SERIEs (2012) 3:209–226
DOI 10.1007/s13209-011-0053-8

ORIGINAL ARTICLE

Freedom of choice: John Stuart Mill and the tree of life

Jorge Alcalde-Unzu · Miguel A. Ballester · Jorge Nieto

Received: 15 November 2010 / Accepted: 3 March 2011 / Published online: 29 March 2011

Abstract This essay deals with the notion and content of freedom of choice proposing a new set up and a new family of measures for this concept which is, indeed, an ethical value of paramount importance in a well ordered and open society. Following some ideas of John Stuart Mill, we propose that freedom of choice has to be understood not in a single stage of choice, but in the ordered collection of choices that a person can make in her life. We then suggest to represent a life in a tree structure, where each node represents a state of life and the edges between nodes will represent possible decisions in life. In this new framework, we propose a set of axioms that imply the following family of measures of lifetime's freedom of choice: the lifetime's freedom of choice has to be evaluated by a weighted sum of all possible states of life an individual might visit, with weights representing the number of decisions the individual took to reach that state.

Keywords Freedom of choice · Tree of life · Autonomy

JEL Classification D63 · D71

J. Alcalde-Unzu · J. Nieto (✉)
Department of Economics, Public University of Navarre,
31006 Pamplona, Spain
e-mail: jnieto@unavarra.es

J. Alcalde-Unzu
e-mail: jorge.alcalde@unavarra.es

M. A. Ballester
Department of Economics, Universitat Autonoma de Barcelona,
08193 Barcelona, Spain
e-mail: miguelangel.ballester@uab.es

1 Introduction

This essay deals with the notion and content of freedom of choice which is, indeed, an ethical value of paramount importance in a well ordered and open society. The rapidly growing modern literature on this issue includes, among its pioneers, names as outstanding as Friedman and Friedman (1980), Sen (1988), Pattanaik and Xu (1990), and many others. See Barbera et al. (2004) for an extensive survey and a formal description that connects the problem to other relevant economic questions. However, the question of freedom of choice and its relevance for Economics was already present in Smith (1776), whose most famous book was a close defense of individual freedom as a key instrument for the Wealth of Nations. However, in the long list of classical normative writers the real champion on this subject is Mill (1859), who wrote a short but extraordinary piece entitled *On Liberty*, and published in 1859. Most of modern writers on freedom of choice explicitly acknowledge their debt to the intuitions and conclusions contained in Mill's book. Mill (1859) set forth this basic principle in the following words:

> This, then, is the appropriate region of human liberty. It comprises, first, the inward domain of consciousness [...]. Secondly, the principle requires liberty of tastes and pursuits; of framing the plan of our life to suit our own character; of doing as we like, subject to such consequences as may follow [...]. The only freedom which deserves the name, is that of pursuing our own good in our own way, so long as we do not attempt to deprive others of theirs, or impede their efforts to obtain it (pp. 15–16).

In some seminal papers, Sen (1988, 1991) established that freedom of choice has two distinct values: intrinsic and instrumental. The latter means that, freedom of choice enables individuals and nations to manage their own capabilities in such a way as to achieve a better outcome, and thus, higher well-being. Freedom of choice, therefore, has an instrumental value in that, given more opportunities, a person has a better chance of achieving states of life that are closer to her preferences. This instrumental value has been widely recognized and is, in fact, the underlying value in Adam Smith's *Wealth of Nations* (Smith 1776), in Milton Friedman's *Free to Choose* (Friedman and Friedman 1980) and also in John Roemer's *Free to lose* (Roemer 1988). It is also the only value that classic microeconomics assigns to the consumer choosing a basket of goods from a budget set. When the consumer compares two budget sets, she looks only at the basket providing the highest utility in each of the sets. This is the idea behind the notion of indirect utility function. Hence, she is indifferent between the whole budget set and the best of its alternatives, even when this best alternative is presented with no other possible choice.[1]

The intrinsic value of freedom of choice means that, by having freedom to choose from a set of possibilities and by the mere act of choosing, individuals are better-off. John Stuart Mill put it in this way:

[1] For an axiomatic treatment of this idea, see Nehring and Puppe (1996) or Ballester et al. (2004).

> If it were felt that the free development of individuality is one of the leading essentials of well-being [...] there would be no danger that liberty should be undervalued [...]. But the evil is, that individual spontaneity is hardly recognised by the common modes of thinking as having any intrinsic worth, or deserving any regard on its own account. (p. 53)

A seminal paper that deals with the intrinsic value of freedom is Pattanaik and Xu (1990). To isolate the intrinsic value from the instrumental one, it is assumed that there are no individual preferences on the alternatives of choice (they are either unknown or irrelevant). Their axiom *Indifference between no-choice situations* establishes that all singletons (sets of only one alternative) have the same null value, because they all offer no choice, independently of the alternative included in each singleton. Combined with other properties, this axiom excludes from the analysis of the intrinsic value of freedom all information regarding the preferences of the individuals or the quality of the alternatives. As a consequence, their proposal is to measure the intrinsic value of freedom of choice by counting the number of alternatives available in each choice set.[2]

This paper adopts the line taken by Pattanaik and Xu (1990), which is to study the intrinsic value of freedom of choice. Hence, freedom of choice of an individual has to be judged by looking at the choices that this individual may face throughout her adult states (or, at least, a significant part of it). However, we extend their line of reasoning by suggesting that freedom of choice has to be understood not at a single stage of choice, but in the ordered sequence of choices that may arise over a lifetime. Indeed, human life is merely a sequential or progressive entity, as John Stuart Mill writes:

> I regard utility as the ultimate appeal on all ethical questions; but it must be utility in the largest sense, grounded on the permanent interests of a man as a progressive being (p. 14).

Freedom of choice must be therefore measured by looking at the whole structure of choice. In other words, the choices we make today determine the choices available to us tomorrow, so that a decision maker's freedom of choice has to be measured by looking at the structure of decisions throughout an entire lifetime. In the following quote, Mill himself defends the act of choice as a way to improve the autonomy of an individual. We therefore need to contemplate choices in a comprehensive way.

> The human faculties of perception, judgment, discriminative feeling, mental activity, and even moral preference, are exercised only in making a choice. He who does anything because it is the custom makes no choice. He gains no practice either in discerning or in desiring what is best. The mental and moral, like the muscular powers, are improved only by being used. (p. 55).

Our proposal, therefore, is to represent a life as a tree structure, where each node represents a state of life and the edges between nodes will represent possible

[2] There are also intermediate perspectives that try to combine intrinsic and instrumental values in a unique criterion to compare opportunity sets (see for instance Bossert et al. 1994; Dutta and Sen 1996; Alcalde-Unzu and Ballester 2005).

evolutions or transitions in life. In this new set up, we propose a set of axioms that imply the following family of measures of lifetime's freedom of choice: a lifetime's freedom of choice has to be evaluated by a weighted sum of all the possible states of life an individual might visit, with weights representing the number of decisions the individual took to reach that state.

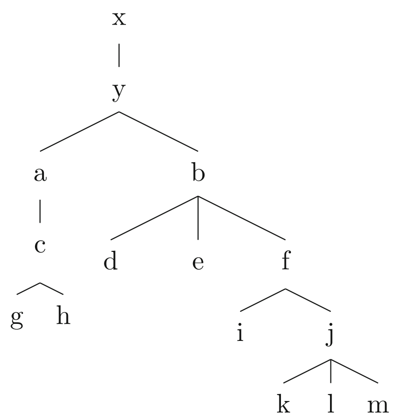
The remainder of the paper, in which we give a formal structure to these ideas and present our main result, is organized as follows: Sect. 2 describes the notation, definitions, axioms and results. Section 3 contains some concluding remarks. The proofs of all the results are given in the Appendix.

2 Trees of life and freedom of choice

The possible **states of life** are represented by an infinite set X. We describe a **life** as a graph-tree structure (T, E_T). The nodes of the graph are a non-empty finite set of states of life, $T \subset X$. The edges of the graph, $E_T \subset T \times T$, represent possible evolutions or transitions in life. That is, y is an **evolution** from x whenever xE_Ty.[3] The graph (T, E_T) representing life has the structure of a tree, i.e., there exists a state $o(T, E_T) \in T$ which is the **origin of life** and for every other state in life, there is a unique path from the origin to that state.

Consider the following example of a life (T, E_T), with nodes $T = \{x, y, a, b, c, d, e, f, g, h, i, j, k, l, m\}$ and edges $E_T = \{(x, y), (y, a), (y, b), (a, c), (c, g), (c, h), (b, d), (b, e), (b, f), (f, i), (f, j), (j, k), (j, l), (j, m)\}$. The following graph is a representation of this tree, where evolutions should be read from top to bottom. Observe that the origin of (T, E_T) is $o(T, E_T) = x$.

LIFE (T, E_T)



Our objective is to measure the freedom of choice a life offers. Formally, we discuss how to construct a complete and transitive binary relation $\succsim$ on the set of all lives. We consider that any tree formed by states of life of the set X is

[3] We will also use notation $(x, y) \in E_T$ when useful.

a possible life and, therefore, our objective is to construct a criterion $\succsim$ defined over the set of all possible trees that we can formally construct with the set X. The interpretation of $(T, E_T) \succsim (S, E_S)$ is that life (T, E_T) offers at least the same freedom of choice than life (S, E_S). Binary relations $\succ$ and $\sim$ are defined as usual.

We have assumed that any tree formed by states of life of the set X is a possible life. This is an important assumption in the model. It can be argued that if the description of the states of life is done with great detail, some states of life, by definition, should not appear in any tree before other states of life. However, we believe that our assumption is the most natural one when states of life are described in terms of the basic components of well-being (as for instance, in the capabilities approach or a similar model). Under that view, a tree would describe all the possible paths lived by the agent in terms of achieved functionings/capabilities. By considering all possible hypothetical societies and life patterns in these societies, one may argue that no tree should be discarded a priori.[4]

We now discuss our axioms. Our first axiom aims to capture the idea that freedom of choice should measure the part of life remaining after the individual has exercised some autonomy. In particular, suppose that two lives start with possibly different sequences of compulsory states of life but all states and possible evolutions *after the first choice in life* are the same. In this case, we consider that both lives offer the same amount of freedom of choice.[5] In order to formally define the property we need to describe the first state of life that offers freedom of choice (if any), and the part of life that comes after that first state.

Given a life (T, E_T), a state $x \in T$ is **irrelevant** if there is a unique evolution from x. Otherwise, we say that x is relevant. We also say that a relevant state x **offers freedom of choice** if there are at least two possible evolutions from x, and that it is **final** if there are none. If a life (T, E_T) possesses at least one state offering freedom of choice, we call the first of these states the **effective origin** of life, denoted by $eo(T, E_T)$.

Given a life (T, E_T) and a state of life $x \in T$, we call the collection of lives that span from x the **future** of x in (T, E_T).[6] We say that the collection of states different from x in the path connecting the origin of life to state x is the set of **past states** of x in (T, E_T).

Indifference of No-choice Origins For every pair of lives (T, E_T), (S, E_S) where the futures of $eo(T, E_T)$ and $eo(S, E_S)$ are the same or $eo(T, E_T)$ and $eo(S, E_S)$ do not exist: $(T, E_T) \sim (S, E_S)$.

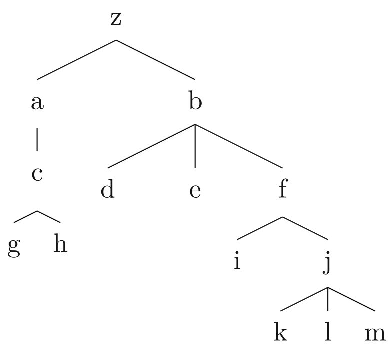
4 If the model is used for one particular application where this assumption is restrictive, the reader may want to know that some reformulation of the axioms and results should be executed. Stronger assumptions would be needed to reproduce our initial propositions in this paper.

5 To some extent, this axiom builds upon the classical idea that freedom of choice can only be exercised if there exists more than one alternative. Thus, we should not consider that part of life before any choice can be made.

6 Notice that if x has k evolutions, there are exactly k lives spanning from x with their origins in those k evolutions. In particular, if x is a final state, its future is empty.

The following example describes Indifference of No-choice Origins. Imagine that, in our initial example, we maintain the entire portion of life beyond the effective origin (which was state y). For instance, we replace the evolution from x to y with a different origin and effective origin, which is now state z. That is, consider life (S, E_S) given by:

LIFE (S, E_S)



Indifference of No-choice Origins claims that both lives (T, E_T) and (S, E_S) should both be judged as having the same freedom of choice.

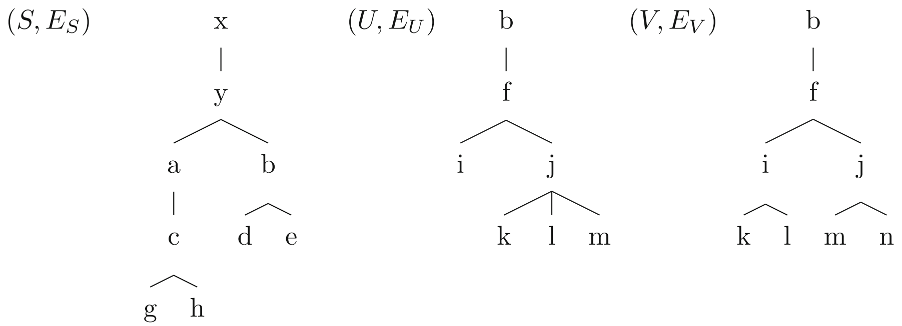
Our second axiom aims to capture the idea that two lives differing only in some portions that are themselves lives should be judged by how much freedom the sub-lives offer. Alternatively, the property can be interpreted as follows. Suppose that, on reaching a certain stage of life, an individual discovers that the expected evolution beyond that state no longer holds; that it has been replaced with a new evolution and a different future. Our axiom claims that we do not need to recompute freedom of choice for the entire life; it is sufficient to evaluate the changes in our new future. Freedom of choice in life increases (decreases) if the newly opened future offers more (less) freedom than expected from the original future.

Separability of the Future Let (S, E_S), (U, E_U), (V, E_V) be lives such that $S \cap U = S \cap V = o(U, E_U) = o(V, E_V)$. If $o(U, E_U) \neq eo(U, E_U)$ and $o(V, E_V) \neq eo(V, E_V)$, then $(U, E_U) \succsim (V, E_V) \iff (S \cup U, E_S \cup E_U) \succsim (S \cup V, E_S \cup E_V)$.[7]

Lives $(S \cup U, E_S \cup E_U)$ and $(S \cup V, E_S \cup E_V)$ coincide everywhere except in the futures associated with state $o(U, E_U) = o(V, E_V)$. Once these lives reach this state, the two lives differ in only one possible evolution (since $o(U, E_U) \neq eo(U, E_U)$ and $o(V, E_V) \neq eo(V, E_V)$). Separability of the Future requires us to rank these two lives by means of lives (U, E_U) and (V, E_V).

The following example describes Separability of the Future. The union of lives (S, E_S) and (U, E_U) defines the life in our initial example, (T, E_T). Now suppose that the future beyond f is no longer as predicted in (T, E_T), but as given by (V, E_V). Separability of the Future tells us to rank (T, E_T) and the union of (S, E_S) and (V, E_V) in the same way that we would rank (U, E_U) and (V, E_V).

[7] To simplify notation, we often omit brackets when describing singleton sets.



Indifference of No-choice Origins and Separability of the Future impose basic conditions on how we should measure freedom of choice. However, the following propositions show that the combination of the two axioms has a powerful effect. Under these two axioms: (1) we should measure freedom of choice in a neutral way, since we only need to consider the volume and distribution of states in life, and not the specific labeling of these states, and (2) we can suppress any irrelevant state from life without compromising freedom of choice.

Given a bijective self-map μ over the set of states X, let $\mu[(T, E_T)] = (\mu(T), \mu(E_T))$ where naturally $\mu(x)\mu(E_T)\mu(y) \Leftrightarrow xE_Ty$.

Proposition 1 *Let $\succsim$ satisfy Indifference of No-choice Origins and Separability of the Future. For every bijective self-map μ over the set of states X and for every life $(T, E_T) : (T, E_T) \sim \mu[(T, E_T)]$.*

The intuition behind Proposition 1 is as follows. Take one state x in life (T, E_T) and suppose that we wish to replace it with a new one y. Indifference of No-choice Origins guarantees that this change leaves freedom unaltered if x is in the past of (or coincides with) the effective origin of (T, E_T). If x is in the future of the effective origin, Separability of the Future plays a substantial role in proving the claim. In particular, consider the state a for which x is an evolution, and construct the life consisting of the evolution from a to x and the entire future of x. In this life, x is in the past of (or coincides with) the effective origin, and we can apply Indifference of No-choice Origins and replace x with y with no alteration to the freedom of this sub-life. Separability of the Future therefore allows replacement of the sub-life without compromising freedom of choice. Repeated application of this trick allows us to replace any state as desired.

Given a life (T, E_T), we denote by $(\hat{T}, E_{\hat{T}})$ the life obtained by eliminating all irrelevant states. That is, let $\{i_1, \ldots, i_k\}$ be the irrelevant states, which are evolutions from states $\{[a_1], a_2, \ldots, a_k\}$ and evolve into states $\{e_1, e_2, \ldots, e_k\}$. Then, $\hat{T} = T \setminus \{i_1, \ldots, i_k\}$, and $E_{\hat{T}}$ is obtained by eliminating from E_T the evolutions $\{([a_1], i_1), (i_1, e_1), (a_2, i_2), (i_2, e_2), \ldots, (a_k, i_k), (i_k, e_k)\}$ and inserting the evolutions $\{([a_1], e_1)(a_2, e_2), \ldots, (a_k, e_k)\}$.[8]

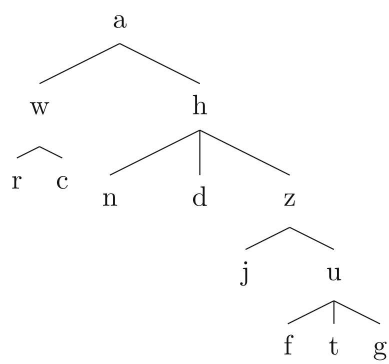
[8] If $o(T, E_T)$ is an irrelevant state, it would not be an evolution from any other state. To simplify the notation, we have represented this idea through $[a_1]$, which can be read as a_1 if x_1 is not the origin and ignored if x_1 is the origin.

Proposition 2 *Let $\succsim$ satisfy Indifference of No-choice Origins and Separability of the Future. For every life $(T, E_T) : (T, E_T) \sim (\hat{T}, E_{\hat{T}})$.*

The intuition behind Proposition 2 is as follows. Choose one irrelevant state i in life (T, E_T) and suppose that we wish to remove it. Indifference of No-choice Origins guarantees that this removal leaves freedom unaltered if i is in the past of $eo(T, E_T)$. If i is in the future of $eo(T, E_T)$, Separability of the Future can be applied in a similar trick to the one in Proposition 1.[9] Let us reconsider the state a from which i evolves, and again, the life composed by the evolution from a to i and all the future of i. In this life, i is in the past of the effective origin and thus, can be removed. Separability of the Future therefore allows replacement of the sub-life without compromising freedom of choice. Repeated application of this trick allows us to conclude the proof.

Propositions 1 and 2 can be illustrated by means of the following example. It merely consists of the elimination of irrelevant states in our initial example (T, E_T), together with a bijection μ that is mapping the initial states to different states. That is, with Propositions 1 and 2 we have that life (T, E_T) provides the same freedom of choice as life $\mu[(\hat{T}, E_{\hat{T}})]$, with $\mu(y) = a$, $\mu(b) = h$, $\mu(c) = w$, $\mu(d) = n$, $\mu(e) = d$, $\mu(f) = z$, $\mu(g) = r$, $\mu(h) = c$, $\mu(i) = j$, $\mu(j) = u$, $\mu(k) = f$, $\mu(l) = t$ and $\mu(m) = g$.

A LIFE $\mu[(\hat{T}, E_{\hat{T}})]$



Our third axiom describes the idea that *equivalent changes* over two lives should be considered an expansion or reduction of freedom for both of them. The condition for changes to be considered equivalent is that those sub-lives removed from or added to lives had or will have exactly the same past. Formally, let lives (T, E_T), (S, E_S) and transformed lives $f(T, E_T)$, $f(S, E_S)$ be such that all are without irrelevant states. The transformation f is **equivalent** for lives (T, E_T) and (S, E_S) if it involves: (1) The removal of the same pieces of the future of nodes $x_1, x_2, \ldots, x_k \in T \cap S$ that share the same past in (T, E_T) and (S, E_S), and (2) the addition of the same new pieces of the future to nodes $y_1, y_2, \ldots, y_l \in T \cap S$ that share the same past in (T, E_T) and (S, E_S).

[9] Note that if i is irrelevant in (T, E_T), $i \neq eo(T, E_T)$.

Independence of Equivalent Transformations For every pair of lives (T, E_T) and (S, E_S) and for every equivalent transformation f for these lives: $f(T, E_T) \succsim (T, E_T) \Leftrightarrow f(S, E_S) \succsim (S, E_S)$.

The next axiom is a classical Archimedean property adapted to our context. It basically states that the quality of different states/decisions is comparable, and thus, the freedom of choice measure will not be lexicographic.

Archimedean Difference Let $(V, E_V), (W, E_W)$ be two lives and let $\{(T_i, E_{T_i}), (S_i, E_{S_i})\}_{i\in\mathbb{N}}$ be sequences of lives such that $(T_i, E_{T_i}) \succ (S_i, E_{S_i})$, and $T_i \cap T_j = S_i \cap S_j = T_i \cap V = S_i \cap W = \{o^*\}$ for all $i, j \in \mathbb{N}$, where o^* is the origin of all these lives. Then, there exists $k \in \mathbb{N}$ such that

$$\left(V \cup \bigcup_{i=1}^{k} T_i, E_V \cup \bigcup_{i=1}^{k} E_{T_i}\right) \succsim \left(W \cup \bigcup_{i=1}^{k} S_i, E_W \cup \bigcup_{i=1}^{k} E_{S_i}\right).$$

In essence, the property says that the fact that (W, E_W) might offer more freedom of choice than (V, E_V) can be reversed at some point by the addition of a sufficient number of better states or decisions to (V, E_V) than to (W, E_W). Our final axiom establishes a minimal condition of monotonicity. It reflects the idea that the addition of one extra choice to any non-final state of life should be always judged as an increment of freedom of choice.

Monotonicity For every life (T, E_T), for every non-final state $x \in T$ and for every $z \in X \setminus T : (T \cup \{z\}, E_T \cup \{(x, z)\}) \succ (T, E_T)$.

Propositions 1 and 2 provide specific instructions on how to rank lives in terms of freedom of choice if we accept Indifference of No-choice Origins and Separability of the Future. In what follows, we refine these results by adding Independence of Equivalent Transformations, Archimedean Difference and Monotonicity to our set of axioms. We show in Proposition 3 that Independence of Equivalent Transformations, together with the two initial axioms, links any life (T, E_T) to a highly structured life.

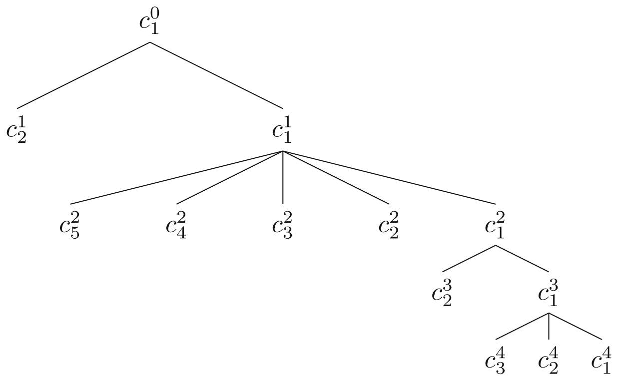
Given a life (T, E_T) denote by $[d_m(T, E_T)]$ the sequence of non-negative integers that describes the number of relevant states in life (T, E_T) with exactly m past states offering freedom of choice. Let M be the last positive integer (if any) such that $d_m(T, E_T) > 0$ (indeed, if $d_m(T, E_T) > 0$, it must be that $d_m(T, E_T) \geq 2$). A **canonical life** associated with (T, E_T) is a life (C, E_C) where: (1) $C = (c_1^0, c_1^1, \ldots, c_{d_1(T,E_T)}^1, \ldots, c_1^M, \ldots, c_{d_M(T,E_T)}^M)$ is a set of $\sum_{m=0}^{M} d_m(T, E_T)$ states and (2) evolutions are of the form $c_1^i E_C c_j^{i+1}$ for every $0 \leq i < M$ and for every $1 \leq j \leq d_{i+1}(T, E_T)$. Proposition 3 shows that a ranking satisfying Indifference of No-choice Origins, Separability of the Future and Independence of Equivalent Transformations allows us to write every tree in any of its canonical forms.

Proposition 3 *Let $\succsim$ satisfy Indifference of No-choice Origins, Separability of the Future and Independence of Equivalent Transformations. For every life (T, E_T) and for every canonical life (C, E_C) associated with $(T, E_T) : (T, E_T) \sim (C, E_C)$.*

The intuition behind Proposition 3 is as follows. Given Proposition 2, we can consider $(\hat{T}, E_{\hat{T}})$. Let us take two states x, y with the same past and imagine that we wish to move all the future of x to y. If this transformation generates an increase (respectively, a reduction) of freedom, equivalent transformations in other trees will also generate an increase (resp. a reduction) of freedom. Think about the hypothetical life completely equal to $(\hat{T}, E_{\hat{T}})$ except that x has the entire future of both x and y, and y is final. Moving a piece of the future from x to y will be a transformation equivalent to the one above, since all the pasts of involved states are the same. Hence, the life where y has the future of x and x has the future of y generates an increase (resp. a reduction) of freedom. However, these two increases (resp. reductions) are incompatible with Proposition 1. Repeated application of this trick allows us to conclude the proof.

Proposition 3 can be illustrated using the life (T, E_T) of our initial example. In precise terms, it states that (T, E_T) provides the same freedom of choice as any of its canonical lives, such as the following:

A LIFE (C, E_C)



We are now ready to describe the main result of the paper. Proposition 3 links our ranking of freedom of choice to the description of how many states of life have level m (meaning there are m states offering freedom in its past). The set of axioms used there together with Archimedean Difference and Monotonicity guarantee that we have to additively consider each of these states, possibly with a different weight for states having different level. These weights cannot have any real value, however. In particular: (1) weights have to be strictly positive for any $m > 0$ and zero for $m = 0$, and (2) the ratio of the weights of two consecutive levels (with $m > 0$) has to be constant.

Theorem 1 *A criterion* $\succsim$ *satisfies Indifference of No-choice Origins, Separability of the Future, Independence of Equivalent Transformations, Archimedean Difference and Monotonicity if and only if there exists* $\Delta \in \mathbb{R}_{++}$ *such that for every pair of lives* $(T, E_T), (S, E_S)$*:*

$$(T, E_T) \succsim (S, E_S) \Leftrightarrow \sum_{m=1}^{\infty} \Delta^m d_m(T, E_T) \geq \sum_{m=1}^{\infty} \Delta^m d_m(S, E_S).$$

The intuition behind Theorem 1 is as follows. We use the classical result reported by Krantz et al. (1971) to construct a scale over the set of sequences with integer values and only a finite number of components different from zero. We can interpret one of these sequences as a transformation from one life to another, and each element α^m of a sequence as the number of states of dimension m added to (if α^m is positive) or removed from (if α^m is negative) the original life. To prove the existence of a scale representation, we work with canonical lives and prove the classical invariance and Archimedean properties by means of our axioms. We then only need to describe the structure of the weights in the scale representation. Monotonicity guarantees that these weights are strictly positive for $m > 0$ and Proposition 2 guarantees that the weight for $m = 0$ is 0. Finally, Separability of the Future plays another key role in determining the weights. Notice that two lives must be judged equally if we insert them into another tree (and thus shift all the states towards higher levels). This implies that the ratio of weights must be constant.

We also show that the axioms used in Theorem 1 are independent.

Proposition 4 *Indifference of No-choice Origins, Separability of the Future, Independence of Equivalent Transformations, Archimedean Difference and Monotonicity are independent.*

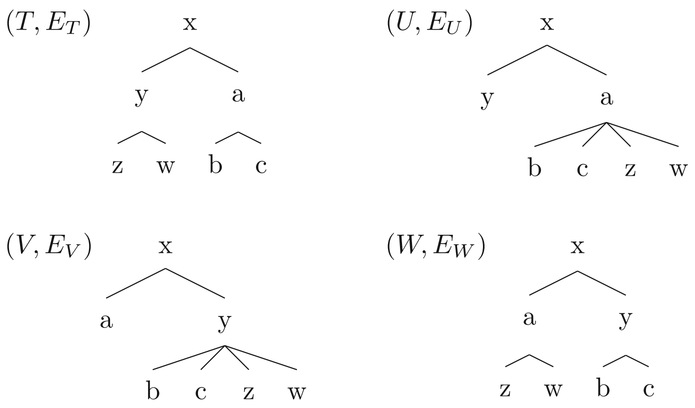
3 Extensions

In this paper we have proposed a family of measures to compare the degree of freedom of choice that individuals have in a society where a life is described by the sequential accessibility to different states of life. The idea of accessibility has been defined in this paper as a binary concept, i.e., either an agent has access to state y from state x or not. Clearly, a more general model would describe accessibility as a continuous variable, where access to y from x is maybe open for everybody but with different degrees/probabilities of access. To take this into account, our model should be adapted by introducing degree levels at each of the edges of the tree defining a life. In this case, the axioms introduced in this paper should also be adapted. We consider this point an interesting line for further research.

As the main result of our analysis, we have characterized a family of measures to evaluate the freedom of choice over a lifetime. Each of the measures evaluates freedom of choice in an additive form, weighting every feasible state of life according to the number of decisions the individual needs to take to reach that state. As noted in the main section, these weights are not completely free, and are in fact determined by only one parameter, $\Delta \in \mathbb{R}_{++}$. We can interpret this parameter as the importance we attach to the moment at which the agent takes the decision.

Determination of the appropriate value for parameter Δ is an interesting but difficult problem. On the one hand, it can be argued that initial decisions have consequences over longer periods or more stages of life, which strengthens the case for values of Δ smaller than 1, which would assign greater value to the possibility of having freedom in the early stages of life. On the other hand, it can also be argued that the autonomy of the individual grows with the effective decisions already taken, which would strengthen the case of values of Δ greater than 1, which would assign greater value to the possibility of having freedom in more advanced stages of life.

The first measure of freedom of choice in a static framework in the literature is the cardinality-based criterion of Pattanaik and Xu (1990). One may argue that a natural translation of such idea to our dynamic framework would be to count the number of possible sequences of states available to the individual. That would correspond to counting the number of paths in the tree starting in the origin and finishing in any final node, or equivalently, to counting the number of final nodes of the tree. This measure is not included in the family characterized in Theorem 1 because it fails to satisfy the property of Independence of Equivalent Transformations. To see why, consider the following trees:



By the cardinality-based criterion $\succsim_C$, we have that $(U, E_U) \sim_C (V, E_V) \succ_C (T, E_T) \sim_C (W, E_W)$. However, it is easy to see that the transformation from (T, E_T) to (U, E_U) is equivalent to the transformation from (V, E_V) to (W, E_W) and in the former case the transformation increases freedom (according to $\succsim_C$), although in the later, it decreases freedom.

We may consider the following alternative definition of an equivalent transformation. Let lives (T, E_T), (S, E_S) and transformed lives $f(T, E_T)$, $f(S, E_S)$ be such that all are without irrelevant states. The transformation f is *equivalent-2* for lives (T, E_T) and (S, E_S) if it involves: (1) The removal of nodes $x_1, x_2, \ldots, x_k \in T \cap S$ (and their future, which is also the same in both lives) that share the same past in (T, E_T) and (S, E_S) and (2) the addition of the same new pieces of the future to nodes $y_1, y_2, \ldots, y_l \in T \cap S$ that share the same past in (T, E_T) and (S, E_S).

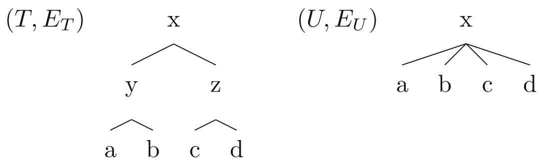
Replacing Independence of Equivalent Transformations by Independence of Equivalent Transformations-2 results into the characterization of a slightly different set of rankings. Given a life (T, E_T) denote by $[g_n(T, E_T)]$ the sequence of non-negative integers that describes the number of final states in (T, E_T) with exactly n past states offering freedom of choice.

Theorem 2 *A criterion $\succsim$ satisfies Indifference of No-choice Origins, Separability of the Future, Independence of Equivalent Transformations-2, Archimedean Difference and Monotonicity if and only if there exists $\Gamma \in \mathbb{R}_{++}$ such that for every pair of lives (T, E_T), (S, E_S):*

$$(T, E_T) \succsim (S, E_S) \Leftrightarrow \sum_{n=1}^{\infty} \Gamma^n g_n(T, E_T) \geq \sum_{n=1}^{\infty} \Gamma^n g_n(S, E_S).$$

Theorem 2 establishes a new family of rankings that weigh the number of final states depending on the number of decisions that the individual has to take in each of the possible paths. The weights have the same structure to those of Theorem 1: all weights have to be strictly positive (except for those possible final states without a decision, for which it is 0), and (2) the ratio of the weights of two consecutive levels (with $n > 0$) has to be constant.[10]

The different rankings of the family differ in the value of the parameter Γ. Determination of the appropriate value for Γ is probably easier than the determination of Δ in the family characterized in Theorem 1 because probably values of $\Gamma < 1$ do not seem appropriate. To see why, consider the following trees:



It seems intuitive that (T, E_T) offers more freedom of choice than (U, E_U) because the four possible sequences in (T, E_T) are richer (in terms of freedom of choice) than the four possible sequences in (U, E_U). However, if we select a parameter of $\Gamma < 1$, we would obtain a criterion that judges (U, E_U) as providing more freedom of choice than (T, E_T). On the other hand, values of $\Gamma > 1$ evaluates (T, E_T) as having more freedom of choice than (U, E_U), and the adaptation of the cardinality-based criterion to this dynamic context defined above (that coincides with the criterion of the family for a value of the parameter $\Gamma = 1$) evaluates these two trees as indifferent in terms of freedom of choice.[11]

Acknowledgments We would like to thank Salvador Barbera, for expanding our *tree of life*, guiding our learning and thereby giving us the chance to choose to participate in the scientific community of scholars working on Social Choice and Welfare Economics. We all feel indebted to Salvador. Financial support from the Spanish Ministry of Education through grants ECO2008-04756, ECO2009-11213, ECO2009-12836, Ramon y Cajal program, FEDER, and the Barcelona Economics Program of CREA is gratefully acknowledged.

[10] Since the proof is very similar to that of Theorem 1, we omit it. It is also easy to prove that the axioms used in Theorem 2 are also independent.

[11] All criteria of the family characterized in Theorem 1 establishes that (T, E_T) provides more freedom of choice than (U, E_U).

Appendix

Proposition 1 *Let $\succsim$ satisfy Indifference of No-choice Origins and Separability of the Future. For every bijective self-map μ over the set of states X and for every life* $(T, E_T) : (T, E_T) \sim \mu[(T, E_T)]$.

Proof Let $\succsim$ satisfy Indifference of No-choice Origins and Separability of the Future. We first prove that for every life (T, E_T), for every $x \in T$ and for every $y \in X \setminus T$, $(T, E_T) \sim (T_{x,y}, E_{T_{x,y}})$, where $T_{x,y}$ denotes the set of states where we just replaced node x with node y and $E_{T_{x,y}}$ denotes the set of edges where we replaced edges including x with similar edges including y.

To see this, consider a life (T, E_T), $x \in T$ and $y \in X \setminus T$. If (T, E_T) has no effective origin, Indifference of No-choice Origins proves the claim. Suppose that $eo(T, E_T)$ exists. If x is a past state of $eo(T, E_T)$ (respectively, $x = eo(T, E_T)$) then y is a past state of $eo(T_{x,y}, E_{T_{x,y}})$ (respectively, $y = eo(T_{x,y}, E_{T_{x,y}})$. Clearly, the futures of the effective origins of these two lives will coincide. The application of Indifference of No-choice Origins guarantees that $(T, E_T) \sim (T_{x,y}, E_{T_{x,y}})$.

If x is in the future of $eo(T, E_T)$, consider the state a such that x is an evolution from a. Consider the lives $(Z, T_Z) = (T(x) \cup \{a\}, E_{T(x)} \cup \{a, x\})$ and $(Z_{x,y}, E_{Z_{x,y}})$, where $(T(x), E_{T(x)})$ denotes the life composed by x and all the future of x in life (T, E_T). Since $eo(Z, E_Z)$ is either x or belongs to the future of x, Indifference of No-choice Origins guarantees that $(Z, E_Z) \sim (Z_{x,y}, E_{Z_{x,y}})$.

Now consider the life $(T \setminus T(x), E_T \setminus E_{T(x)})$, which is non-empty because x is in the future of $eo(T, E_T)$. Clearly, $(T \setminus T(x)) \cap Z = (T \setminus T(x)) \cap Z_{x,y} = a = o(Z, E_Z) = o(Z_{x,y}, E_{Z_{x,y}})$ and a is an irrelevant origin in the latter two lives. Hence, we can apply Separability of the Future obtaining $(T, E_T) \sim (T_{x,y}, E_{T_{x,y}})$. This proves our initial claim.

Now, simply notice that lives are finite graphs and hence, we can apply our claim by replacing, one by one, every state and its associated evolutions with the corresponding state and its associated evolutions, and the proposition is proved. □

Proposition 2 *Let $\succsim$ satisfy Indifference of No-choice Origins and Separability of the Future. For every life* $(T, E_T) : (T, E_T) \sim (\hat{T}, E_{\hat{T}})$.

Proof Let $\succsim$ satisfy Indifference of No-choice Origins and Separability of the Future. We first claim that for every life (T, E_T) and for every irrelevant state i in (T, E_T) (evolution from $[a]$ and evolving into e), $(T, E_T) \sim (T \setminus \{i\}, E_T \setminus \{([a], i), (i, e)\} \cup \{([a], e)\})$.

Consider life (T, E_T) and the irrelevant state i. If i is a past state of $eo(T, E_T)$, the application of Indifference of No-choice Origins proves our claim.

If i belongs to the future of $eo(T, E_T)$, then consider the lives $(T(i) \cup \{a\}, E_{T(i)} \cup \{(a, i)\})$ and $(T(e) \cup \{a\}, E_{T(e)} \cup \{(a, e)\})$. Clearly, since a and i are irrelevant in the former life and a is irrelevant in the latter life, the futures of the effective origins of these two lives coincide. Hence, Indifference of No-choice Origins guarantees that $(T(i) \cup \{a\}, E_{T(i)} \cup \{(a, i)\}) \sim (T(e) \cup \{a\}, E_{T(e)} \cup \{(a, e)\})$.

Now consider the life $(T \setminus T(i), E_T \setminus E_{T(i)})$, which is non-empty because i is in the future of $eo(T, E_T)$. Clearly, $(T \setminus T(i)) \cap (T(i) \cup \{a\}) = (T \setminus T(i)) \cap (T(e) \cup \{a\}) =$

$a = o(T(i) \cup \{a\}, E_{T(i)} \cup \{(a, i)\}) = o(T(e) \cup \{a\}, E_{T(e)} \cup \{(a, e)\})$ and a is an irrelevant origin in the latter two lives. Hence, we can apply Separability of the Future and we obtain $(T, E_T) \sim (T \setminus \{i\}, E_T \setminus \{(a, i), (i, e)\} \cup \{(a, e)\}$. This proves our initial claim.

The end of the proof is immediate from the fact that lives are finite graphs, and hence we can apply the previous claim repeatedly, removing the irrelevant states one by one and modifying the set of evolutions accordingly. □

Proposition 3 *Let $\succsim$ satisfy Indifference of No-choice Origins, Separability of the Future and Independence of Equivalent Transformations. For every life (T, E_T) and for every canonical life (C, E_C) associated with $(T, E_T) : (T, E_T) \sim (C, E_C)$.*

Proof Let $\succsim$ satisfy Indifference of No-choice Origins, Separability of the Future and Independence of Equivalent Transformations. We first claim that for every life without irrelevant states (T, E_T) and for every two states, x and y, in (T, E_T) sharing the same past, we have $(T, E_T) \sim (T, E_S)$, where (T, E_S) is the life where the only changes in evolutions are such that the future of x in (T, E_S) is empty and the future of y in (T, E_S) is the union of futures of x and y in (T, E_T).

To see this, consider the auxiliary lives (T, E_U), (T, E_V) where the only changes in evolutions are as follows: (1) the future of y in (T, E_U) is empty and the future of x in (T, E_U) is the union of the futures of both x and y in (T, E_T) and (2) the future of x in (T, E_V) is the future of y in (T, E_T) and the future of x in (T, E_V) is the future of y in (T, E_T).

Now suppose by contradiction that $(T, E_T) \succ (T, E_S)$. The transformation from (T, E_T) to (T, E_S) only involves cancelling the future of x in (T, E_T) and adding this future to y. Notice, however, that this is exactly the same transformation from (T, E_U) to (T, E_V), and states x and y have exactly the same past in both (T, E_T) and (T, E_U). Since all these lives are composed by relevant states, Independence of Equivalent Transformations guarantees that $(T, E_U) \succ (T, E_V)$.

However, Proposition 1 implies that $(T, E_S) \sim (T, E_U)$ and $(T, E_V) \sim (T, E_T)$ and by transitivity, $(T, E_S) \succ (T, E_T)$. This is obviously a contradiction. By analogous reasoning we must reject $(T, E_S) \succ (T, E_T)$ and thus it must be that $(T, E_T) \sim (T, E_S)$ as we initially claimed.

The end of the proof is now immediate. Proposition 2 guarantees the conversion of any life (T, E_T) into a sub-life without irrelevant states $(\hat{T}, E_{\hat{T}})$ respecting freedom of choice. Since lives are finite graphs, we can apply our previous claim by modifying the futures in $(\hat{T}, E_{\hat{T}})$ one by one, respecting freedom of choice, until we reach a canonical life with states in $\hat{T}$ is reached. Proposition 1 and transitivity thus guarantee that any other canonical life associated with (T, E_T) will also provide the same freedom of choice as (T, E_T). □

Theorem 1 *A criterion $\succsim$ satisfy Indifference of No-choice Origins, Separability of the Future, Independence of Equivalent Transformations, Archimedean Difference and Monotonicity if and only if there exists $\Delta \in \mathbb{R}_{++}$ such that for every pair of lives (T, E_T), (S, E_S):*

$$(T, E_T) \succsim (S, E_S) \Leftrightarrow \sum_{m=1}^{\infty} \Delta^m d_m(T, E_T) \geq \sum_{m=1}^{\infty} \Delta^m d_m(S, E_S).$$

Proof First, it is easy to see that all the criteria of the family satisfies the axioms. To see the other implication, consider a criterion $\succsim$ satisfying Indifference of No-choice Origins, Separability of the Future, Independence of Equivalent Transformations, Archimedean Difference and Monotonicity. Then, we are going to denote by Λ the set of sequences $[\lambda_m]$ with integer values and only a finite number of components different from zero. Consider the following binary relation R defined over Λ: for any $[\alpha_m], [\beta_m] \in \Lambda$, $[\alpha_m] R [\beta_m]$ if and only if there exist lives (U, E_U) and (V, E_V) without irrelevant states such that $[d_m(U, E_U) - d_m(V, E_V)] = [\alpha_m - \beta_m]$ and $(U, E_U) \succsim (V, E_V)$.

We first claim that R is a complete and transitive binary relation over Λ. To see this, let $[\alpha_m], [\beta_m] \in \Lambda$ and consider lives (T, E_T), (S, E_S), (U, E_U) and (V, E_V) such that $[d_m(T, E_T) - d_m(S, E_S)] = [d_m(U, E_U) - d_m(V, E_V)] = [\alpha_m - \beta_m]$. We show that (T, E_T) and (S, E_S) are compared in the same vein as (U, E_U) and (V, E_V) and hence R is well-defined. By Proposition 3, we can assume that both (T, E_T) and (U, E_U) are in canonical form, and also, that they share all the non-final states of one of the two lives. By Proposition 1, we can also assume that we will remove and add the same futures to these two lives. Notice that, under these conditions, the transformations performed in both (T, E_T) and (U, E_U) are equivalent, and thus, by Independence of Equivalent Transformations, (T, E_T) and (S, E_S) are ranked exactly as (U, E_U) and (V, E_V). This shows that R is well-defined. Clearly, from the completeness and transitivity of $\succsim$ and the fact that X is infinite, we have completeness and transitivity of R.

Second, we claim that $(\Lambda, R, +)$, where $+$ is the usual addition operator on sequences, is a closed extensive structure.[12] On top of the completeness and transitivity of R, we obviously have associativity of $+$. Moreover, for every $[\alpha_m], [\beta_m], [\gamma_m] \in \Lambda$ with $[\alpha_m]\, R\, [\beta_m]$, there exist lives (T, E_T) and (U, E_U) with $(U, E_U) \succsim (T, E_T)$ such that $[d_m(U, E_U) - d_m(T, E_T)] = [\alpha_m - \beta_m] = [(\alpha_m + \gamma_m) - (\beta_m + \gamma_m)]$ and thus, $[\alpha_m] + [\gamma_m]\, R\, [\beta_m] + [\gamma_m]$ and $[\gamma_m] + [\alpha_m]\, R\, [\gamma_m] + [\beta_m]$. Finally, consider $[\alpha_m], [\beta_m], [\gamma_m], [\delta_m] \in \Lambda$ such that $[\alpha_m]\, P\, [\beta_m]$. Then, by definition of R, there exist lives (V, E_V), (W, E_W) such that $[d_m(V, E_V) - d_m(W, E_W)] = [\gamma_m - \delta_m]$. Given that X is infinite, we can define sequences of lives $\{(T_i, E_{T_i})\}_{i \in \mathbb{N}}$, $\{(S_i, E_{S_i})\}_{i \in \mathbb{N}}$ such that: (1) $[d_m(T_i, E_{T_i}) - d_m(S_i, E_{S_i})] = [\alpha_m - \beta_m]$ for all $i \in \mathbb{N}$, and (2) $T_i \cap T_j = S_i \cap S_j = T_i \cap V = S_i \cap W = \{o^*\}$ for all $i, j \in \mathbb{N}$, where o^* is the origin of all these lives. Then, by Archimedean Difference, we know that there exists $k \in \mathbb{N}$ such that $(V \cup \bigcup_{i=1}^{k} T_i, E_V \cup \bigcup_{i=1}^{k} E_{T_i}) \succsim (W \cup \bigcup_{i=1}^{k} S_i, E_W \cup \bigcup_{i=1}^{k} E_{S_i})$. It is easy to see that $[d_m(V \cup \bigcup_{i=1}^{k} T_i, E_V \cup \bigcup_{i=1}^{k} E_{T_i}) - d_m(W \cup \bigcup_{i=1}^{k} S_i, E_W \cup \bigcup_{i=1}^{k} E_{S_i})] = k \cdot [\alpha_m] + [\gamma_m] - (k \cdot [\beta_m] + [\delta_m])$ and hence, $(k \cdot [\alpha_m] + [\gamma_m]) R (k \cdot [\beta_m] + [\delta_m])$, as desired. Thus, $(\Lambda, R, +)$ is a closed extensive structure.

[12] See Krantz et al. (1971) for a proper treatment of closed extensive structures.

Now, Theorem 1 in Krantz et al. (1971, p. 74), guarantees that there exists a real-valued function f over Λ such that for all $[\alpha_m], [\beta_m] \in \Lambda$: (1) $[\alpha_m]R[\beta_m] \Leftrightarrow f([\alpha_m]) \geq f([\beta_m])$, and (2) $f([\alpha_m] + [\beta_m]) = f([\alpha_m]) + f([\beta_m])$. Additionally, another function g satisfies conditions (a) and (b) if and only if there exists a strictly positive real value t such that $g = t \cdot f$.

By definition of R, we have obviously that $(T, E_T) \succsim (S, E_S) \Leftrightarrow [d_m(T, E_T)]R[d_m(S, E_S)] \Leftrightarrow f([d_m(T, E_T)]) \geq f([d_m(S, E_S)] \Leftrightarrow \sum_{m=1}^{\infty} w_m d_m(T, E_T) \geq \sum_{m=1}^{\infty} w_m d_m(S, E_S)$ where w_m is the value of f in the sequence with all values equal to zero except component m with value 1.

We finish by describing the structure of weights w_m. Clearly, we can deduce from Monotonicity that $w_m > 0$ for all $m > 0$. Similarly, by Proposition 2 we have that $w_0 = 0$. Now consider the life $(S, E_S) = (\{x, y, z\}, \{(x, y), (x, z)\})$ and two lives $(U, E_U), (V, E_V)$ such that $U \cap \{x, y, z\} = V \cap \{x, y, z\} = \{y\} = o(U, E_U) = o(V, E_V)$. By Separability of the Future, $(S \cup U, E_S \cup E_U) \succsim (S \cup V, E_S \cup E_V) \Leftrightarrow (U, E_U) \succsim (V, E_V)$. This is equivalent to $2w_1 + \sum_{m=1}^{\infty} w_{m+1} d_m(U, E_U) \geq 2w_1 + \sum_{m=1}^{\infty} w_{m+1} d_m(V, E_V) \Leftrightarrow \sum_{m=1}^{\infty} w_m d_m(U, E_U) \geq \sum_{m=1}^{\infty} w_m d_m(V, E_V)$. Since this is true for every pair of lives $(U, E_U), (V, E_V)$, it must be that $\frac{w_{m+1}}{w_m} = \Delta$ for all $m > 0$. Thus, we have proved the result. □

Proposition 4 *Indifference of No-choice Origins, Separability of the Future, Independence of Equivalent Transformations, Archimedean Difference and Monotonicity are independent.*

Proof We need some additional notation. Given a tree (T, E_T), we will denote by $z(T, E_T)$ the number of final nodes in (T, E_T). Consider the following rankings that compare any two trees $(T, E_{\hat{T}}), (S, E_{\hat{S}})$:

$$(T, E_T) \succsim_A (S, E_S) \Leftrightarrow \sum_{m=0}^{\infty} d_m(T, E_T) \geq \sum_{m=0}^{\infty} d_m(S, E_S).$$

$$(T, E_T) \succsim_B (S, E_S) \Leftrightarrow \sum_{m=1}^{\infty} m \cdot d_m(T, E_T) \geq \sum_{m=1}^{\infty} m \cdot d_m(S, E_S).$$

$$(T, E_T) \succsim_C (S, E_S) \Leftrightarrow z(T, E_T) \geq z(S, E_S).$$

$$(T, E_T) \succsim_D (S, E_S) \Leftrightarrow \text{ There does not exist } k \in \mathbb{N} \text{ such that}$$
$$d_l(S, E_{\hat{S}}) = d_l(T, E_{\hat{T}}) \text{ for all } l > k \text{ and } d_k(S, E_{\hat{S}}) > d_k(T, E_{\hat{T}})$$

$$(T, E_T) \sim_E (S, E_S) \text{ for all } (T, E_T), (S, E_S).$$

Rankings $\succsim_A, \succsim_B, \succsim_C, \succsim_D$ and $\succsim_E$ satisfy all the axioms except Indifference of No-choice Origins, Separability of the Future, Independence of Equivalent Transformations, Archimedean Difference and Monotonicity, respectively. □

References

Alcalde-Unzu J, Ballester MA (2005) Some remarks on ranking opportunity sets and arrow impossibility theorems: correspondence results. J Econ Theory 124:116–123

Ballester MA, de Miguel JR, Nieto J (2004) Set comparisons in a general domain: the indirect utility criterion. Math Soc Sci 48:139–150

Barbera S, Bossert W, Pattanaik P (2004) Ranking sets of objects. In: Barbera S, Hammond P, Seidl C (eds) Handbook of utility theory, vol 2

Bossert W, Pattanaik P, Xu Y (1994) Ranking opportunity sets: an axiomatic approach. J Econ Theory 63:326–345

Dutta B, Sen A (1996) Ranking opportunity sets and arrow impossibility theorems: correspondence results. J Econ Theory 71:90–101

Friedman M, Friedman R (1980) Free to choose: a personal statement. Harv Educ Rev 72(2):177–205

Krantz DH, Luce D, Suppes P, Tversky A (1971) Foundations of measurement. Additive and polynomial representations. Academic Press, New York

Mill JS (1859) On liberty. John W. Parker and Son/Oxford University Press, Oxford

Nehring K, Puppe C (1996) Continuous extensions of an order on a set to the power set. J Econ Theory 68:456–479

Pattanaik P, Xu Y (1990) On ranking opportunity sets in terms of freedom of choice. Recherches Economiques de Louvain 56:383–390

Roemer J (1988) Free to lose: an introduction to Marxist economic philosophy. Harvard University Press, Cambridge

Sen A (1988) Freedom of choice: concept and content. Eur Econ Rev 32:269–294

Sen A (1991) Welfare, preference and freedom. J Econom 50:15–29

Smith A (1776) An inquiry into the Nature and Causes of the Wealth of Nations. In: Cannan E (ed) Methuen and Co., Ltd., London