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On sequences with prescribed metric discrepancy behavior

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Abstract An important result of H. Weyl states that for every sequence $(a_n)_{n\geq 1}$ of distinct positive integers the sequence of fractional parts of $(a_n\alpha)_{n\geq 1}$ is uniformly distributed modulo one for almost all α. However, in general it is a very hard problem to calculate the precise order of convergence of the discrepancy D_N of $(\{a_n\alpha\})_{n\geq 1}$ for almost all α. By a result of R. C. Baker this discrepancy always satisfies $ND_N = \mathcal{O}(N^{\frac{1}{2}+\varepsilon})$ for almost all α and all $\varepsilon > 0$. In the present note for arbitrary $\gamma \in (0, \frac{1}{2}]$ we construct a sequence $(a_n)_{n\geq 1}$ such that for almost all α we have $ND_N = \mathcal{O}(N^{\gamma})$ and $ND_N = \Omega(N^{\gamma-\varepsilon})$ for all $\varepsilon > 0$, thereby proving that any prescribed metric discrepancy behavior within the admissible range can actually be realized.

Keywords Discrepancy theory · Metric number theory

Mathematics Subject Classification 11K38 · 11J83

1 Introduction

Weyl [12] proved that for every sequence $(a_n)_{n\geq 1}$ of distinct positive integers the sequence $(\{a_n\alpha\})_{n\geq 1}$ is uniformly distributed modulo one for almost all reals α. Here, and in the sequel, $\{\cdot\}$ denotes the fractional part function. The speed of convergence

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towards the uniform distribution is measured in terms of the discrepancy, which—for an arbitrary sequence $(x_n)_{n\geq 1}$ of points in $[0,1)$—is defined by

$$D_N = D_N(x_1,\ldots,x_N) = \sup_{0\leq a<b\leq 1}\left|\frac{\mathcal{A}_N([a,b))}{N} - (b-a)\right|,$$

where $\mathcal{A}_N([a,b)) := \#\{1\leq n\leq N \mid x_n \in [a,b)\}$. For a given sequence $(a_n)_{n\geq 1}$ it is usually a very hard and challenging problem to give sharp estimates for the discrepancy D_N of $(\{a_n\alpha\})_{n\geq 1}$ valid for almost all α. For general background on uniform distribution theory and discrepancy theory see for example the monographs [6,9].

A famous result of Baker [3] states that for any sequence $(a_n)_{n\geq 1}$ of distinct positive integers for the discrepancy D_N of $(\{a_n\alpha\})_{n\geq 1}$ we have

$$N D_N = \mathcal{O}(N^{\frac{1}{2}}(\log N)^{\frac{3}{2}+\varepsilon}) \quad \text{as } N\to\infty \tag{1}$$

for almost all α and for all $\varepsilon > 0$.

Note that (1) is a general upper bound which holds for *all* sequences $(a_n)_{n\geq 1}$; however, for some specific sequences the precise typical order of decay of the discrepancy of $(\{a_n\alpha\})_{n\geq 1}$ can differ significantly from the upper bound in (1). The fact that (1) is essentially optimal (apart from logarithmic factors) as a general result covering all possible sequences can for example be seen by considering so-called lacunary sequences $(a_n)_{n\geq 1}$, i.e., sequences for which $\frac{a_{n+1}}{a_n}\geq 1+\delta$ for a fixed $\delta > 0$ and all n large enough. In this case for D_N we have

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N\to\infty}\frac{N D_N}{\sqrt{2N\log\log N}} \leq c_\delta$$

for almost all α (see [10]), which shows that the exponent $1/2$ of N on the right-hand side of (1) cannot be reduced for this type of sequence. For more information concerning possible improvements of the logarithmic factor in (1), see [5].

Quite recently in [2] it was shown that also for a large class of sequences with polynomial growth behavior Baker's result is essentially best possible. For example, the following result was shown there: let $f\in\mathbb{Z}[x]$ be a polynomial of degree larger or equal to 2. Then for the discrepancy D_N of $(\{f(n)\alpha\})_{n\geq 1}$ for almost all α and for all $\varepsilon > 0$ we have

$$N D_N = \Omega(N^{\frac{1}{2}-\varepsilon}).$$

On the other hand there is the classical example of the Kronecker sequence, i.e., $a_n = n$, which shows that the actual metric discrepancy behavior of $(\{a_n\alpha\})_{n\geq 1}$ can differ vastly from the general upper bound in (1). Namely, for the discrepancy of the sequence $(\{n\alpha\})_{n\geq 1}$ for almost all α and for all $\varepsilon > 0$ we have

$$N D_N = \mathcal{O}(\log N(\log\log N)^{1+\varepsilon}), \tag{2}$$

which follows from classical results of Khintchine in the metric theory of continued fractions (for even more precise results, see [11]). The estimate (2) of course also holds for $a_n = f(n)$ with $f \in \mathbb{Z}[x]$ of degree 1. In [2] further examples for $(a_n)_{n\geq 1}$ were given, where $(a_n)_{n\geq 1}$ has polynomial growth behavior of arbitrary degree, such that for the discrepancy of $(\{a_n\alpha\})_{n\geq 1}$ we have

$$ND_N = \mathcal{O}((\log N)^{2+\varepsilon})$$

for almost all α and for all $\varepsilon > 0$; see there for more details.

These results may seduce to the hypothesis that for all choices of $(a_n)_{n\geq 1}$ for the discrepancy of $(\{a_n\alpha\})_{n\geq 1}$ for almost all α we either have

$$ND_N = \mathcal{O}(N^{\varepsilon}) \tag{3}$$

or

$$ND_N = \Omega(N^{\frac{1}{2}-\varepsilon}). \tag{4}$$

This hypothesis, however, is wrong as was shown in [1]: let $(a_n)_{n\geq 1}$ be the sequence of those positive integers with an even sum of digits in base 2, sorted in increasing order; that is $(a_n)_{n\geq 1} = (3, 5, 6, 9, 10, \dots)$. Then for the discrepancy of $(\{a_n\alpha\})_{n\geq 1}$ for almost all α we have

$$ND_N = \mathcal{O}(N^{\kappa+\varepsilon})$$

and

$$ND_N = \Omega(N^{\kappa-\varepsilon})$$

for all $\varepsilon > 0$, where κ is a constant with $\kappa \approx 0.404$. Interestingly, the precise value of κ is unknown; see [8] for the background.

The aim of the present paper is to show that the example above is not a singular counter-example, but that indeed "everything" between (3) and (4) is possible. More precisely, we will show the following theorem.

Theorem 1 *Let $0 < \gamma \leq \frac{1}{2}$. Then there exists a strictly increasing sequence $(a_n)_{n\geq 1}$ of positive integers such that for the discrepancy of the sequence $(\{a_n\alpha\})_{n\geq 1}$ for almost all α we have*

$$ND_N = \mathcal{O}(N^{\gamma})$$

and

$$ND_N = \Omega(N^{\gamma-\varepsilon})$$

for all $\varepsilon > 0$.

2 Proof of the theorem

For the proof we need an auxiliary result which easily follows from classical work of Behnke [4].

Lemma 1 *Let* $(e_k)_{k\geq 1}$ *be a strictly increasing sequence of positive integers. Let* $\varepsilon > 0$. *Then for almost all* α *there is a constant* $K(\alpha, \varepsilon) > 0$ *such that for all* $r \in \mathbb{N}$ *there exist* $M_r \leq e_r$ *such that for the discrepancy of the sequence* $(\{n^2\alpha\})_{n\geq 1}$ *we have*

$$M_r D_{M_r} \geq K(\alpha, \varepsilon) \sqrt{\frac{e_r}{(\log e_r)^{1+\varepsilon}}}.$$

Proof For $\alpha \in \mathbb{R}$ let $a_k(\alpha)$ denote the kth continued fraction coefficient in the continued fraction expansion of α. Then it is well-known that for almost all α we have $a_k(\alpha) = \mathcal{O}\left(k^{1+\varepsilon}\right)$ for all $\varepsilon > 0$. Let $\varepsilon > 0$ be given and let α and $c(\alpha, \varepsilon)$ be such that

$$a_k(\alpha) \leq c(\alpha, \varepsilon) k^{1+\varepsilon} \tag{5}$$

for all $k \geq 1$.

Let q_l the lth best approximation denominator of α. Then

$$q_{l+1} \leq (c(\alpha, \varepsilon) l^{1+\varepsilon} + 1) q_l. \tag{6}$$

Since $q_l \geq 2^{\frac{l}{2}}$ in any case, we have $l \leq \frac{2 \log q_l}{\log 2}$, and we obtain

$$q_{l+1} \leq c_1(\alpha, \varepsilon) q_l (\log q_l)^{1+\varepsilon}, \tag{7}$$

for an appropriate constant $c_1(\alpha, \varepsilon)$. In [4] it was shown in Satz XVII that for every real α we have

$$\left|\sum_{n=1}^{N} e^{2\pi i n^2 \alpha}\right| = \Omega(N^{\frac{1}{2}}).$$

Indeed, if we follow the proof of this theorem we find that even the following was shown: for every α and for every best approximation denominator q_l of α there exists an $Y_l < \sqrt{q_l}$ such that $\left|\sum_{n=1}^{Y_l} e^{2\pi i n^2 \alpha}\right| \geq c_{\mathrm{abs}} \sqrt{q_l}$. Here c_{abs} is a positive absolute constant (not depending on α).

Let now $r \in \mathbb{N}$ be given and let l be such that $q_l \leq e_r < q_{l+1}$, and let $M_r := Y_l$ from above. Then by (6) and (7) we obtain, for an appropriate constant $c_2(\alpha, \varepsilon)$,

$$\begin{aligned} \left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right| &\geq c_{\text{abs}} \sqrt{q_l} \\ &\geq c_2(\alpha, \varepsilon) \sqrt{\frac{q_{l+1}}{(\log q_l)^{1+\varepsilon}}} \\ &\geq c_2(\alpha, \varepsilon) \sqrt{\frac{e_l}{(\log e_l)^{1+\varepsilon}}}. \end{aligned}$$

By the fact that (see Chapter 2, Corollary 5.1 of [9])

$$M_r D_{M_r} \geq \frac{1}{4} \left| \sum_{n=1}^{M_r} e^{2\pi i n^2 \alpha} \right|,$$

which is a special case of Koksma's inequality, the result follows. □

Now we are ready to prove the main theorem.

Proof of Theorem 1 Let $(m_j)_{j \geq 1}$ and $(e_j)_{j \geq 1}$ be two strictly increasing sequences of positive integers, which will be determined later. We will consider the following strictly increasing sequence of positive integers, which will be our sequence $(a_n)_{n \geq 1}$:

$$\begin{aligned} &1, 2, 3, \ldots, \underbrace{m_1}_{=:A_1}, \\ &A_1 + 1^2, A_1 + 2^2, A_1 + 3^2, A_1 + 4^2, \ldots, \underbrace{A_1 + {e_1}^2}_{:=B_1}, \\ &B_1 + 1, B_1 + 2, B_1 + 3, \ldots, \underbrace{B_1 + m_2}_{=:A_2}, \\ &A_2 + 1^2, A_2 + 2^2, A_2 + 3^2, A_2 + 4^2, \ldots, \underbrace{A_2 + {e_2}^2}_{=:B_2}, \\ &B_2 + 1, B_2 + 2, B_2 + 3, \ldots, \underbrace{B_2 + m_3}_{=:A_3}, \\ &A_3 + 1^2, A_3 + 2^2, A_3 + 3^2, A_3 + 4^2, \ldots, \underbrace{A_3 + {e_3}^2}_{=:B_3}, \\ &\vdots \end{aligned}$$

Furthermore, let

$$F_s := \sum_{i=1}^{s} m_i + \sum_{i=1}^{s-1} e_i \quad \text{and} \quad E_s := \sum_{i=1}^{s} m_i + \sum_{i=1}^{s} e_i.$$

The sequence $(a_n)_{n\geq 1}$ is constructed in such a way that it contains sections where it grows like $(n)_{n\geq 1}$ as well as sections where it grows like $(n^2)_{n\geq 1}$. By this construction we exploit both the strong upper bounds for the discrepancy of $(\{n\alpha\})_{n\geq 1}$ and the strong lower bounds for the discrepancy of $(\{n^2\alpha\})_{n\geq 1}$, in an appropriately balanced way, in order to obtain the desired discrepancy behavior of the sequence $(\{a_n\alpha\})_{n\geq 1}$. In our argument we will repeatedly make use of the fact that

$$D_N(x_1, \ldots, x_N) = D_N(\{x_1 + \beta\}, \ldots, \{x_N + \beta\}) \tag{8}$$

for arbitrary $x_1, \ldots, x_N \in [0, 1]$ and $\beta \in \mathbb{R}$, which allows us to transfer the discrepancy bounds for $(\{n\alpha\})_{n\geq 1}$ and $(\{n^2\alpha\})_{n\geq 1}$ directly to the shifted sequences $(\{(M + n)\alpha\})_{n\geq 1}$ and $(\{(M + n^2)\alpha\})_{n\geq 1}$ for some integer M.

Let α be such that it satisfies (5) with $\varepsilon = \frac{1}{2}$. Then it is also well-known (see for example [9]) that for the discrepancy D_N of the sequence $(\{n\alpha\})_{n\geq 1}$ we have

$$N D_N \leq \bar{c}_1(\alpha) (\log N)^{\frac{3}{2}} \tag{9}$$

for all $N \geq 2$.

By the above mentioned general result of Baker, that is by (1), we know that for almost all α for the discrepancy D_N of the sequence $(\{n^2\alpha\})_{n\geq 1}$ we have

$$N D_N \leq c_3(\alpha, \varepsilon) N^{\frac{1}{2}} (\log N)^{\frac{3}{2}+\varepsilon}$$

for all $\varepsilon > 0$ and for all $N \geq 2$, for an appropriate constant $c_3(\alpha, \varepsilon)$. Actually an even slightly sharper estimate was given for the special case of the sequence $(\{n^2\alpha\})_{n\geq 1}$ by Fiedler et al. [7], who proved that

$$N D_N \leq c_4(\alpha, \varepsilon) N^{\frac{1}{2}} (\log N)^{\frac{1}{4}+\varepsilon} \tag{10}$$

for almost all α and for all $\varepsilon > 0$ and all $N \geq 2$.

Assume that α satisfies (10) with $\varepsilon = \frac{1}{8}$. Then

$$N D_N \leq \bar{c}_2(\alpha) N^{\frac{1}{2}} (\log N)^{\frac{3}{8}} \tag{11}$$

for all $N \geq 2$. Now for such α and for arbitrary N we consider the discrepancy D_N of the sequence $(\{a_n\alpha\})_{n\geq 1}$.

Case 1 Let $N = F_l$ for some l. Then $N D_N \leq E_{l-1} D_{E_{l-1}} + (N - E_{l-1}) D_{E_{l-1}, F_l}$, where $D_{x,y}$ denotes the discrepancy of the point set $(\{a_n\alpha\})_{n=x+1,x+2,\ldots,y}$. Hence by (8), (9) and by the trivial estimate $D_{B_{l-1}} \leq 1$ we have

$$\begin{aligned} ND_N &\le E_{l-1} + \overline{c}_1(\alpha)(\log m_l)^{\frac{3}{2}} \\ &\le 2(\log m_l)^2 \\ &\le 2(\log N)^2 \end{aligned}$$

for all l large enough, provided that [condition (i)] m_l is chosen such that $(\log m_l)^2 \ge E_{l-1}$.

Case 2 Let $F_l < N \le E_l$ for some l. Then by Case 1 and by (8) and (11) we have for l large enough that

$$\begin{aligned} ND_N &\le F_l D_{F_l} + (N - F_l) D_{F_l,N} \\ &\le 2(\log F_l)^2 + \overline{c}_2(\alpha)(N - F_l)^{\frac{1}{2}} (\log(N - F_l))^{\frac{3}{8}}. \end{aligned}$$

Note that $0 < N - F_l < e_l$.

We choose [condition (ii)]

$$e_l := \left\lceil \frac{{F_l}^{2\gamma}}{\log\left({F_l}^{2\gamma}\right)} \right\rceil. \tag{12}$$

Note that conditions (i) and (ii) do not depend on α. Now assume that l is so large that $2(\log F_l)^2 < \frac{{F_l}^{\gamma}}{2}$. Then

$$\frac{F_l^{\gamma}}{2} \le 2(\log F_l)^2 + (e_l \log e_l)^{\frac{1}{2}} \le 2F_l^{\gamma}$$

and (note that $\gamma \le \frac{1}{2}$)

$$F_l < N \le E_l = F_l + e_l \le 2F_l. \tag{13}$$

Hence

$$\begin{aligned} ND_N &\le \max(1, \overline{c}_2(\alpha))\, 2F_l^{\gamma} \\ &\le \max(1, \overline{c}_2(\alpha))\, 2N^{\gamma}. \end{aligned}$$

Case 3 Let $E_l < N < F_{l+1}$ for some l. Then by Case 2 and by (8) and (9) we have

$$\begin{aligned} ND_N &\le E_l D_{E_l} + (N - E_l) D_{E_l,N} \\ &\le 2\max(1, \overline{c}_2(\alpha)) E_l^{\gamma} + \overline{c}_1(\alpha)(\log(N - E_l))^2 \\ &\le 3\max(1, \overline{c}_2(\alpha)) N^{\gamma} \end{aligned}$$

for N large enough.

It remains to show that for every $\varepsilon > 0$ we have $ND_N \ge N^{\gamma-\varepsilon}$ for infinitely many N. Let l be given and let $M_l \le e_l$ with the properties given in Lemma 1. Let $N := F_l + M_l$. Then by Lemma 1, Case 1, (8), (12) and (13) for l large enough we have

$$\begin{aligned} N D_N &\geq M_l D_{F_l,N} - F_l D_{F_l} \\ &\geq K(\alpha,\varepsilon)\sqrt{\frac{e_l}{(\log e_l)^{1+\varepsilon}}} - 2(\log m_l)^2 \\ &\geq \frac{F_l^\gamma}{(\log F_l)^3} \\ &\geq N^{\gamma-\varepsilon}. \end{aligned}$$

This proves the theorem. □

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References

1. Aistleitner, C., Hofer, R., Larcher, G.: On parametric Thue-Morse sequences and lacunary trigonometric products. arXiv:1502.06738
2. Aistleitner, C., Larcher, G.: Metric results on the discrepancy of sequences $(a_n\alpha)_{n\geq 1}$ modulo one for integer sequences $(a_n)_{n\geq 1}$ of polynomial growth. Mathematika (To appear). arXiv:1507.00207
3. Baker, R.C.: Metric number theory and the large sieve. J. Lond. Math. Soc. (2) **24**(1), 34–40 (1981)
4. Behnke, H.: Zur Theorie der diophantischen Approximationen. Abh. Math. Sem. Hambg. **3**, 261–318 (1924)
5. Berkes, I., Philipp, W.: The size of trigonometric and Walsh series and uniform distribution mod 1. J. Lond. Math. Soc. (2) **50**(3), 454–464 (1994)
6. Drmota, M., Tichy, R.F.: Sequences, discrepancies and applications. In: Lecture Notes in Mathematics, vol. 1651. Springer, Berlin (1997)
7. Fiedler, H., Jurkat, W., Körner, O.: Asymptotic expansions of finite theta series. Acta Arith. **32**, 129–146 (1977)
8. Fouvry, E., Mauduit, C.: Sommes des chiffres et nombres presque premiers. Math. Ann. **305**(3), 571–599 (1996)
9. Kuipers, L., Niederreiter, H.: Uniform Distribution of Sequences. Wiley, New York (1974)
10. Philipp, W.: Limit theorems for lacunary series and uniform distribution mod 1. Acta Arith. **26**(3), 241–251 (1974/1975)
11. Schoissengeier, J.: A metrical result on the discrepancy of $(n\alpha)$. Glasg. Math. J. **40**(3), 393–425 (1998)
12. Weyl, H.: Über die Gleichverteilung von Zahlen modulo Eins. Math. Ann. **77**, 313–352 (1916)