gives

$$\lim_{N \to \infty} \langle I_N(f), g\rangle = \lim_{N \to \infty} \sum_{\lambda \in F_N} \langle D_\lambda^\star f, D_\lambda^\star g\rangle = \langle f, g\rangle.$$

□

Theorem 3 *Let $(T_\lambda : \lambda \in P)$ be a family of operators such that for some $\delta > 0$ and $p_0 \in (1, 2)$*

$$\|T_\lambda\|_{L^1 \to L^1} \lesssim 1, \tag{3.6}$$

$$\big\|T_\mu T_\lambda^\star\big\|_{L^2 \to L^2} \lesssim q^{-\delta|\mu-\lambda|} \quad \text{and} \quad \big\|T_\mu^\star T_\lambda\big\|_{L^2 \to L^2} \lesssim q^{-\delta|\mu-\lambda|}, \tag{3.7}$$

$$\big\|D_\lambda T_\mu D_{\lambda'}\big\|_{L^2 \to L^2} \lesssim q^{-\delta|\lambda-\mu|} q^{-\delta|\lambda'-\mu|}, \tag{3.8}$$

$$\Big\|\sup_{\lambda \in P} |T_\lambda f_\lambda|\Big\|_{L^{p_0}} \lesssim \Big\|\sup_{\lambda \in P} |f_\lambda|\Big\|_{L^{p_0}}. \tag{3.9}$$

Then for any $p \in (p_0, 2]$ the sum $\sum_{\lambda \in P} T_\lambda$ converges unconditionally in the strong operator topology for operators on $L^p(\Omega_0)$.

Proof First, recall that the Cotlar–Stein Lemma (see e.g. [16]) states that (3.7) implies the unconditional convergence of $\sum_{\lambda \in P} T_\lambda$ in the strong operator topology on $L^2(\Omega_0)$. Let $(F_N : N \in \mathbb{N})$ be an arbitrary increasing and exhaustive sequence of finite subsets of P. For $N > 0$, we set

$$V_N = \sum_{\mu \in F_N} T_\mu, \qquad I_N = \sum_{\lambda \in F_N} D_\lambda D^\star_\lambda.$$

By (3.6), (3.7) and interpolation, each T_μ is bounded on L^p for $p \in [1, 2]$ and the same holds for the finite sum V_N. We consider $f \in L^p(\Omega_0)$ for $p \in (p_0, 2)$. By Proposition 3.2 and Theorem 2, we

$$\begin{aligned}
&\big\|V_M I_N(f)\big\|_{L^p} \\
&\quad\lesssim \big\|\mathcal{S}\big(V_M I_N(f)\big)\big\|_{L^p} = \Big\|\Big(\sum_{\mu\in F_M}\sum_{\lambda'\in F_N} D_\lambda T_\mu D_{\lambda'} D^\star_{\lambda'} f : \lambda \in P\Big)\Big\|_{L^p(\ell^2)} \\
&\quad= \Big\|\Big(\sum_{\gamma,\gamma'\in P} \mathbf{1}_{F_N}(\lambda+\gamma+\gamma')\mathbf{1}_{F_M}(\lambda+\gamma) D_\lambda T_{\lambda+\gamma} D_{\lambda+\gamma+\gamma'} D^\star_{\lambda+\gamma+\gamma'} f : \lambda \in P\Big)\Big\|_{L^p(\ell^2)} \\
&\quad\leq \sum_{\gamma,\gamma'\in P} \Big\| \Big(\mathbf{1}_{F_N}(\lambda+\gamma+\gamma')\mathbf{1}_{F_M}(\lambda+\gamma) D_\lambda T_{\lambda+\gamma} D_{\lambda+\gamma+\gamma'} D^\star_{\lambda+\gamma+\gamma'} f : \lambda \in P\Big) \Big\|_{L^p(\ell^2)}.
\end{aligned}$$

Finally, by change of variables, we get

$$\big\|V_M I_N(f)\big\|_{L^p} \lesssim \sum_{\gamma,\gamma'\in P} \Big\| \Big(D_{\lambda+\gamma+\gamma'} T_{\lambda+\gamma} D_\lambda D^\star_\lambda f : \lambda \in F_N\Big) \Big\|_{L^p(\ell^2)}.$$

Assuming there is $\delta_p > 0$ such that

$$\Big\|\Big(D_{\lambda+\gamma+\gamma'} T_{\lambda+\gamma} D_\lambda f_\lambda : \lambda \in P\Big)\Big\|_{L^p(\ell^2)} \lesssim q^{-\delta_p(|\gamma|+|\gamma'|)} \|(f_\lambda : \lambda \in P)\|_{L^p(\ell^2)} \tag{3.10}$$

we can estimate

$$\begin{aligned}
\big\|V_M I_N(f)\big\|_{L^p} &\lesssim \sum_{\gamma,\gamma'\in P} q^{-\delta_p(|\gamma|+|\gamma'|)} \Big\|\Big(D^\star_\lambda f : \lambda \in F_N\Big)\Big\|_{L^p(\ell^2)} \\
&\lesssim \Big\|\Big(\sum_{\lambda\in F_N} (D^\star_\lambda f)^2\Big)^{1/2}\Big\|_{L^p}.
\end{aligned} \tag{3.11}$$

Theorem 2, Proposition 3.2 and (3.11) imply that the V_M are uniformly bounded on L^p.

For the proof of (3.10), we consider an operator $\mathcal{T}$ defined for $\mathbf{f} \in L^p\big(\pi, \ell^2(P)\big)$ by

$$\mathcal{T}\vec{f} = \Big(D_{\lambda+\gamma+\gamma'} T_{\lambda+\gamma} D_\lambda f_\lambda : \lambda \in P\Big).$$

Since $\|D_\lambda\|_{L^1\to L^1} \lesssim 1$ and $\big\|T_\mu\big\|_{L^1\to L^1} \lesssim 1$, we have

$$\big\|\mathcal{T}\vec{f}\,\big\|_{L^1(\ell^1)} \lesssim \big\|\vec{f}\,\big\|_{L^1(\ell^1)}.$$

Also, by (3.8), we can estimate

$$\|\mathcal{T}\vec{f}\,\|_{L^2(\ell^2)}^2 = \sum_{\lambda\in P} \|D_{\lambda+\gamma+\gamma'}T_{\lambda+\gamma}D_\lambda f_\lambda\|_{L^2}^2 \lesssim q^{-\delta(|\gamma|+|\gamma'|)} \sum_{\lambda\in P} \|f_\lambda\|_{L^2}^2.$$

Therefore, using interpolation between $L^1\big(\pi, \ell^1(P)\big)$ and $L^2\big(\pi, \ell^2(P)\big)$ we obtain that there is $\delta' > 0$ such that

$$\|\mathcal{T}\vec{f}\,\|_{L^{p_0}(\ell^{p_0})} \lesssim q^{-\delta'(|\gamma|+|\gamma'|)}\|\vec{f}\,\|_{L^{p_0}(\ell^{p_0})}.$$

Because $|D_\lambda g| \lesssim L^*R^*(|g|)$, and because Theorem 1 says that L^* and R^* are bounded on L^{p_0}, we know that $(D_\lambda : \lambda \in P)$ is bounded on $L^{p_0}(\pi, \ell^\infty(P))$. Of course the same holds for $\big(D_{\lambda+\gamma+\gamma'} : \lambda \in P\big)$. Hence, by (3.9), we get

$$\|\mathcal{T}\vec{f}\,\|_{L^{p_0}(\ell^\infty)} \lesssim \|\vec{f}\,\|_{L^{p_0}(\ell^\infty)}.$$

Next, interpolating between $L^{p_0}(\pi, \ell^{p_0}(P))$ and $L^{p_0}(\pi, \ell^\infty(P))$ gives a $\delta'' > 0$ such that

$$\|\mathcal{T}\vec{f}\,\|_{L^{p_0}(\ell^2)} \lesssim q^{-\delta''(|\gamma|+|\gamma'|)}\|\vec{f}\,\|_{L^{p_0}(\ell^2)}.$$

Finally, interpolating between $L^{p_0}\big(\pi, \ell^2(P)\big)$ and $L^2\big(\pi, \ell^2(P)\big)$, we obtain (3.10).

To complete the proof, we are going to show that $(V_N f : N \in \mathbb{N})$ is a Cauchy sequence in $L^p(\Omega_0)$. Let us consider $g \in L^p(\Omega_0) \cap L^2(\Omega_0)$. Setting

$$a = \frac{2(p-p_0)}{4-p-p_0}, \quad \text{and} \quad \tilde{p} = \frac{p+p_0}{2}$$

and using the log-convexity of the L^q-norms, we get

$$\|V_M g - V_N g\|_{L^p}^p \le \|V_M g - V_N g\|_{L^2}^a \|V_M g - V_N g\|_{L^{\tilde{p}}}^{p-a}.$$

Since $(V_N g : N \in \mathbb{N})$ converges in $L^2(\Omega_0)$ and is uniformly bounded on $L^{\tilde{p}}(\Omega_0)$ it is a Cauchy sequence in $L^p(\Omega_0)$. For an arbitrary $f \in L^p(\Omega_0)$ use the density of g's as above. We have

$$\|V_M f - V_N f\|_{L^p} \lesssim \|f-g\|_{L^p} + \|V_N g - V_M g\|_{L^p}.$$

Thus, $(V_N f : N \in \mathbb{N})$ also converges, and this finishes the proof of the theorem. □

4 Double Differences

The martingale transforms are expressed in terms of double differences defined for a martingale $f = (f_\lambda : \lambda \in P)$ as

$$d_\lambda f = f_\lambda - f_{\lambda-\lambda_1} - f_{\lambda-\lambda_2} + f_{\lambda-\lambda_1-\lambda_2}.$$

4.1 Martingale Transforms

The following proposition is our key tool.

Proposition 4.1 *Let $f \in L^2(\Omega_0)$ and $\lambda \in P$. If $f_{\lambda-j\lambda_1} = 0$ for $j \in \mathbb{N}$ then for each $k \geq j$*

$$\left\| \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}] \right\|_{L^2} \leq 2q^{-(k-j+1)/2} \| f_\lambda \|_{L^2}.$$

Analogously, for λ_1 and λ_2 exchanged.

Proof Suppose $j = 1$. We are going to show that if $f_{\lambda-\lambda_1} = 0$ then for all $k \geq 1$

$$\left\| \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}] \right\|_{L^2} \leq q^{-k/2} \| f_\lambda \|_{L^2}. \tag{4.1}$$

Indeed, if $k = 1$ then by (2.1) of Lemma 2.2

$$\begin{aligned} \left\| \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-\lambda_1+\lambda_2}] \right\|_{L^2}^2 &= \langle \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-\lambda_1+\lambda_2} | \mathcal{F}_\lambda], f_\lambda \rangle \\ &= q^{-1} \| f_\lambda \|_{L^2}^2 - q^{-1} \left\| \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-\lambda_1} \vee \mathcal{F}_{\lambda-\lambda_2}] \right\|_{L^2}^2. \end{aligned}$$

If $k > 1$, we use Lemma 2.3 to write

$$\mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}] = \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-(\lambda_1-\lambda_2)} | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}].$$

Since, by Lemma 2.4,

$$\mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-(\lambda_1-\lambda_2)} | \mathcal{F}_{\lambda-\lambda_1-(\lambda_1-\lambda_2)}] = \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-\lambda_1} | \mathcal{F}_{\lambda-\lambda_1-(\lambda_1-\lambda_2)}] = 0$$

we can use induction to obtain

$$\begin{aligned} \left\| \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-(\lambda_1-\lambda_2)} | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}] \right\|_{L^2} &\leq q^{-(k-1)/2} \left\| \mathbb{E}[f_\lambda | \mathcal{F}_{\lambda-(\lambda_1-\lambda_2)}] \right\|_{L^2} \\ &\leq q^{-k/2} \| f_\lambda \|_{L^2}. \end{aligned}$$

Let us consider $j > 1$. For each $i = 0, 1, \ldots, j-1$, we set

$$g_i = f_{\lambda-i\lambda_1} - f_{\lambda-(i+1)\lambda_1}.$$

By Lemma 2.4 and (4.1), we have

$$\begin{aligned}\left\|\mathbb{E}[g_i|\mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}]\right\|_{L^2} = \left\|\mathbb{E}[g_i|\mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)-i\lambda_2}]\right\|_{L^2} &\leq q^{-(k-i)/2}\|g_i\|_{L^2}\\ &\leq q^{-(k-i)/2}\|f_\lambda\|_{L^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\mathbb{E}[f_\lambda|\mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}]\right\|_{L^2} &\leq \sum_{i=0}^{j-1}\left\|\mathbb{E}[g_i|\mathcal{F}_{n-k(\lambda_1-\lambda_2)}]\right\|_{L^2}\\ &\leq \sum_{i=0}^{j-1} q^{-(k-i)/2}\|f_\lambda\|_{L^2}\end{aligned}$$

which finishes the proof since

$$\sum_{i=0}^{j-1} q^{i/2} \leq 2q^{(j-1)/2}.$$

□

We have the following

Proposition 4.2 *For any $\lambda, \lambda', \mu \in P$ and $m \geq 1$*

$$\begin{aligned}\left\|D_\lambda d_\mu^m D_{\lambda'}\right\|_{L^2\to L^2} &\lesssim q^{-|\mu-\lambda|/4}q^{-|\mu-\lambda'|/4},\\ \left\|d_\lambda^m d_\mu^m\right\|_{L^2\to L^2} &\lesssim q^{-|\lambda-\mu|/2}.\end{aligned}$$

Proof We observe that for $f \in L^2(\Omega_0)$, $d_\mu f \in L^2(\pi, \mathcal{F}_\mu)$ and

$$\mathbb{E}[d_\mu f|\mathcal{F}_\nu] = 0 \tag{4.2}$$

whenever $\langle\nu, \alpha_0\rangle \leq \langle\mu, \alpha_0\rangle - 2$. For the proof it is enough to analyze the case $\nu = \mu - 2\lambda_2$. By Lemma 2.4, we can write

$$\mathbb{E}[f_{\mu-\lambda_1}|\mathcal{F}_{\mu-2\lambda_2}] = \mathbb{E}[f_{\mu-\lambda_1}|\mathcal{F}_{\mu-\lambda_1-\lambda_2}|\mathcal{F}_{\mu-2\lambda_2}] = \mathbb{E}[f_{\mu-\lambda_1-\lambda_2}|\mathcal{F}_{\mu-2\lambda_2}].$$

Suppose $\lambda = i\lambda_1 + j\lambda_2$. Let us consider $R_j d_\mu$. If $j \geq \langle\mu, \alpha_2\rangle + 1$ then $R_j d_\mu f = 0$. For $j \leq \langle\mu, \alpha_2\rangle - 2$, in view of (4.2) we can use Proposition 4.1 to estimate

$$\left\|R_j d_\mu f\right\|_{L^2} \lesssim q^{-\langle\mu-\lambda,\alpha_2\rangle/2}\left\|d_\mu f\right\|_{L^2}. \tag{4.3}$$

Next, if $\langle\lambda, \alpha_0\rangle \geq \langle\mu, \alpha_0\rangle + 2$ then $D_\lambda d_\mu f = 0$, because $d_\mu f$ is $\mathcal{F}_\mu$-measurable. For $\langle\lambda, \alpha_0\rangle \leq \langle\mu, \alpha_0\rangle - 4$ and $\langle\lambda, \alpha_2\rangle \leq \langle\mu, \alpha_2\rangle$, by Lemma 2.5, we can write

$D_\lambda d_\mu f = L_i g$ where

$$g = \mathbb{E}[R_j d_\mu f | \mathcal{F}_\nu]$$

and $\nu = (\langle \mu, \alpha_0 \rangle - j)\lambda_1 + j\lambda_2$. By Lemma 2.5, we have

$$R_j d_\mu f = \mathbb{E}[d_\mu f | \mathcal{F}_\nu] - \mathbb{E}[d_\mu f | \mathcal{F}_{\nu+\lambda_1-\lambda_2}].$$

We notice that by Lemma 2.4 and (4.2)

$$\mathbb{E}[d_\mu f | \mathcal{F}_\nu | \mathcal{F}_{\nu-2\lambda_1}] = \mathbb{E}[d_\mu f | \mathcal{F}_{\mu-2\lambda_2} | \mathcal{F}_{\nu-2\lambda_1}] = 0.$$

Similarly, one can show

$$\mathbb{E}[d_\mu f | \mathcal{F}_{\nu+\lambda_1-\lambda_2} | \mathcal{F}_{\nu-2\lambda_1}] = 0.$$

Therefore, $\mathbb{E}[g | \mathcal{F}_{\nu-2\lambda_1}] = 0$. Now, by Proposition 4.1, we obtain

$$\|L_i g\|_{L^2} \lesssim q^{-\langle \nu-\lambda, \alpha_0 \rangle/2} \left\| R_j d_\mu f \right\|_{L^2}. \tag{4.4}$$

Combining (4.4) with (4.3), we get

$$\left\| D_\lambda d_\mu f \right\|_{L^2} \lesssim q^{-\langle \mu-\lambda, \alpha_0 \rangle/2} q^{-\langle \mu-\lambda, \alpha_2 \rangle/2} \left\| d_\mu f \right\|_{L^2} \tag{4.5}$$

since $\langle \nu, \alpha_0 \rangle = \langle \mu, \alpha_0 \rangle$. By analogous reasoning one can show the corresponding norm estimates for $D^\star_{\lambda'} d_\mu$. Hence, taking adjoint

$$\left\| d_\mu D_{\lambda'} f \right\|_{L^2} \lesssim q^{-\langle \mu-\lambda', \alpha_0 \rangle/2} q^{-\langle \mu-\lambda', \alpha_2 \rangle/2} \| f \|_{L^2}. \tag{4.6}$$

Finally, (4.5) and (4.6) allow us to conclude the proof of the first inequality.

For the second, we may assume $0 \leq \langle \mu - \lambda, \alpha_0 \rangle \leq 1$. Suppose $\langle \mu - \lambda, \alpha_0 \rangle = 0$ and $\langle \mu - \lambda, \alpha_2 \rangle \geq 2$. Since $d_\mu f \in L^2(\pi, \mathcal{F}_\mu)$, by (4.2) and Proposition 4.1

$$\left\| \mathbb{E}[d_\mu f | \mathcal{F}_\lambda] \right\|_{L^2} \lesssim q^{-\langle \mu-\lambda, \alpha_2 \rangle/2} \left\| d_\mu f \right\|_{L^2}.$$

Similarly, we deal with the case $\langle \mu - \lambda, \alpha_0 \rangle = 1$. We can assume $\langle \mu - \lambda, \alpha_2 \rangle \geq 1$. By Lemma 2.4, we have

$$\mathbb{E}[d_\mu f | \mathcal{F}_\lambda] = \mathbb{E}[d_\mu f | \mathcal{F}_{\mu-\lambda_2} | \mathcal{F}_\lambda] = \mathbb{E}[f_{\lambda-\lambda_1-\lambda_2} - f_{\lambda-\lambda_1} | \mathcal{F}_\lambda].$$

Hence, by Proposition 4.1,

$$\left\| \mathbb{E}[d_\mu f | \mathcal{F}_\lambda] \right\|_{L^2} \lesssim q^{-\langle \mu-\lambda, \alpha_2 \rangle/2} \| f \|_{L^2}.$$

□

Let $(a_\lambda : \lambda \in P)$ be an uniformly bounded *predictable* family of functions, i.e. each function a_λ is measurable with respect to $\mathcal{F}_{\lambda-\lambda_1-\lambda_2}$ and

$$\sup_{\omega \in \Omega_0} |a_\lambda(\omega)| \leq M.$$

Predictability is the condition needed to ensure that $d_\lambda\big(a_\lambda f\big) = a_\lambda d_\lambda f$. By Theorems 1 and 3, Proposition 4.2 and duality when $p > 2$, we get

Theorem 4 *For each $p \in (1, \infty)$ and $m \in \mathbb{N}$ the series*

$$\sum_{\lambda \in P} a_\lambda d_\lambda^m$$

converges unconditionally in the strong operator topology for the operators on $L^p(\Omega_0)$, and defines the operator with norm bounded by a constant multiply of

$$\sup_{\lambda \in P} \sup_{\omega \in \Omega_0} |a_\lambda(\omega)|.$$

4.2 Martingale Square Function

For a martingale $f = (f_\lambda : \lambda \in P)$ there is the natural square function defined by

$$Sf = \Big(\sum_{\lambda \in P} (d_\lambda f)^2\Big)^{1/2}.$$

Although S does not preserve L^2 norm, we have

Theorem 5 *For every $p \in (1, \infty)$ there is $C_p > 0$ such that*

$$C_p^{-1} \|f\|_{L^p} \leq \|Sf\|_{L^p} \leq C_p \|f\|_{L^p}.$$

Proof We start from proving the identity

$$d_\lambda^4 - d_\lambda^3 - q^{-1} d_\lambda^2 + q^{-1} d_\lambda = 0. \tag{4.7}$$

Let us notice that

$$d_\lambda \mathbb{E}_\lambda = d_\lambda, \qquad d_\lambda \mathbb{E}_{\lambda-\lambda_1-\lambda_2} = 0,$$
$$d_\lambda \mathbb{E}_{\lambda-\lambda_2} = -\mathbb{E}_{\lambda-\lambda_1}\mathbb{E}_{\lambda-\lambda_2} + \mathbb{E}_{\lambda-\lambda_1-\lambda_2}, \qquad d_\lambda \mathbb{E}_{\lambda-\lambda_1} = -\mathbb{E}_{\lambda-\lambda_2}\mathbb{E}_{\lambda-\lambda_1} + \mathbb{E}_{\lambda-\lambda_1-\lambda_2}.$$

Therefore, consecutively we have

$$\begin{aligned} d_\lambda^2 &= d_\lambda + \mathbb{E}_{\lambda-\lambda_1}\mathbb{E}_{\lambda-\lambda_2} + \mathbb{E}_{\lambda-\lambda_1}\mathbb{E}_{\lambda-\lambda_1} - 2\mathbb{E}_{\lambda-\lambda_1-\lambda_2}, \\ d_\lambda^3 &= d_\lambda^2 - \mathbb{E}_{\lambda-\lambda_1}\mathbb{E}_{\lambda-\lambda_2}\mathbb{E}_{\lambda-\lambda_1} - \mathbb{E}_{\lambda-\lambda_2}\mathbb{E}_{\lambda-\lambda_1}\mathbb{E}_{\lambda-\lambda_2} + 2\mathbb{E}_{\lambda-\lambda_1-\lambda_2}, \\ d_\lambda^4 &= d_\lambda^3 + (\mathbb{E}_{\lambda-\lambda_1}\mathbb{E}_{\lambda-\lambda_2})^2 + (\mathbb{E}_{\lambda-\lambda_2}\mathbb{E}_{\lambda-\lambda_1})^2 - 2\mathbb{E}_{\lambda-\lambda_1-\lambda_2}. \end{aligned} \tag{4.8}$$

Hence, by Lemma 2.2,

$$d_\lambda^4 = d_\lambda^3 + q^{-1}\mathbb{E}_{\lambda-\lambda_1}\mathbb{E}_{\lambda-\lambda_2} + q^{-1}\mathbb{E}_{\lambda-\lambda_2}\mathbb{E}_{\lambda-\lambda_1} - 2q^{-1}\mathbb{E}_{\lambda-\lambda_1-\lambda_2}$$

which together with (4.8) implies (4.7).

Next, we consider an operator $\mathcal{T}$ defined for a function $f \in L^p(\Omega_0)$ by

$$\mathcal{T} f = (d_\lambda f : \lambda \in P)\,.$$

We also need an operator $\widetilde{\mathcal{T}}$ acting on $g \in L^{p'}(\Omega_0)$ as

$$\widetilde{\mathcal{T}} g = \Big(-q d_\lambda^3 g + q d_\lambda^2 g + d_\lambda g : \lambda \in P\Big)\,.$$

We observe that by two-parameter Khinchine's inequality and Theorem 4 we have

$$\big\|\mathcal{T} f\big\|_{L^p(\ell^2)} \lesssim \|f\|_{L^p}, \quad \text{and} \quad \big\|\widetilde{\mathcal{T}} g\big\|_{L^{p'}(\ell^2)} \lesssim \|g\|_{L^{p'}}.$$

The dual operator $\mathcal{T}^\star : L^{p'}\big(\pi, \ell^2(\mathbb{Z}^2)\big) \to L^{p'}(\Omega_0)$ is given by

$$\mathcal{T}^\star \vec{g} = \sum_{\lambda \in P} d_\lambda g_\lambda.$$

Since $\widetilde{\mathcal{T}} g \in L^{p'}\big(\pi, \ell^2(\mathbb{Z}^2)\big)$, by (4.7) and Theorem 4,

$$\mathcal{T}^\star \widetilde{\mathcal{T}} g = \sum_{\lambda \in P} d_\lambda g = g$$

Therefore, by Cauchy–Schwarz and Hölder inequalities

$$\langle f, g\rangle = \langle f, \mathcal{T}^\star \widetilde{\mathcal{T}} g\rangle \leq \big\|\mathcal{T} f\big\|_{L^p(\ell^2)} \big\|\widetilde{\mathcal{T}} g\big\|_{L^{p'}(\ell^2)} \lesssim \big\|\mathcal{T} f\big\|_{L^p(\ell^2)} \|g\|_{L^{p'}}$$

and since $\|\mathcal{T} f\|_{L^p(\ell^2)} = \|Sf\|_{L^p}$ the proof is finished.

Finally, the method of the proof of Theorem 3, together with Theorems 4 and 5 shows the following

Theorem 6 *Let $(T_\lambda : \lambda \in P)$ be a family of operators such that for some $\delta > 0$ and $p_0 \in (1, 2)$*

$$\|T_\lambda\|_{L^1\to L^1} \lesssim 1,$$
$$\left\|T_\mu T_\lambda^\star\right\|_{L^2\to L^2} \lesssim q^{-\delta|\mu-\lambda|} \quad \text{and} \quad \left\|T_\mu^\star T_\lambda\right\|_{L^2\to L^2} \lesssim q^{-\delta|\mu-\lambda|},$$
$$\left\|d_\lambda T_\mu d_{\lambda'}\right\|_{L^2\to L^2} \lesssim q^{-\delta|\lambda-\mu|}q^{-\delta|\lambda'-\mu|},$$
$$\Big\|\sup_{\lambda\in P} |T_\lambda f_\lambda|\Big\|_{L^{p_0}} \lesssim \Big\|\sup_{\lambda} |f_\lambda|\Big\|_{L^{p_0}}.$$

Then for any $p \in (p_0, 2]$ the sum $\sum_{\lambda\in P} T_\lambda$ converges unconditionally in the strong operator topology for the operators on $L^p(\Omega_0)$.

Appendix: About Ω_0 and Heisenberg Group

In some cases Ω_0 can be identified with a Heisenberg group over a nonarchimedean local field. Let us recall, that F is a *nonarchimedean local field* if it is a topological field[1] that is locally compact, second countable, non-discrete and totally disconnected. Since F together with the additive structure is a locally compact topological group it has a Haar measure μ that is unique up to multiplicative constant. Observe that for each $x \in F$, the measure $\mu_x(B) = \mu(xB)$ is also a Haar measure. We set

$$|x| = \frac{\mu_x(B)}{\mu(B)},$$

where B is any measurable set with finite and positive measure. By $\mathcal{O} = \{x \in F : |x| \leq 1\}$, we denote the ring of integers in F. We fix $\pi \in \mathfrak{p} - \mathfrak{p}^2$, where

$$\mathfrak{p} = \big\{x \in F : |x| < 1\big\}.$$

We are going to sketch the construction of a building associated to $\mathrm{GL}(3, F)$. For more details, we refer to [14]. A lattice is a subset $L \subset F^3$ of the form

$$L = \mathcal{O}v_1 + \mathcal{O}v_2 + \mathcal{O}v_3,$$

where $\{v_1, v_2, v_3\}$ is a basis of F^3. We say that two lattices L_1 and L_2 are equivalent if and only if $L_1 = aL_2$ for some nonzero $a \in F$. Then $\mathscr{X}$, the building of $\mathrm{GL}(3, F)$,

[1] A *topological field* is an algebraic field with a topology making addition, multiplication and multiplicative inverse a continuous mappings.

is the set of equivalence classes of lattices in F^3. For $x, y \in \mathscr{X}$ there are a basis $\{v_1, v_2, v_3\}$ of F^3 and integers $j_1 \le j_2 \le j_3$ such that (see [14, Proposition 3.1])

$$x = \mathcal{O}v_1 + \mathcal{O}v_2 + \mathcal{O}v_3, \quad \text{and} \quad y = \pi^{j_1}\mathcal{O}v_1 + \pi^{j_2}\mathcal{O}v_2 + \pi^{j_3}\mathcal{O}v_3.$$

We say that x and y are joined by an edge if and only if $0 = j_1 \le j_2 \le j_3 = 1$. The subset

$$\mathscr{A} = \left\{\pi^{j_1}\mathcal{O}v_1 + \pi^{j_2}\mathcal{O}v_2 + \pi^{j_3}\mathcal{O}v_3 : j_1, j_2, j_3 \in \mathbb{Z}\right\}$$

is called an apartment. A sector in $\mathscr{A}$ is a subset of the form

$$\mathcal{S} = \left\{x + \pi^{j_1}\mathcal{O}v_1 + \pi^{j_2}\mathcal{O}v_2 + \pi^{j_3}\mathcal{O}v_3 : j_{\sigma(1)} \le j_{\sigma(2)} \le j_{\sigma(3)}, j_1, j_2, j_3 \in \mathbb{Z}\right\},$$

where σ is a permutation of $\{1, 2, 3\}$ and $x \in \mathscr{A}$. Thus, a subsector of $\mathcal{S}$ is

$$\left\{x + \pi^{k_1+j_1}\mathcal{O}v_1 + \pi^{k_2+j_2}\mathcal{O}v_2 + \pi^{k_3+j_3}\mathcal{O}v_3 : j_{\sigma(1)} \le j_{\sigma(2)} \le j_{\sigma(3)}, j_1, j_2, j_3 \in \mathbb{Z}\right\},$$

for some $0 \le k_{\sigma(1)} \le k_{\sigma(2)} \le k_{\sigma(3)}$. Finally, two sectors

$$\mathcal{S} = \left\{x + \pi^{j_1}\mathcal{O}v_1 + \pi^{j_2}\mathcal{O}v_2 + \pi^{j_3}\mathcal{O}v_3 : j_{\sigma(1)} \le j_{\sigma(2)} \le j_{\sigma(3)}, j_1, j_2, j_3 \in \mathbb{Z}\right\},$$

and

$$\mathcal{S}' = \left\{x' + \pi^{j_1}\mathcal{O}v_1 + \pi^{j_2}\mathcal{O}v_2 + \pi^{j_3}\mathcal{O}v_3 : j_{\sigma'(1)} \le j_{\sigma'(2)} \le j_{\sigma'(3)}, j_1, j_2, j_3 \in \mathbb{Z}\right\},$$

are opposite if $\sigma' \circ \sigma^{-1} = (3\,2\,1)$.

A sector in $\mathscr{X}$ is a sector in one of its apartments. Two sectors in $\mathscr{X}$ are equivalent if and only if its intersection contains a sector. By Ω, we denote the equivalence classes of sectors in $\mathscr{X}$. Let ω_0 and ω_0' be the equivalence class of

$$\mathscr{S}_0 = \left\{\pi^{j_1}\mathcal{O}e_1 + \pi^{j_2}\mathcal{O}e_2 + \pi^{j_3}\mathcal{O}e_3 : j_1 \le j_2 \le j_3, j_1, j_2, j_3 \in \mathbb{Z}\right\},$$

and

$$\mathscr{S}_0' = \left\{\pi^{j_1}\mathcal{O}e_1 + \pi^{j_2}\mathcal{O}e_2 + \pi^{j_3}\mathcal{O}e_3 : j_1 \ge j_2 \ge j_3, j_1, j_2, j_3 \in \mathbb{Z}\right\},$$

respectively. Two sectors $\mathscr{S}$ and $\mathscr{S}'$ are opposite in $\mathscr{X}$ if there are subsectors of $\mathscr{S}$ and $\mathscr{S}'$ opposite in a common apartment. By Ω_0, we denote the equivalence classes of sectors opposite to $\mathscr{S}_0$.

Suppose that $\omega' \in \Omega_0$. Let $\{v_1, v_2, v_3\}$ be a basis of F^3, and $k_1 \le k_2 \le k_3$ and $k_1' \ge k_2' \ge k_3'$ be integers such that

$$\left\{\pi^{j_1+k_1}\mathcal{O}v_1 + \pi^{j_2+k_2}\mathcal{O}v_2 + \pi^{j_3+k_3}\mathcal{O}v_3 : j_1 \le j_2 \le j_3, j_1, j_2, j_3 \in \mathbb{Z}\right\}, \quad (4.9)$$

and

$$\left\{\pi^{j_1+k'_1}\mathcal{O}v_1 + \pi^{j_2+k'_2}\mathcal{O}v_2 + \pi^{j_3+k'_3}\mathcal{O}v_3 : j_1 \geq j_2 \geq j_3,\ j_1, j_2, j_3 \in \mathbb{Z}\right\}, \quad (4.10)$$

belong to ω_0 and ω', respectively. Since the sector (4.9) belongs to ω_0, we have

$$v_1 = b_{11}e_1, \quad v_2 = b_{21}e_1 + b_{22}e_2, \quad v_3 = b_{31}e_1 + b_{32}e_2 + b_{33}e_3,$$

for some $b_{ij} \in F$ such that $b_{11}, b_{22}, b_{33} \neq 0$. Hence, the matrix

$$g = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ 0 & b_{22} & b_{32} \\ 0 & 0 & b_{33} \end{pmatrix},$$

satisfies $ge_j = v_j$. In particular, $g\omega'_0 = \omega'$. Therefore, the group of upper triangular matrices acts transitively on Ω_0. Observe also that the stabilizer of ω'_0 in $\mathrm{GL}(3, F)$ is the group of lower triangular matrices. Thus, the group

$$\left\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in F\right\}$$

acts simply transitively on Ω_0.

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