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RESEARCH ARTICLE

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On the category of modules over some semisimple bialgebras

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Abstract We study the tensor category of modules over a semisimple bialgebra H under the assumption that irreducible H-modules of the same dimension > 1 are isomorphic. We consider properties of Clebsch–Gordan coefficients showing multiplicities of occurrences of each irreducible H-module in a tensor product of irreducible ones. It is shown that, in general, these coefficients cannot have small values.

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الملخص

ندرس الفئة المُوَتِّرِيَّة للحلقيات على جبرية ثنائية نصف سهلة H باشتراط أن جميع حلقيات- H غير القابلة للاختزال ذات نفس البعد > 1 متشاكلة تقابلياً. نعتبر خصائص معاملات كلبش-جوردان مبينين تعدُّد ظهور كل حلقية- H في الضرب الموتري لتلك غير القابلة للاختزال. تم تبيين أنه، بشكل عام، لا يمكن أن تكون قيم تلك المعاملات صغيرة.

1 Introduction

Throughout the paper, the basic field k is algebraically closed and H is a finite dimensional k-bialgebra that is semisimple as an algebra. The restriction that k is algebraically closed implies that any finite dimensional simple k-algebra is a full matrix algebra over k. We shall use the notations for bialgebras and Hopf algebras from [4,5].

An element $g \in H$ is a *group-like element* if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. The set of all group-like elements $G(H)$ of a bialgebra H is a multiplicative monoid. If H is a Hopf algebra with an antipode S, then $G(H)$ is a group, where $g^{-1} = S(g)$ for any $g \in G(H)$.

The dual bialgebra H^* has a natural pairing $\langle -, - \rangle : H^* \otimes H \to k$. The monoid $G = G(H^*)$ of group-like elements in H^* consists just of algebra homomorphisms $H \to k$.

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A semisimple algebra H is a direct sum of full matrix algebras over k. One-dimensional summands are in one-to-one correspondence with algebra homomorphisms $H \to k$. Hence, under our assumptions, H as a k-algebra has a semisimple direct decomposition

$$H = \left(\bigoplus_{g \in G} k\, e_g \right) \oplus \left(\bigoplus_{j=1}^{n} \mathrm{Mat}(d_j, k) \right), \tag{1.1}$$

where n, d_j are natural numbers and $\{e_g,\ g \in G\}$ is a system of central orthogonal idempotents in H corresponding to the one-dimensional direct summands. For $h \in H$ and $g \in G$ we have $he_g = e_g h = \langle g, h\rangle e_g$.

As in [1], we here deal with the case when

$$1 < d_1 < d_2 < \cdots < d_n, \tag{1.2}$$

which just means that irreducible H-modules of the same dimension > 1 are isomorphic.

The main result of the paper [1] is the following:

Theorem 1.1 *Let H be a semisimple Hopf algebra with decomposition (1.1), $n \geqslant 1$, such that (1.2) holds. Suppose that at least one single matrix constituent is a Hopf ideal in H. Then it is the last summand* $\mathrm{Mat}(d_n, k)$.

In the present paper, for a bialgebra H, we consider properties of the *Clebsch–Gordan coefficients*, that is, the multiplicities of occurrences of irreducible H-modules in semisimple decompositions of tensor products of irreducible ones. These play a substantial role in representation theory of groups and their applications to physics.

More general than in [1], we consider the case of a bialgebra H not assuming that it is a Hopf algebra. In Theorem 4.5, under some restrictions on the Clebsch–Gordan coefficients, it is shown that $n \leqslant 2$ in (1.1). In Theorem 4.6, for the case $n = 2$, we compare the number of one-dimensional summands in (1.1) and the sizes of matrix components. Further properties of Clebsch–Gordan coefficients are found in Theorem 4.7. In the last section we consider the comodule structure of H.

2 Bialgebra structure of H and H^*

We consider comultiplication and counit in the bialgebra H having as algebra a decomposition (1.1). The counit $\varepsilon : H \to k$ has the form

$$\varepsilon(x) = \begin{cases} \delta_{g,1}, & x = e_g, \\ 0, & x \in \mathrm{Mat}(d_i, k). \end{cases} \tag{2.1}$$

For each one-dimensional H-module $E_g = ke_g$ related to $g \in G$,

$$he_g = \langle g, h\rangle e_g, \quad h \in H. \tag{2.2}$$

For further information on the bialgebra structure of H some additional properties of the dual bialgebra H^* are needed.

The semisimple bialgebra H over an algebraically closed field k has the decomposition (1.1). If char $k = 0$ and H is a Hopf algebra, then, by the Larson–Radford theorem [4, Theorem 7.4.6], the dual Hopf algebra H^* is also semisimple. Recall that some additional information on semisimple Hopf algebras in positive characteristic can be found in [6].

Consider one of the main samples of bialgebras, namely a monoid algebra $F = kG$ of a finite monoid G. In this case $\Delta(g) = g \otimes g$ for any $g \in G$. It means that G is the monoid of group-like elements of F.

It is well-known that the dual bialgebra F^* is a direct sum of one-dimensional ideals $\oplus_{g\in G} ke_g$. Here $\{e_g \mid g \in G\}$ is the dual base for the base $\{g \mid g \in G\}$ of F. In particular, F^* is semisimple.

However, its dual bialgebra $F^{**} = F$ is not necessarily semisimple. For example, take the three-element commutative monoid $G = \{1, a, b\}$ with the identity element 1 such that $ab = b^2 = a^2 = b$. Then the one-dimensional space $k(a - b)$ in the monoid algebra $F = kG$ is annihilated by a, b. Hence it is a nilpotent ideal and the monoid algebra kG is not semisimple.

We shall now expand these structural observations to the case of the bialgebra H from (1.1).

Consider in each matrix component $\mathrm{Mat}(d_i, k)$, the non-degenerated symmetric bilinear form

$$\langle x, y\rangle = \mathrm{tr}\left(x \cdot {}^t y\right). \tag{2.3}$$

In the case of a Hopf algebra we consider the form $\langle x, y\rangle = \mathrm{tr}(x \cdot S(y))$ where S is the antipode [3]. We shall prove results from [3, Section 3] on Hopf algebras for the bialgebra case.

Using the form (2.3), we can identify the space $\mathrm{Mat}(d_i, k)$ with its dual space. Then the base of $\mathrm{Mat}(d_i, k)$ consisting of matrix units $E^{(i)}_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, d_i$, is self-dual, namely

$$\langle E^{(i)}_{\alpha\beta}, E^{(i)}_{\gamma\tau}\rangle = \mathrm{tr}\left(E^{(i)}_{\alpha\beta} E^{(i)}_{\tau\gamma}\right) = \delta_{\beta\tau}\,\mathrm{tr}\left(E^{(i)}_{\alpha\gamma}\right) = \delta_{\beta\tau}\delta_{\alpha\gamma}.$$

Thus, as a vector space, H^* has a direct decomposition

$$H^* = kG \oplus \mathrm{Mat}(d_1, k) \oplus \cdots \oplus \mathrm{Mat}(d_n, k).$$

The counit ε^* in H^* is defined as $\varepsilon(f) = f(1)$ for any $f \in H^*$, where 1 is the unit of H, and $1 = \sum_{g\in G} e_g + E^{(1)} + \cdots + E^{(n)} \in H$. Direct calculations, as in [3], show $\varepsilon(g) = 1, \ \ \varepsilon(x) = \mathrm{tr}(x)$, if $g \in G, \ x \in \mathrm{Mat}(d_i, k)$. The comultiplication Δ^* in H^* is defined by $\langle \Delta^*(f), a \otimes b\rangle = \langle f, ab\rangle$, for all $a, b \in H$.

Proposition 2.1 *The following conditions are satisfied:*

(i) *For* $g \in G$, $\Delta^*(g) = g \otimes g$.
(ii) *For the matrix unit* $E^{(i)}_{\alpha\beta}$ *from the* i*-th matrix component,*

$$\Delta^*\left(E^{(i)}_{\alpha\beta}\right) = \sum_\gamma E^{(i)}_{\alpha\gamma} \otimes E^{(i)}_{\gamma\beta}.$$

Proof Let

$$a = \sum_{g\in G} \tau_g g + \sum_{\substack{i=1,\ldots,n;\\ \alpha\beta=1,\ldots,d_i}} E^{(i)}_{\alpha\beta} a^{(i)}_{\alpha\beta}, \quad b = \sum_{g\in G} \xi_g g + \sum_{\substack{i=1,\ldots,n;\\ \gamma,\lambda=1,\ldots,d_i}} E^{(i)}_{\gamma\lambda} b^{(i)}_{\gamma\lambda}, \tag{2.4}$$

where $\tau_g, \xi_g, a^{(i)}_{\alpha\beta}, b^{(i)}_{\gamma\lambda} \in k$. Then

$$ab = \sum_{g\in G} \tau_g \xi_g g + \sum_{\substack{i=1,\ldots,n;\\ \alpha,\lambda=1,\ldots,d_i}} E^{(i)}_{\alpha\lambda} \left(\sum_{\beta=1}^{d_i} a^{(i)}_{\alpha\beta} b^{(i)}_{\beta\lambda}\right).$$

So, if $g \in G$, then $\langle \Delta^*(g), a \otimes b\rangle = \langle g, ab\rangle = \tau_g \xi_g = \langle g, a\rangle\langle g, b\rangle = \langle g \otimes g, a \otimes b\rangle$, hence $\Delta^*(g) = g \otimes g$.

Now

$$\left\langle \Delta^*\left(E^{(i)}_{\alpha\lambda}\right), a \otimes b\right\rangle = \langle E^{(i)}_{\alpha\lambda}, ab\rangle = \sum_{\beta=1}^{d_i} a^{(i)}_{\alpha\beta} b^{(i)}_{\beta\lambda} = \sum_{\beta=1}^{d_i} \langle E^{(i)}_{\alpha\beta}, a\rangle \langle E^{(i)}_{\beta\lambda}, b\rangle$$
$$= \left\langle \sum_{\beta=1}^{d_i} E^{(i)}_{\alpha\beta} \otimes E^{(i)}_{\beta\lambda}, a \otimes b \right\rangle,$$

and this means $\Delta^*(E^{(i)}_{\alpha\lambda}) = \sum_{\beta=1}^{d_i} E^{(i)}_{\alpha\beta} \otimes E^{(i)}_{\beta\lambda}$. □

Proposition 2.2 *If* $p, q \in G$, *then* $p * q = pq$. *Suppose that* H *is a Hopf algebra. If* $x \in \mathrm{Mat}(d_i, k)$, *then* $p * x = p \rightharpoonup x$, $x * p = x \leftharpoonup p$.

Proof Suppose that a is from (2.4). Then by (2.6)

$$\langle p * q, a\rangle = \sum_{g,h,f\in G,\ hf=g} \tau_g \langle p, e_h\rangle\langle q, e_f\rangle = \tau_{pq} = \langle pq, a\rangle$$

and therefore $p * q = pq$.

In the case of Hopf algebras we can prove the last formulas as in [3]. □

Now we shall consider some new properties of the bialgebra H from (1.1). The bialgebra H is a left and right H^*-module algebra with respect to actions $f \rightharpoonup x,\ x \leftharpoonup f$ of $f \in H^*$ on $x \in H$, [5, Example 4.1.10], that is, for $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$,

$$f \rightharpoonup x = \sum_x x_{(1)}\langle f, x_{(2)}\rangle, \quad x \leftharpoonup f = \sum_x \langle f, x_{(1)}\rangle x_{(2)}. \tag{2.5}$$

For $f \in G$, the maps $f \rightharpoonup$, $\leftharpoonup f$ are algebra endomorphisms of H preserving the identity element 1 of H, and $1 = \sum_{f\in G} e_f + \sum_{i\geqslant 1} E^{(i)}$, where $E^{(i)}$ is the identity matrix of $\mathrm{Mat}(d_i, k)$.

As shown in [2, Propposition 1.3, Corollary 1.2],

$$\Delta(e_g) = \sum_{p,q\in G,\ pq=g} e_p \otimes e_q + \sum_{i=1}^{n} \mathcal{D}_{g,i};$$

$$\Delta(x) = \sum_{g\in G} \big((g \rightharpoonup x) \otimes e_g + e_g \otimes (x \leftharpoonup g)\big) + \sum_{i,j=1}^{n} \Delta_{ij}^t(x), \tag{2.6}$$

where $\mathcal{D}_{g,i} \in \mathrm{Mat}(d_i, k)^{\otimes 2}$ and, $\Delta_{ij}^t(x) \in \mathrm{Mat}(d_i, k) \otimes \mathrm{Mat}(d_j, k)$, for $i, j = 1, \dots, n$.

With respect to the natural pairing $\langle -, -\rangle$, the elements $g \in G \subset H^*$ are dual to the elements $e_g,\ g \in G$, and each matrix component is annihilated by elements of G.

Proposition 2.3 (1) *The element e_1 is the left and the right integral in H.*
(2) *For $g, f \in G$, $g \rightharpoonup e_f$ is equal either to zero or to the sum of all $e_p,\ p \in G$, such that $pg = f$.*
(3) *An element $g \in G$ is invertible if and only if $g \rightharpoonup e_1 \neq 0$.*
(4) *For $g \in G$,*

$$g \rightharpoonup \left(\sum_{f\in G} e_f\right) = \sum_{f\in G} e_f, \quad g \rightharpoonup \left(\sum_i E^{(i)}\right) = \left(\sum_i E^{(i)}\right),$$

where the $E^{(i)}$ denote the identity matrix in $\mathrm{Mat}(d_i, k)$.

Proof (1) For $h \in H$, $he_1 = \langle 1, h\rangle e_1 = \varepsilon(h)e_1$ by (2.1) and (2.2).
(2) Using the first equation in (2.6), we obtain

$$g \rightharpoonup e_f = \sum_{p,q\in G,\ pq=f} e_p\langle g, e_q\rangle = \sum_{p\in G,\ pg=f} e_p.$$

(3) By (2), the element $g \rightharpoonup e_1 \neq 0$ if and only if there exists an element $p \in G$ such that $pg = 1$. It means that $p = g^{-1}$.
(4) Let $g \in G$. The map $h \mapsto (g \rightharpoonup h)$ is an algebra endomorphism of H preserving the unit element $1 = \sum_{f\in G} e_f + \sum_{i\geqslant 1} E^{(i)}$, where $E^{(i)}$ is the identity matrix of $\mathrm{Mat}(d_i, k)$. Each full matrix algebra $\mathrm{Mat}(d_i, k)$ is simple and therefore it is mapping either to zero or injectively into H. Hence we obtain the required equality by (2). □

Theorem 2.4 *Let α be a unit preserving endomorphism of the semisimple algebra $R = \oplus_{i=1}^n \mathrm{Mat}(d_i, k)$, where $1 < d_1 < d_2 < \cdots < d_n$. Suppose that each integer d_j is not a linear combination of $d_1, \dots, d_{j-1}$ with non-negative integer coefficients. Then α is an automorphism of R preserving each matrix component.*

Proof We shall proceed by induction on n. If $n = 1$, then α is an endomorphism of the full matrix algebra preserving the unit element. Hence α is injective and therefore it is surjective.

Suppose that the theorem is proved for $n-1$. Since $d_n > d_j$ for any $j < n$ we can conclude that $\mathrm{Mat}(d_n, k)$ is stable under α. By induction, α induces an automorphism on $R/\mathrm{Mat}(d_n, k)$. So without loss of generality we can assume that α is identical modulo $\mathrm{Mat}(d_n, k)$. It means that if $x \in \mathrm{Mat}(d_j, k),\ j < n$, then $\alpha(x) = x + \beta_j(x)$, where $\beta_j : \mathrm{Mat}(d_j, k) \to \mathrm{Mat}(d_n, k)$ is an algebra homomorphism, not necessarily preserving the unit element.

Suppose first that $\alpha(E^{(n)}) \neq 0$. Then α induces an automorphism of $\mathrm{Mat}(d_n, k)$ and therefore $\alpha(E^{(n)}) = E^{(n)}$. If $x \in \mathrm{Mat}(d_j, k),\ j < n$ then $xE^{(n)} = 0$ in R and therefore

$$0 = \alpha(x)\alpha\left(E^{(n)}\right) = \left(x + \beta_j(x)\right)E^{(n)} = \beta_j(x)E^{(n)} = \beta_j(x).$$

Hence, in this case, α is an automorphism and the proof is complete.

Suppose that $\mathrm{Mat}(d_n, k)$ is contained in the kernel of α. Then $E^{(n)} = \beta_1(E^{(1)}) + \cdots + \beta_{n-1}(E^{(n-1)})$ because α preserves the unit element of R. Note that $\beta_i(x)\beta_j(y) = 0$ if $i \neq j$, so the elements $\beta_1(E^{(1)}), \ldots, \beta_{n-1}(E^{(n-1)})$ form an orthogonal system of idempotents of sizes $t_1, \ldots, t_{n-1}$, respectively, and therefore $t_1 + \cdots + t_{n-1} = d_n$.

By the Noether–Skolem and centralizer theorems, we can conclude that $\mathrm{Mat}(t_j, k) \simeq \beta_j(\mathrm{Mat}(d_j, k)) \otimes \mathrm{Mat}(s_j, k)$ for some non-negative integer s_j. Hence $t_j = d_j s_j$ and therefore $d_n = t_1 + \cdots + t_{n-1} = d_1 s_1 + \cdots + d_{n-1}s_{n-1}$, a contradiction. □

Note that the restriction on the numbers in Theorem 2.4 is satisfied if, for each j, the greatest common divisor of $d_1, \ldots d_j$ is smaller than the greatest common divisor of $d_1, \ldots, d_{j-1}$.

3 The category of modules

Let H be, as above, a semisimple bialgebra with direct sum decomposition (1.1) such that (1.2) is satisfied. In what follows we shall in addition assume that either G is a group or $d_1, \ldots, d_n$ are as in Theorem 2.4. In both cases, for each $g \in G$, the map $g \rightharpoonup$ induces an algebra automorphism of every matrix component in (1.1).

The tensor product $M \otimes N$ of two left H-modules M, N is again a left H-module by putting, for $h \in H$ and $\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)}$,

$$h(x \otimes y) := \sum_h h_{(1)}x \otimes h_{(2)}y, \quad x \in M,\ y \in N. \tag{3.1}$$

Let M_i be the irreducible H-module associated with matrix component $\mathrm{Mat}(d_i, k)$. The module M_i is annihilated by each element $e_g,\ g \in G$, and by any $\mathrm{Mat}(d_j, k),\ j \neq i$.

Note that if $h \in \mathrm{Mat}(d_i, k)$ and $x \in M_p,\ y \in M_q$, then by (3.1) we have

$$h(x \otimes y) = \Delta^i_{pq}(h) \cdot (x \otimes y), \tag{3.2}$$

where $\Delta^i_{pq}(h) \cdot (x \otimes y)$ is the componentwise action on the tensor product.

As in [1, Formula (9), Lemma 3.1] we can prove:

Proposition 3.1 *Let $h \in H,\ g \in G$ and $\mathcal{D}_{g,i}$ from (2.6). If $x, y \in M_i$ then $h(\mathcal{D}_{g,i} \cdot (x \otimes y)) = \langle g, h\rangle \mathcal{D}_{g,i} \cdot (x \otimes y)$ and $\mathcal{D}^2_{g,i} = \mathcal{D}_{g,i}$.*

Proof We have

$$\begin{aligned} h\left(\mathcal{D}_{g,i} \cdot (x \otimes y)\right) &= \left(\Delta(h)\mathcal{D}_{g,i}\right) \cdot (x \otimes y) \\ &= \left(\Delta(h)\Delta(e_g)\right) \cdot (x \otimes y) = \Delta(he_g) \cdot (x \otimes y) = \langle g, h\rangle \mathcal{D}_{g,i} \cdot (x \otimes y)\,. \end{aligned}$$

The last statement holds because e_g is an idempotent. □

The next fact is well known for Hopf algebras [1]. In virtue of Theorem 2.3 it holds for bialgebras H satisfying the above restrictions.

Proposition 3.2 *Let H be a bialgebra with a direct decomposition (1.1) such that (1.2) holds. Suppose M to be an irreducible H-module, $\dim M > 1$. Let E_g be the one-dimensional H-module associated with an element $g \in G$. Then $M \otimes E_g$ and $E_g \otimes M$ are irreducible H-modules and*

$$M \otimes E_g \simeq E_g \otimes M \simeq M.$$

For any square matrix X denote its transpose by ${}^tX$. Let M_i be as above the irreducible H-module of dimension d_i Then the dual space $M_i^* = \mathrm{Hom}_k(M_i, k)$ is a left H-module. In fact, let $f \in M_i^*$, $h \in \mathrm{Mat}(d_i, k)$ and $x \in M_i$. Put $\langle h \cdot f, x\rangle = \langle f, {}^th \cdot x\rangle$. Then for $h_1, h_2 \in \mathrm{Mat}(d_i, k)$,

$$\begin{aligned}\langle h_1 h_2 \cdot f,\ x\rangle = \langle f,\ {}^t(h_1 h_2) \cdot x\rangle &= \langle f,\ {}^th_2\, {}^th_1 \cdot x\rangle \\ &= \langle h_2 \cdot f,\ {}^th_1 \cdot x\rangle = \langle h_1 \cdot (h_2 \cdot f),\ x\rangle.\end{aligned}$$

Using [4, Lemma 7.5.10, p. 322] as in [1, Proposition 1.7], we obtain

Proposition 3.3 *Let M_i, M_j be irreducible left H-modules of dimensions > 1. Then $\dim \mathrm{Hom}_H(M_i \otimes M_j, E_\varepsilon) = \delta_{ij}$.*

Proposition 3.4 *Denote by A the direct sum $\oplus_{g\in G} E_g$ of all one-dimensional H-modules E_g, $g \in G$. Then there is a direct sum decomposition*

$$M_i \otimes M_j = \delta_{ij} A \oplus \left(\oplus_{t=1}^n m_{ij}^t M_t\right), \tag{3.3}$$

where $m_{ij}^t = \dim_k \mathrm{Hom}_H(M_i \otimes M_j, M_t) \geqslant 0$. In particular,

$$\begin{aligned}\dim(M_i \otimes M_j) = d_i d_j &= \delta_{ij}|G| + \sum_{t=1}^n m_{ij}^t d_t \\ &= \dim\left(\delta_{ij} A \oplus \left(\oplus_{t=1}^n m_{ij}^t M_t\right)\right)\end{aligned} \tag{3.4}$$

and $|G| \leqslant d_1^2$.

Proposition 3.4 generalizes [1, Corollary 1.8, Theorem 1.9] from Hopf algebras to the case of bialgebras with the mentioned properties.

Using Proposition 3.1, we can prove as in [1, Lemma 3.1]:

Corollary 3.5 *Let $\mu : E_g \to M_i \otimes M_i$ be an embedding of H-modules from Proposition 3.4. Then $\mu(E_g) = \mathcal{D}_{g,i}(M_i \otimes M_i)$.*

The next affirmation follows from associativity of tensor products of H-modules.

Theorem 3.6 ([1]) *The multiplicities m_{ij}^t defined in Proposition 3.4 satisfy the Eq. (3.4) and the equations*

$$m_{ij}^s = m_{js}^i, \quad \delta_{ij}\delta_{ls}|G| + \sum_{t=1}^n m_{ij}^t m_{ts}^l = \delta_{js}\delta_{li}|G| + \sum_{t=1}^n m_{js}^t m_{it}^l,$$

for all $i, j, s, l = 1, \dots, n$. In particular, $m_{ij}^s = m_{js}^i = m_{si}^j$ and

$$\delta_{ij}\delta_{ls}|G| + \sum_{t=1}^n m_{ti}^j m_{ts}^l = \delta_{js}\delta_{li}|G| + \sum_{t=1}^n m_{st}^j m_{it}^l.$$

If $i, j, p = 1, \dots, n$, then $m_{ij}^p \leqslant d_{\min(i,j,p)}$.

Furthermore, if H is a Hopf algebra, then $m_{pq}^i = m_{qp}^i$ for all $i, p, q = 1, \dots, n$, that is, $M_i \otimes M_j \simeq M_j \otimes M_i$ for all $i, j = 1, \dots, n$.

Denote by $R_t,\ 1 \leqslant t \leqslant n$, the square matrix of size n whose (i, j)th entry is equal to m_{ij}^t. Then R_r is a non-negative integer matrix. By Theorem 3.6, each matrix R_t is symmetric. Now the equality (3.4) and the statement of Theorem 3.6 can be rewritten as

$$\begin{aligned} \left[{}^t R_j, R_l\right] &= |G|(E_{lj} - E_{jl}), \\ \sum_t d_t R_t &= \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} (d_1 \dots d_n) - |G| E_n, \end{aligned} \tag{3.5}$$

where E_n and E_{lj} are the identity matrix and the matrix units of size n. If H is a Hopf algebra, then each matrix R_i is symmetric.

For later use consider the case $n = 2$. In view of Theorem 3.6 put

$$a = m_{11}^1,\ b = m_{12}^1 = m_{21}^1 = m_{11}^2,\ c = m_{22}^1 = m_{12}^2 = m_{21}^2,\ d = m_{22}^2, \tag{3.6}$$

which all are non-negative integers. Then

$$R_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad R_2 = \begin{pmatrix} b & c \\ c & d \end{pmatrix}. \tag{3.7}$$

Now the first equation in (3.5) can be rewritten as

$$b^2 + c^2 - ac - bd = |G|, \tag{3.8}$$

and the second equation in (3.5) as

$$\begin{aligned} d_1 a + d_2 b &= d_1^2 - |G|, \\ d_1 b + d_2 c &= d_1 d_2, \\ d_1 c + d_2 d &= d_2^2 - |G|. \end{aligned} \tag{3.9}$$

4 Properties of coefficients

In this section we shall consider properties of the Clebsch–Gordan coefficients m_{ij}^t in the decomposition (3.3) for a bialgebra H with decomposition (1.1) and with additional properties from Sect. 3.

Proposition 4.1 *Let H be a bialgebra as above and M_p, M_q irreducible H-modules of dimensions greater than* 1, *such that $M_p \otimes M_q$ and $M_q \otimes M_p$ are irreducible H-modules. Then the order of the monoid G is equal to* 1. *If H is a Hopf algebra then $M_p \otimes M_q \simeq M_q \otimes M_p$.*

Proof Suppose the H-module $M_p \otimes M_q$ is irreducible for some indices $p, q = 1, \dots, n$. Then $p \neq q$ by Proposition 3.4. So $M_p \otimes M_q \simeq M_i$ for some index $i = 1, \dots, n$. It means that $m_{pq}^i = 1 = m_{iq}^p$. Note that the indices i, p, q are distinct because $d_i = d_p d_q > d_p, d_q$. In particular $n \geqslant 3$.

Associativity of the tensor product of modules yields by Theorem 3.6, since $m_{pq}^i = 1 = m_{qi}^p$,

$$\begin{aligned} M_p \otimes M_q \otimes M_q &\simeq M_p \otimes \left(A \oplus \left(\oplus_t m_{qq}^t M_t\right)\right) \\ &\simeq \left(M_p \otimes A\right) \oplus \left[\oplus_t m_{qq}^t \left(M_p \otimes M_t\right)\right] \\ &\simeq |G| M_p \oplus m_{qq}^p A \oplus \left[\left(\oplus_{t,s} m_{qq}^t m_{pt}^s M_s\right)\right]; \\ M_p \otimes M_q \otimes M_q &\simeq M_i \otimes M_q = M_p \oplus \left[\oplus_{t \neq p} m_{iq}^t M_t\right]. \end{aligned}$$

Comparing coefficients in M_p, we obtain $|G| + \sum_t m_{qq}^t m_{pt}^p = 1$. Hence $|G| = 1$. □

Consider other cases when tensor products of some irreducible H-modules have similar almost trivial decompositions.

Proposition 4.2 *Let* $1 \leqslant i \neq j \leqslant n$. *Suppose that there exists a unique index* t *such that* $m_{ij}^t \geqslant 1$. *Then* $t \geqslant \max(i, j)$.

Proof By the assumption,

$$M_i \otimes M_j \simeq m_{ij}^t M_t. \tag{4.1}$$

Theorem 3.6 and (4.1) imply

$$m_{ij}^t = \frac{\dim M_i \cdot \dim M_j}{\dim M_t} = \frac{d_i \cdot d_j}{d_t} \leqslant d_{\min(i,j,t)} \leqslant d_i.$$

Hence $d_j \leqslant d_t$ which means that $j \leqslant t$. Similarly $i \leqslant t$. □

Proposition 4.3 *Suppose that* (4.1) *holds for some* $t \neq i$ *and*

$$M_t \otimes M_i \simeq m_{ti}^{t'} M_{t'}, \tag{4.2}$$

for some index t'. *Then* $t = t' = j > i$ *and* $m_{ij}^t = m_{ji}^t = d_i$.

Proof By Proposition 4.2 and the assumption, $t \geqslant \max(i, j)$. Since $t > i$ we can apply Theorem 3.6 and get $m_{ij}^t = m_{ti}^j > 0$. So $t' = j$ by the assumption and $M_t \otimes M_i \simeq m_{ti}^j M_j$. Applying Proposition 4.2 we obtain $j \geqslant \max(t, i) = t \geqslant j$ and therefore $t = j > i$ because $j \neq i$. Comparing dimensions we complete the proof. □

Proposition 4.4 *Let* i *be an index with the property: for every index* $j \neq i$, *there exists a unique index* t *such* $m_{ij}^t > 0$ *and if* $t \neq i$, *then also* (4.2) *holds for some index* t'. *Then:*

(1) if $j \neq i$, *then* $M_i \otimes M_j \simeq d_{\min(i,j)} M_{\max(i,j)}$;
(2) $M_i \otimes M_i \simeq A \oplus d_1 M_1 \oplus \cdots \oplus d_{i-1} M_{i-1} \oplus m_{ii}^i M_i$;
(3) $d_i^2 = |G| + d_1^2 + \cdots + d_{i-1}^2 + m_{ii}^i d_i$; *in particular, if* $i = 1$, *then the order of the monoid* G *is divisible by* d_1;
(4) $\Delta(\mathrm{Mat}(d_i, k)) \subseteq H \otimes \mathrm{Mat}(d_i, k) + \mathrm{Mat}(d_i, k) \otimes H + \left(\sum_{j \geqslant i} \mathrm{Mat}(d_j, k)^{\otimes 2}\right)$.

Proof (1) Suppose that $j > i$. Then $t \geqslant \max(i, j) = j > i$ by Proposition 4.2 and $t = j$, $m_{ij}^j = m_{jj}^i = d_i$. If $j < i$, then, by Proposition 4.3, the case $t \neq i$ is impossible. Hence $j < i$ implies $t = i$ and $m_{ij}^i = d_j$. So in all cases (1) is proved. Moreover, for any $j \neq i$,

$$m_{ij}^s = \begin{cases} d_{\min(i,j)}, & s = \max(i, j); \\ 0, & \text{otherwise}. \end{cases} \tag{4.3}$$

(2) By Theorem 3.6, there is an H-module decomposition

$$M_i \otimes M_i \simeq A \oplus \left(\oplus_j m_{ii}^j M_s\right).$$

Note that $m_{ii}^j = m_{ij}^i$. Hence, by (4.3), the inequality $m_{ii}^j > 0$ implies $i = \max(i, j) > j$ and in this case $m_{ij}^i = d_j$. Hence we obtain the required decomposition of $M_i \otimes M_i$.

(3) Comparing dimensions in the decomposition from (2) we can obtain the required equality. In particular if $i = 1$, then $d_1^2 = |G| + m_{11}^1 d_1$ and therefore $|G|$ is divisible by d_1.

(4) Take any indices $p, q = 1, \ldots, n$ such that $\Delta_{pq}^i \neq 0$ in (2.6). Combining (2.6), (3.2) and Proposition 4.4, properties (1), (2), we see that $\mathrm{Mat}(d_i, k)$ annihilates $M_p \otimes M_q$ if either $i \neq \max(p, q)$ where $p \neq q$ or $p = q < i$. By (3.2) it means that (4) is satisfied. □

Theorem 4.5 *Let* H *be a bialgebra with decomposition* (1.1) *such that* (1.2) *is satisfied and either* G *is a group or* $d_1, \ldots, d_n$ *are as in* Theorem 2.4. *Suppose that* H *satisfy the assumptions of* Proposition 4.4 *for some index* i. *If* $i = 1$, *then* $J = \oplus_{j \geqslant 2} \mathrm{Mat}(d_j, k)$ *is a bi-ideal in* H. *If* $i = n$, *then* $\mathrm{Mat}(d_i, k)$ *is a bi-ideal of* H.

Proof Let $i = 1$ and $\Delta^j_{pq} \neq 0$ for some $j \geqslant 2$ where either $p = 1$ or $q = 1$. The case $p = q = 1$ is impossible by Proposition 4.4, (1) and (2). Hence either p or q is greater than 1. Hence J is a bi-ideal.

Suppose that $i = n$ and $\Delta^n_{pq} \neq 0$ for some p, q. If either $p < n$ or $q < n$, then, by Proposition 4.4, (1), $n = \max(p, q)$ and therefore either $p = n$ or $q = n$. In both cases,

$$\Delta\left(\mathrm{Mat}(d_n, k)\right) \subseteq H \otimes \mathrm{Mat}(d_n, k) \oplus \mathrm{Mat}(d_n, k) \otimes H.$$

□

Theorem 4.6 *Let H be a Hopf algebra with decomposition (1.1). If the number n of full matrix algebras of size > 1 in (1.1) is equal to 2, then the greatest common divisor D of sizes d_1, d_2 of matrices is greater than 1. The order of the group G is divisible by D.*

Proof As it is noticed in [7] the order $|G|$ of the group G divides d_1^2 and d_2^2. Suppose that d_1, d_2 are coprime. Using the notations (3.6), we see in the second equation in (3.9) that b is divisible by d_2 and c is divisible by d_1, namely $b = d_2 u_1$, $c = d_1 u_2$ for some non-negative integers u_1, u_2. So this equation can be rewritten as $u_1 + u_2 = 1$. It follows immediately that there is an alternative,

$$\text{either } u_1 = 1,\ u_2 = 0, \text{ or } u_1 = 0,\ u_2 = 1.$$

Suppose first that $u_1 = 1$, $u_2 = 0$. Then $b = d_2$, $c = 0$ and the first equation in (3.9) has the form $d_1 a + d_2^2 = d_1^2 - |G|$. This is impossible because $d_2 > d_1$ but the left hand side is greater or equal to d_2^2 while the right hand side is smaller than d_1^2.

Suppose now that $u_1 = 0$, $u_2 = 1$. Then $b = 0, c = d_1$ and the first equation in (3.9) has the form $d_1 a = d_1^2 - 1$ which is impossible since $d_1 > 1$. □

Theorem 4.7 *Let H be a semisimple bialgebra with decomposition (1.1) where $n \geqslant 2$. Then $m^t_{n-1,n} \geqslant 2$ for some index $t = 1, \ldots, n$ in (3.3).*

Proof Suppose that $m^t_{n-1,n} \leqslant 1$ for all $t = 1, \ldots, n$. Then, in equation (3.3), we have $d_{n-1} d_n \leqslant d_1 + \cdots + d_n$. Dividing by d_n we get by (1.2),

$$d_{n-1} \leqslant \frac{d_1}{d_n} + \cdots + \frac{d_{n-1}}{d_n} + 1 < n$$

On the other hand, (1.2) implies that $d_i \geqslant i + 1$ for any i and in particular $d_{n-1} > n$, a contradiction. □

5 The category of (H, H)-bimodules

Let, as above H, be the semisimple bialgebra with decomposition (1.1). By (3.1) the comultiplication $\Delta : H \to H \otimes H$ is also a homomorphism of (H, H)-bimodules. So it is interesting to look at the structure of (H, H)-bimodules.

Note that any (H, H)-bimodule can be considered as a left module over $H \otimes H^{op}$ where H^{op} is defined on the same vector space as H by the new multiplication $x \cdot y = yx$. Clearly H^{op} is a semisimple algebra with a similar decomposition (1.1). Its irreducible modules are dual modules E^*_g, $g \in G$, and $M^*_1, \ldots, M^*_n$. The action of $h \in H^{op}$ on E^*_g and on M^*_i is the following. If $f \in E^*_g$ then $\langle fh, e_g \rangle = \langle g, h \rangle \langle f, e_g \rangle$. If $f \in M^*_i$ and $x \in M_i$ then $\langle fh, x \rangle = \langle f, hx \rangle$. By Proposition 1.5 [1], each M^*_i is an irreducible H^{op}-module.

Now H^{op} is a bialgebra with comultiplication $\Delta^{op} = \Delta$ and a counit $\varepsilon^{op} = \varepsilon$.

Consider the bialgebra $H \otimes H^{op}$. It is a semisimple bialgebra whose simple ideals are tensor products of simple ideals of H and of H^{op}. It means that irreducible $H \otimes H^{op}$-modules are just tensor products

$$E_g \otimes E^*_f, \quad E_g \otimes M^*_i, \quad M_j \otimes E^*_g, \quad M_i \otimes M^*_j, \qquad f, g \in G.$$

The one-dimensional bimodule $E_g \otimes E^*_f$ has a base $e_g \otimes e_f$ such that

$$h\left(e_g \otimes e_f\right) r = \langle g, h \rangle \langle f, r \rangle \left(e_g \otimes e_f\right),$$

for all $h, r \in H$.

By Proposition 3.2 and Proposition 1.5 [1], the bimodule $E_g \otimes M_i^*$ can be identified with M_i where $hxr = \langle g, h\rangle \cdot {}^t r \cdot x$ for all $h, r \in H$ and $x \in M_i$.

The bimodule $M_j \otimes E_g^*$ can be identified with M_i where $hxr = hx\langle g, r\rangle$ for all $h, r \in H$ and $x \in M_i$.

Finally, the bimodule $M_i \otimes M_j^*$ is identified with $M_i \otimes M_j$ where $hxr = hx \cdot {}^t r$ for all $h, r \in H$ and $x \in M_i$.

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