



Input-to-state stability of Lur'e systems

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Abstract An input-to-state stability theory, which subsumes results of circle criterion type, is developed in the context of continuous-time Lur'e systems. The approach developed is inspired by the complexified Aizerman conjecture.

Keywords Absolute stability · Circle criterion · Complexified Aizerman conjecture · Input-to-state stability · Lur'e systems · Stability radius

1 Introduction

We will be concerned with controlled Lur'e systems of the form

$$\dot{x} = Ax + Bf(Cx) + B_e v, \quad (1.1)$$

where A , B , B_e and C are matrices of appropriate formats, f is a locally Lipschitz nonlinearity and v denotes the input or forcing. Obviously, system (1.1) can be thought of as a feedback system, namely the linear controlled and observed system

$$\dot{x} = Ax + Bu + B_e v, \quad y = Cx$$

with nonlinear output feedback $u = f(y)$.

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Lur'e systems are a common and important class of nonlinear systems and there is a large body of work on the absolute stability theory of these systems: see, for example [6, 7, 16, 19, 27, 28]. Traditionally, Lyapunov approaches to the stability theory of systems of the form (1.1) consider unforced Lur'e systems (i.e., $v = 0$ in (1.1)), whilst Lur'e systems with forcing (usually acting through B , that is, $B_e = B$) have been studied using the input–output framework initiated by Sandberg and Zames in the 1960s, see, for example [27]. More recently, forced Lur'e systems have been analysed in the context of input-to-state stability (ISS) theory, see [1, 2, 12, 13] (and [22] for discrete-time systems). In [1], an ISS result is obtained for Lur'e systems (1.1) under the assumptions that $B_e = B$, the underlying linear system has the positive real property and the nonlinearity (which may have superlinear growth) satisfies a suitable cone condition. Partial extensions of the classical Popov and circle criteria to an ISS setting can be found in [2] and [12, 13], respectively. The concept of ISS (for a general controlled nonlinear system) appears first in [23] published in 1989. The theory of ISS which has been subsequently developed, provides a natural stability framework for nonlinear systems with inputs, merging, in a sense, Lyapunov and input–output approaches to stability (the latter initiated by Sandberg and Zames in the 1960s). We refer the reader to [3, 25] for overviews of ISS theory.

In this paper, we derive an ISS result which is reminiscent of the complexified Aizerman conjecture [9, 10] (see [7, 17, 18, 27] for details on the original *real* Aizerman conjecture). More precisely, let K be a matrix of appropriate format and assume that every *complex* matrix in the ball $\{F : \|F - K\| < r\}$, where $r > 0$, is a stabilizing output feedback gain for the linear system (A, B, C) . The main result of the paper (Theorem 3.2) guarantees that, under this hypothesis, the nonlinear system (1.1) is ISS for every locally Lipschitz nonlinearity f for which there exists a \mathcal{K}_∞ function α such that

$$\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|) \quad \text{for all } \xi. \quad (1.2)$$

As a corollary (see Corollary 3.10), we derive a clear-cut ISS version of the circle criterion: it is shown that, under conditions very similar to those of the circle criterion, the Lur'e system (1.1) is ISS. In particular, Corollary 3.10 contains earlier ISS versions [12, 13] of the circle criterion as special cases. Moreover, a further corollary (Corollary 3.11) shows that the conditions of the usual textbook version of the circle criterion for global asymptotic stability (see [7, 16, 27]) are actually sufficient for ISS.

Finally, we mention that if A is not Hurwitz and f is bounded (for example, if f is of “saturation” type), then the nonlinearity is not “powerful” enough to counteract large (but bounded) inputs (at least if $\text{im } B \subset \text{im } B_e$) and the Lur'e system (1.1) is not ISS (see [20] and Proposition 3.4 in the current paper). Correspondingly, it is not difficult to show that if A is not Hurwitz, f is bounded and every complex output feedback gain in the ball $\{F : \|F - K\| < r\}$ is stabilizing, then there does not exist $\alpha \in \mathcal{K}_\infty$ such that (1.2) holds (see Proposition 3.4).

1.1 Notation and terminology

As usual, \mathbb{R} and \mathbb{C} denote the fields of real and complex numbers, respectively. We set $\mathbb{R}_+ := [0, \infty)$.

In the following, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. For $K \in \mathbb{C}^{m \times p}$ and $r > 0$, we define the open ball in $\mathbb{F}^{m \times p}$ with centre K and radius r :

$$\mathbb{B}_{\mathbb{F}}(K, r) := \{M \in \mathbb{F}^{m \times p} : \|M - K\| < r\}.$$

For $M \in \mathbb{C}^{n \times m}$, let M^* denote the Hermitian transposition of M (transposition if M is real). The open right-half of the complex plane \mathbb{C} is denoted by \mathbb{C}_+ . The Hardy space of all bounded holomorphic functions $\mathbb{C}_+ \rightarrow \mathbb{C}^{p \times m}$ is denoted by $H^\infty(\mathbb{C}^{p \times m})$. The norm of a function $H \in H^\infty(\mathbb{C}^{p \times m})$ is given by

$$\|H\|_{H^\infty} = \sup_{s \in \mathbb{C}_+} \|H(s)\|,$$

where $\|\cdot\|$ is the operator norm on $\mathbb{C}^{p \times m}$ induced by the 2-norms on \mathbb{C}^m and \mathbb{C}^p .

Let $A \in \mathbb{C}^{n \times n}$ be Hurwitz (that is, all eigenvalues of A have negative real parts), let $B \in \mathbb{C}^{n \times m}$ and $C \in \mathbb{C}^{p \times n}$. The structured stability radius of A with respect to the perturbation structure given by B and C is defined by

$$r_{\mathbb{F}}(A; B, C) := \inf\{\|\Delta\| : \Delta \in \mathbb{F}^{m \times p} \text{ and } A + B\Delta C \text{ is not Hurwitz}\}.$$

The number $r_{\mathbb{C}}(A; B, C)$ is said to be the complex stability radius, whilst $r_{\mathbb{R}}(A; B, C)$ is called the real stability radius, see [8, 10]. Note that, even if A, B and C are real, the perturbation Δ in the definition of $r_{\mathbb{C}}(A; B, C)$ is in $\mathbb{C}^{m \times p}$.

Finally, we recall the definitions of certain classes of comparison functions. Let \mathcal{K} denote the set of all continuous functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\varphi(0) = 0$ and φ is strictly increasing. Moreover,

$$\mathcal{K}_\infty := \{\varphi \in \mathcal{K} : \varphi(s) \rightarrow \infty \text{ as } s \rightarrow \infty\}.$$

We denote by \mathcal{KL} the set of functions $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the following properties: $\psi(\cdot, t) \in \mathcal{K}$ for every $t \geq 0$, and $\psi(s, \cdot)$ is non-increasing with $\lim_{t \rightarrow \infty} \psi(s, t) = 0$ for every $s \geq 0$. Note that, following [24–26], continuity is not imposed in the above definition of a \mathcal{KL} -function. It is known that a discontinuous \mathcal{KL} -function can be bounded from above by a continuous \mathcal{KL} -function, see [24, Proposition 7]. For more details on comparison functions, we refer the reader to [15].

2 Preliminaries

Set $\Sigma := \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$. With a triple $(A, B, C) \in \Sigma$, we associate the following controlled and observed linear system

$$\dot{x} = Ax + Bu, \quad y = Cx. \tag{2.1}$$

The transfer function (matrix) G of (2.1) (or of the triple (A, B, C)) is given by

$$G(s) = C(sI - A)^{-1}B.$$

The closed-loop system obtained by application of linear feedback of the form $u = Ky + v$ to (2.1), where $K \in \mathbb{R}^{m \times p}$ and $v \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$, is described by the triple $(A + BKC, B, C) \in \Sigma$. The associated transfer function is

$$G^K(s) := C(sI - A - BKC)^{-1}B = G(s)(I - KG(s))^{-1}.$$

We denote the set of stabilizing output feedback matrices for (A, B, C) by $\mathbb{S}_{\mathbb{F}}(A, B, C)$, that is,

$$\mathbb{S}_{\mathbb{F}}(A, B, C) := \{K \in \mathbb{F}^{m \times p} : A + BKC \text{ is Hurwitz}\},$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and we will be speaking of real or complex stabilizing output feedback matrices, respectively. Moreover, defining

$$\mathbb{S}_{\mathbb{F}}(G) := \{K \in \mathbb{F}^{m \times p} : G^K \in H^\infty(\mathbb{C}^{p \times m})\},$$

we have that

$$\mathbb{S}_{\mathbb{F}}(A, B, C) \subseteq \mathbb{S}_{\mathbb{F}}(G). \tag{2.2}$$

If $\mathbb{S}_{\mathbb{F}}(A, B, C) \neq \emptyset$, then (A, B, C) is stabilizable and detectable and equality holds in (2.2).

The following lemma provides some simple properties of linear output feedback.

Lemma 2.1 *Let $(A, B, C) \in \Sigma$ with transfer function G , let $K \in \mathbb{C}^{m \times p}$ and let $r > 0$.*

- (a) $\mathbb{S}_{\mathbb{C}}(G) - K = \mathbb{S}_{\mathbb{C}}(G^K)$.
- (b) $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ if, and only if, $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G^K)$.
- (c) $(G^K)^L = G^{K+L}$ for all $L \in \mathbb{C}^{m \times p}$.
- (d) $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$ if, and only if, $\|G^K\|_{H^\infty} \leq 1/r$.

Assume that, in Lemma 2.1, the matrix K is real, that is, $K \in \mathbb{R}^{m \times p}$. Then statements (a) and (b) and the sufficiency part of statement (d) remain valid if $\mathbb{B}_{\mathbb{C}}$ and $\mathbb{S}_{\mathbb{C}}$ are replaced by $\mathbb{B}_{\mathbb{R}}$ and $\mathbb{S}_{\mathbb{R}}$, respectively. However, the condition $\mathbb{B}_{\mathbb{R}}(K, r) \subseteq \mathbb{S}_{\mathbb{R}}(G)$ does not imply that $\|G^K\|_{H^\infty} \leq 1/r$.

Proof of Lemma 2.1 The proofs of statements (a)–(c) are straightforward and are therefore omitted.

We proceed to prove statement (d). Assuming that $\|G^K\|_{H^\infty} \leq 1/r$, it is clear that $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(G^K)$ (by the “small-gain theorem”). Hence, by statement (b), $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G)$.

We prove the reverse implication by contraposition. To this end, assume $\|G^K\|_{H^\infty} > 1/r$. We have to show that there exists $L \in \mathbb{B}_{\mathbb{C}}(K, r)$ such that $L \notin \mathbb{S}_{\mathbb{C}}(G)$. By assumption, $\|G^K(z)\| > 1/r$ for some $z \in \mathbb{C}_+$. As is well known from matrix theory, there exists $M \in \mathbb{C}^{m \times p}$ with $\|M\| = 1/\|G^K(z)\| < r$ and $\det(I - MG^K(z)) = 0$. Now

$$M(G^K)^M = MG(I - MG^K)^{-1} = (I - MG^K)^{-1} - I,$$

and we conclude that $M(G^K)^M$ has a pole at z . Setting $L := K + M$ and using statement (c), we see that $G^L = G^{K+M} = (G^K)^M$ has a pole at z , showing that $L \notin \mathbb{S}_{\mathbb{C}}(G)$. Obviously, $L \in \mathbb{B}_{\mathbb{C}}(K, r)$, completing the proof of statement (d). \square

Next we state a version of the well-known bounded real lemma which is convenient for our purposes.

Lemma 2.2 *Let $(A, B, C) \in \Sigma$. Assume that A is Hurwitz and that the transfer function G of (A, B, C) satisfies $\|G\|_{H^\infty} \leq 1$. Then there exist a positive semi-definite matrix $P = P^* \in \mathbb{R}^{n \times n}$ and a matrix $L \in \mathbb{R}^{m \times n}$ such that*

$$A^*P + PA = -C^*C - L^*L \text{ and } PB = -L^*.$$

Proof By elementary stability radius theory, $r_{\mathbb{C}}(A; B, C) = 1/\|G\|_{H^\infty} \geq 1$, see [8, 10]. Hence, by [8, Theorem 3.3], there exists a matrix $Q = Q^* \in \mathbb{R}^{n \times n}$ which solves the algebraic Riccati equation

$$A^*Q + QA - C^*C - QBB^*Q = 0.$$

Setting $P := -Q$ and $L := -B^*P$, it follows that P solves the Lyapunov matrix equation

$$A^*P + PA = -C^*C - L^*L. \tag{2.3}$$

Since A is Hurwitz, (2.3) has a unique solution which is given by

$$P = \int_0^\infty e^{A^*t}(C^*C + L^*L)e^{At} dt,$$

see, for example [10, Corollary 3.3.46]. Obviously, the matrix $C^*C + L^*L$ is positive semi-definite and it follows that P is positive semi-definite, completing the proof. \square

In the following, we will consider linear systems of the form

$$\dot{x} = Ax + Bu + B_e v, \quad y = Cx \tag{2.4}$$

where

$$(A, B, B_e, C) \in \Sigma_e := \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m_e} \times \mathbb{R}^{p \times n}$$

It is convenient to define the behaviour $\mathcal{B}(A, B, B_e, C)$ of (2.4) (or of the quadruple (A, B, B_e, C)) by

$$\mathcal{B}(A, B, B_e, C) := \{(v, u, x, y) \in \mathcal{T} : (v, u, x, y) \text{ satisfies (2.4)}\},$$

where

$$\mathcal{T} := L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{m_e}) \times L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m) \times W^{1,1}_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times C(\mathbb{R}_+, \mathbb{R}^p).$$

Obviously, in the above definition of $\mathcal{B}(A, B, B_e, C)$, the solution x of the differential equation in (2.4) has to be understood in the sense of Carathéodory. A triple (v, u, x, y) is in $\mathcal{B}(A, B, B_e, C)$ if, and only if,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}(Bu(s) + B_e v(s))ds \quad \forall t \geq 0$$

and $y = Cx$.

We now use the bounded real lemma to obtain a quadratic form useful in stability analysis.

Proposition 2.3 *Let $(A, B, B_e, C) \in \Sigma_e$ and assume that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$, where $K \in \mathbb{R}^{m \times p}$ and $r > 0$. Then there exists positive semi-definite $P = P^* \in \mathbb{R}^{n \times n}$ with the following property: for every $\alpha \in \mathcal{K}_{\infty}$, there exists $\beta \in \mathcal{K}_{\infty}$, such that, for every $(v, u, x, y) \in \mathcal{B}(A, B, B_e, C)$, the function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by $V(\zeta) := \langle P\zeta, \zeta \rangle$ satisfies*

$$\frac{d}{dt} V(x(t)) \leq -r^2 \|y(t)\|^2 + \|u(t) - Ky(t)\|^2 + \|x(t)\|\alpha(\|x(t)\|) + \beta(\|v(t)\|)$$

for almost every $t \geq 0$.

For the proof of this result, the following simple lemma will be useful.

Lemma 2.4 *If $\alpha \in \mathcal{K}_{\infty}$, then there exists $\beta \in \mathcal{K}_{\infty}$ such that*

$$s_1 s_2 \leq s_1 \alpha(s_1) + \beta(s_2) \quad \forall s_1, s_2 \geq 0.$$

Proof If $s_2 \leq \alpha(s_1)$, then $s_1 s_2 \leq s_1 \alpha(s_1)$; and if $s_2 > \alpha(s_1)$, then $s_1 < \alpha^{-1}(s_2)$, so that $s_1 s_2 < s_2 \alpha^{-1}(s_2)$. Hence $\beta(s_2) := s_2 \alpha^{-1}(s_2)$ satisfies all the requirements. \square

Proof of Proposition 2.3 Set $A_K := A + BK C$, and consider the system (A_K, rB, C) , the transfer function of which is rG^K , where $G(s) = C(sI - A)^{-1}B$. By hypothesis,

$$\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C) = \mathbb{S}_{\mathbb{C}}(G).$$

Hence, A_K is Hurwitz and, furthermore, it follows from statement (d) of Lemma 2.1 that, $r \|G^K\|_{H_{\infty}} \leq 1$. An application of Lemma 2.2 to the system (A_K, rB, C) shows that there exist a positive semi-definite matrix $Q = Q^* \in \mathbb{R}^{n \times n}$ and a matrix $L \in \mathbb{R}^{m \times n}$ such that

$$A_K^* Q + Q A_K = -C^* C - L^* L \quad \text{and} \quad r Q B = -L^*. \tag{2.5}$$

Define the quadratic form U by $U(\zeta) := \langle Q\zeta, \zeta \rangle$ for all $\zeta \in \mathbb{R}^n$. Let $(v, u, x, y) \in \mathcal{B}(A, B, B_e, C)$ be arbitrary. Writing $w := u - Ky$, then, trivially, the quadruple $(v, w, x, y) \in \mathcal{B}(A_K, B, B_e, C)$ and we obtain that, for almost every $t \geq 0$,

$$\begin{aligned} \frac{d}{dt}U(x(t)) &= 2 \langle Qx(t), A_Kx(t) + Bw(t) + B_e v(t) \rangle \\ &= \langle (A_K^* Q + Q A_K)x(t), x(t) \rangle + 2 \langle x(t), Q B w(t) \rangle \\ &\quad + 2 \langle Qx(t), B_e v(t) \rangle. \end{aligned}$$

Setting $c := 2\|Q\|\|B_e\|$ and invoking (2.5), it follows that, for almost every $t \geq 0$,

$$\begin{aligned} \frac{d}{dt}U(x(t)) &\leq -\|Cx(t)\|^2 - \|Lx(t)\|^2 - \frac{2}{r} \langle Lx(t), w(t) \rangle + c\|x(t)\|\|v(t)\| \\ &= -\|y(t)\|^2 - \left\| Lx(t) + \frac{1}{r}w(t) \right\|^2 + \frac{1}{r^2} \|w(t)\|^2 + c\|x(t)\|\|v(t)\|. \end{aligned}$$

By Lemma 2.4, for a given $\alpha \in \mathcal{K}_\infty$, there exists $\beta \in \mathcal{K}_\infty$ such that

$$r^2 c s_1 s_2 \leq s_1 \alpha(s_1) + \beta(s_2) \quad \forall s_1, s_2 \geq 0.$$

Consequently, for almost every $t \geq 0$,

$$\frac{d}{dt}U(x(t)) \leq -\|y(t)\|^2 + \frac{1}{r^2} \left(\|u(t) - Ky(t)\|^2 + \|x(t)\| \alpha(\|x(t)\|) + \beta(\|v(t)\|) \right).$$

The claim now follows with $P := r^2 Q$. □

The next proposition (inspired by [1]) guarantees the existence of another quadratic form which will be useful in the ISS analysis of Lur'e systems

Proposition 2.5 *Let $(A, B, B_e, C) \in \Sigma_e$ and assume that the pair (A, C) is detectable. Then there exists a positive-definite matrix $P = P^* \in \mathbb{R}^{n \times n}$ and $\delta > 0$ such that, for every $(v, u, x, y) \in \mathcal{B}(A, B, B_e, C)$, the function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by $V(\zeta) := \langle P\zeta, \zeta \rangle$ satisfies*

$$\frac{d}{dt}V(x(t)) \leq -\delta \|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2 + \|v(t)\|^2 \quad \text{for a.e. } t \geq 0.$$

Proof By detectability of (A, C) , there exists $H \in \mathbb{R}^{n \times p}$ such that $A + HC$ is Hurwitz. Consequently, there exists a (unique) positive-definite solution $Q = Q^*$ of the Lyapunov equation

$$(A + HC)^* Q + Q(A + HC) = -I, \tag{2.6}$$

see, for example [10, Corollary 3.3.46]. Define the quadratic form U by $U(\zeta) := \langle Q\zeta, \zeta \rangle$ and let $(v, u, x, y) \in \mathcal{B}(A, B, B_e, C)$. Then

$$\frac{d}{dt}U(x(t)) = 2 \langle Qx(t), \dot{x}(t) \rangle \quad \text{for a.e. } t \geq 0.$$

Setting $w := Bu + B_e v$ and invoking (2.6), we conclude that, for almost every $t \geq 0$,

$$\begin{aligned} \frac{d}{dt}U(x(t)) &= \langle Qx(t), (A + HC)x(t) \rangle - \langle Qx(t), HCx(t) \rangle + \langle Qx(t), w(t) \rangle \\ &\quad + \langle (A + HC)x(t), Qx(t) \rangle - \langle HCx(t), Qx(t) \rangle + \langle w(t), Qx(t) \rangle \\ &= -\|x(t)\|^2 - 2\langle Qx(t), Hy(t) \rangle + 2\langle Qx(t), w(t) \rangle. \end{aligned}$$

An application of the Cauchy–Schwarz inequality and subsequent use of the elementary inequality $ab \leq a^2/c^2 + c^2b^2$ (which is valid for all real a, b and $c, c \neq 0$) show that there exist positive constants c_1, c_2, c_3 and c_4 such that, for all $(v, u, x, y) \in \mathcal{B}(A, B, B_e, C)$,

$$\begin{aligned} \frac{d}{dt}U(x(t)) &\leq -c_1 \|x(t)\|^2 + c_2 \|y(t)\|^2 + c_3 \|u(t)\|^2 + c_4 \|v(t)\|^2 \\ &\text{for a.e. } t \geq 0. \end{aligned}$$

Setting $c_5 := 1/\max\{c_2, c_3, c_4\}$, the claim follows with $P = c_5 Q$ and $\delta := c_1 c_5$. \square

3 ISS of Lur’e systems

In this section, we will apply the results provided in Sect. 2 to prove ISS properties for Lur’e systems of the form

$$\dot{x}(t) = Ax + Bf(Cx) + B_e v, \tag{3.1}$$

where $(A, B, B_e, C) \in \Sigma_e, f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ is locally Lipschitz and $v \in L^\infty_{loc}(\mathbb{R}_+, \mathbb{R}^{m_e})$ is the control (forcing, input) function. Obviously, (3.1) can (and should) be thought of as the feedback system given by

$$\dot{x} = Ax + Bu + B_e v, \quad y = Cx; \quad u = f(y).$$

Frequently, we shall refer to (3.1) as the Lur’e system (A, B, B_e, C, f) .

It is convenient to define the behaviour $\mathcal{B}(A, B, B_e, C, f)$ of (3.1) (or of the Lure’e system (A, B, B_e, C, f)) by

$$\begin{aligned} \mathcal{B}(A, B, B_e, C, f) := \left\{ (v, x) \in L^\infty_{loc}(\mathbb{R}_+, \mathbb{R}^{m_e}) \times W^{1,1}_{loc}(\mathbb{R}_+, \mathbb{R}^n) : \right. \\ \left. (v, x) \text{ satisfies (3.1) a.e. on } \mathbb{R}_+ \right\}. \end{aligned}$$

This definition may seem restrictive, since only trajectories defined on the whole half-line \mathbb{R}_+ are included in the behaviour. However, in the following, we will impose an assumption on f which implies that f is linearly bounded, and hence, for every initial condition $x(0) = x^0$ and every $v \in L^\infty_{loc}(\mathbb{R}_+, \mathbb{R}^{m_e})$, there exists a unique absolutely continuous solution of (3.1) which is defined on \mathbb{R}_+ .

The following lemma is obvious and does not require a proof.

Lemma 3.1 *Let $(A, B, B_e, C) \in \Sigma_e$, let $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be locally Lipschitz and let $(v, x) \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{m_e}) \times W^{1,1}_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$. Then $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$ if, and only if, $(v, f \circ Cx, x, Cx) \in \mathcal{B}(A, B, B_e, C)$.*

The Lur’e system (3.1) (or the quintuple (A, B, B_e, C, f)) is said to be input-to-state stable (ISS) if there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$,

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \varphi(\|v\|_{L^\infty(0,t)}) \quad \forall t \geq 0. \tag{3.2}$$

The concept of ISS (for a general controlled nonlinear system) appeared first in [23]. For overviews of ISS theory, we refer the reader to [3, 25].

We say that two functions $V_1, V_2 : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are \mathcal{K}_∞ -equivalent if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(V_1(\zeta)) \leq V_2(\zeta) \leq \alpha_2(V_1(\zeta))$ for all $\zeta \in \mathbb{R}^n$. A continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be an ISS-Lyapunov function for (3.1) (or for (A, B, B_e, C, f)) if V and $\|\cdot\|_{\mathbb{R}^n}$ are \mathcal{K}_∞ -equivalent and there exist $\beta, \gamma \in \mathcal{K}_\infty$ such that, for all $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$,

$$\frac{d}{dt} V(x(t)) \leq -\beta(\|x(t)\|) + \gamma(\|v(t)\|) \quad \text{for a.e. } t \geq 0$$

It is a well-known result in ISS theory (see, for example [25]) that the existence of an ISS-Lyapunov function guarantees ISS.

We are now ready to state and prove the main result of this paper.

Theorem 3.2 *Let $(A, B, B_e, C) \in \Sigma_e$, $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be locally Lipschitz, $r > 0$ and $K \in \mathbb{R}^{m \times p}$. If $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$ and there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\|f(\xi) - K\xi\| \leq r \|\xi\| - \alpha(\|\xi\|) \quad \forall \xi \in \mathbb{R}^p, \tag{3.3}$$

then the Lur’e system (A, B, B_e, C, f) is ISS.

In particular, if A is Hurwitz, then the Lur’e system (A, B, B_e, C, f) is ISS, provided that there exists $\alpha \in \mathcal{K}_\infty$ such that $\|f(\xi)\| \leq r \|\xi\| - \alpha(\|\xi\|)$ for all $\xi \in \mathbb{R}^p$, where $r = r_{\mathbb{C}}(A; B, C)$. This shows that the complex stability radius $r_{\mathbb{C}}(A; B, C)$ provides a measure of the robustness of ISS of the linear system $\dot{x} = Ax + B_e v$ with respect to additive nonlinear perturbations F of the form $F(x) = Bf(Cx)$.

Proof of Theorem 3.2 It is sufficient to show that there exists an ISS-Lyapunov function for (A, B, B_e, C, f) . This will be done by constructing two functions V and W and then showing that $V + W$ is an ISS-Lyapunov function.

Since $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$, it is clear that the system (A, B, C) is stabilizable and detectable. Proposition 2.5 guarantees the existence of a positive definite $Q = Q^* \in \mathbb{R}^{n \times n}$ and a positive $\delta > 0$ such that, for every $(v, u, x, y) \in \mathcal{B}(A, B, B_e, C)$, the function $U_0 : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by $U_0(\zeta) := \langle Q\zeta, \zeta \rangle$ satisfies

$$\frac{d}{dt} U_0(x(t)) \leq -\delta \|x(t)\|^2 + \|y(t)\|^2 + \|u(t)\|^2 + \|v(t)\|^2 \quad \text{for a.e. } t \geq 0.$$

Let $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$. Then, by Lemma 3.1, $(v, f \circ Cx, x, Cx) \in \mathcal{B}(A, B, B_e, C, f)$, and thus

$$\frac{d}{dt}U_0(x(t)) \leq -\delta \|x(t)\|^2 + \|Cx(t)\|^2 + \|f(Cx(t))\|^2 + \|v(t)\|^2 \quad \text{for a.e. } t \geq 0. \tag{3.4}$$

By (3.3),

$$\|f(\xi)\|^2 \leq c_0 \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^p,$$

where $c_0 := 2(\|K\|^2 + r^2)$. Setting

$$U := \frac{1}{1 + c_0}U_0 \quad \text{and} \quad \varepsilon := \frac{\delta}{1 + c_0},$$

it then follows from (3.4) that, for every $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$,

$$\frac{d}{dt}U(x(t)) \leq -\varepsilon \|x(t)\|^2 + \|Cx(t)\|^2 + \|v(t)\|^2 \quad \text{for a.e. } t \geq 0. \tag{3.5}$$

It is convenient to define constants

$$c_1 := r\sqrt{\varepsilon/2}, \quad c_2 := \sqrt{\varepsilon/2}, \quad c_3 := \|C\|^2$$

and to choose positive constants c_4 and c_5 such that

$$c_4\|\zeta\| \leq \sqrt{U(\zeta)} \leq c_5\|\zeta\| \quad \forall \zeta \in \mathbb{R}^n, \tag{3.6}$$

with

$$c_4 = \frac{1}{\sqrt{(1 + c_0)\|Q^{-1}\|}} \quad \text{and} \quad c_5 = \sqrt{\frac{\|Q\|}{1 + c_0}}$$

being a possible choice.

Furthermore, we define $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\mu(s) := \frac{\varepsilon}{4} \min \left\{ c_4^2 s^3, \frac{c_1 c_4 \alpha(c_2 c_4 s / c_5)}{c_3 c_5} \right\} \quad \forall s \geq 0,$$

where α is the \mathcal{K}_∞ -function from (3.3), the existence of which is part of the hypothesis. It is obvious that $\mu \in \mathcal{K}_\infty$. By Proposition 2.3, there exist positive semi-definite $P = P^* \in \mathbb{R}^{n \times n}$ and $\beta \in \mathcal{K}_\infty$ such that, for every $(v, u, x, y) \in \mathcal{B}(A, B, B_e, C)$, the function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined by $V(\zeta) := \langle P\zeta, \zeta \rangle$ satisfies

$$\begin{aligned} \frac{d}{dt}V(x(t)) &\leq -r^2 \|y(t)\|^2 + \|u(t) - Ky(t)\|^2 + \|x(t)\| \mu(\|x(t)\|) \\ &\quad + \beta(\|v(t)\|) \quad \text{for a.e. } t \geq 0 \end{aligned}$$

Let $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$. Then, by Lemma 3.1, $(v, f \circ Cx, x, Cx) \in \mathcal{B}(A, B, B_e, C)$, and thus,

$$\begin{aligned} \frac{d}{dt} V(x(t)) &\leq -r^2 \|Cx(t)\|^2 + \|f(Cx(t)) - KCx(t)\|^2 + \|x(t)\| \mu(\|x(t)\|) \\ &\quad + \beta(\|v(t)\|) \quad \text{for a.e. } t \geq 0. \end{aligned} \tag{3.7}$$

Invoking (3.3), we have

$$\|f(\xi) - K\xi\|^2 - r^2 \|\xi\|^2 \leq -2\alpha(\|\xi\|)r \|\xi\| + \alpha^2(\|\xi\|) \quad \forall \xi \in \mathbb{R}^P.$$

Inequality (3.3) implies in particular that $\alpha(s) \leq rs$ for all $s \geq 0$, and so

$$\|f(\xi) - K\xi\|^2 - r^2 \|\xi\|^2 \leq -r \|\xi\| \alpha(\|\xi\|) \quad \forall \xi \in \mathbb{R}^P.$$

Using this estimate in (3.7), we obtain

$$\begin{aligned} \frac{d}{dt} V(x(t)) &\leq -r \|Cx(t)\| \alpha(\|Cx(t)\|) + \|x(t)\| \mu(\|x(t)\|) \\ &\quad + \beta(\|v(t)\|) \quad \text{for a.e. } t \geq 0. \end{aligned} \tag{3.8}$$

We will now “adjust” U by composing it with a suitable function h , that is, we will be considering

$$W := h \circ U.$$

The function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is given by

$$h(s) = \int_0^s k(\sigma) d\sigma \quad \forall s \geq 0,$$

where $k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined as follows:

$$k(0) := 0 \quad \text{and} \quad k(s) := \min \left\{ s, \frac{c_1 c_4 \alpha(c_2 \sqrt{s} / c_5)}{c_3 \sqrt{s}} \right\} \quad \forall s > 0.$$

Obviously, h is continuously differentiable and

$$0 \leq h'(s) = k(s) \leq \frac{r c_1 c_2 c_4}{c_3 c_5} =: c_6 \quad \forall s \geq 0, \tag{3.9}$$

where we have used again that $\alpha(s) \leq rs$ for all $s \geq 0$.

We claim that

$$h'(U(\zeta))(-\varepsilon \|\zeta\|^2 + \|C\zeta\|^2) \leq -2\|\zeta\| \mu(\|\zeta\|) + r \|C\zeta\| \alpha(\|C\zeta\|) \quad \forall \zeta \in \mathbb{R}^n. \tag{3.10}$$

To avoid breaking the flow of the argument, we relegate the verification of (3.10) to the end of the proof.

Invoking (3.5), it follows that, for every $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$,

$$\frac{d}{dt}W(x(t)) = \frac{d}{dt}h(U(x(t))) \leq h'(U(x(t)))[-\varepsilon \|x(t)\|^2 + \|Cx(t)\|^2 + \|v(t)\|^2] \text{ for a.e. } t \geq 0.$$

Combining this with (3.10) shows that, for every $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$,

$$\frac{d}{dt}W(x(t)) \leq -2\|x(t)\|\mu(\|x(t)\|) + r\|Cx(t)\|\alpha(\|Cx(t)\|) + c_6\|v(t)\|^2 \text{ for a.e. } t \geq 0, \tag{3.11}$$

where we have used (3.9). Defining $\gamma \in \mathcal{K}_\infty$ by $\gamma(s) := \beta(s) + c_6s^2$ for all $s \geq 0$, it follows from (3.8) and (3.11) that, for every $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$,

$$\frac{d}{dt}(V + W)(x(t)) \leq -\|x(t)\|\mu(\|x(t)\|) + \gamma(\|v(t)\|) \text{ for a.e. } t \geq 0. \tag{3.12}$$

Consequently, if $V + W$ and $\|\cdot\|_{\mathbb{R}^n}$ are \mathcal{K}_∞ -equivalent, then $V + W$ is an ISS-Lyapunov function for (A, B, B_e, C, f) . To show that $V + W$ and $\|\cdot\|_{\mathbb{R}^n}$ are \mathcal{K}_∞ -equivalent, note that

$$(V + W)(\zeta) \leq c_7\|\zeta\|^2 = \eta_1(\|\zeta\|) \quad \forall \zeta \in \mathbb{R}^n, \tag{3.13}$$

where $c_7 := \|P\| + c_5^2c_6$ and $\eta_1 \in \mathcal{K}_\infty$ is defined by $\eta_1(s) := c_7s^2$ for all $s \geq 0$. Moreover, noting that $h \in \mathcal{K}_\infty$, it is clear that η_2 , defined by $\eta_2(s) := h(c_4^2s^2)$ for all $s \geq 0$, is also in \mathcal{K}_∞ , and it follows that

$$(V + W)(\zeta) \geq h(U(\zeta)) \geq h(c_4^2\|\zeta\|^2) = \eta_2(\|\zeta\|) \quad \forall \zeta \in \mathbb{R}^n. \tag{3.14}$$

Inequalities (3.13) and (3.14) show that $V + W$ and $\|\cdot\|_{\mathbb{R}^n}$ are \mathcal{K}_∞ -equivalent. We have now established that $V + W$ is an ISS-Lyapunov function for (A, B, B_e, C, f) .

It only remains to prove that (3.10) holds. To this end, using (3.6), we estimate,

$$h'(U(\zeta)) = k(U(\zeta)) \leq \frac{c_1\alpha(c_2\|\zeta\|)}{c_3\|\zeta\|} \quad \forall \zeta \in \mathbb{R}^n, \quad \zeta \neq 0.$$

Consequently,

$$c_3\|\zeta\|^2h'(U(\zeta)) \leq c_1\|\zeta\|\alpha(c_2\|\zeta\|) \quad \forall \zeta \in \mathbb{R}^n. \tag{3.15}$$

We consider two cases.

Case a. If $\|C\zeta\|^2 > \varepsilon\|\zeta\|^2/2$, then it follows from (3.15) and the definition of c_1, c_2 and c_3 that

$$\|C\zeta\|^2h'(U(\zeta)) \leq r\|C\zeta\|\alpha(c_2\|\zeta\|) \leq r\|C\zeta\|\alpha(\|C\zeta\|).$$

Case b. If $\|C\zeta\|^2 \leq \varepsilon\|\zeta\|^2/2$, then trivially,

$$\|C\zeta\|^2 h'(U(\zeta)) \leq \frac{\varepsilon}{2} \|\zeta\|^2 h'(U(\zeta)).$$

Therefore, we conclude

$$\|C\zeta\|^2 h'(U(\zeta)) \leq \max \left\{ \frac{\varepsilon}{2} \|\zeta\|^2 h'(U(\zeta)), r\|C\zeta\|\alpha(\|C\zeta\|) \right\} \quad \forall \zeta \in \mathbb{R}^n. \quad (3.16)$$

Furthermore, using again (3.6), we obtain

$$h'(U(\zeta)) = k(U(\zeta)) \geq \min \left\{ c_4^2 \|\zeta\|^2, \frac{c_1 c_4 \alpha(c_2 c_4 \|\zeta\|/c_5)}{c_3 c_5 \|\zeta\|} \right\} \quad \forall \zeta \in \mathbb{R}^n, \quad \zeta \neq 0,$$

implying that

$$2\|\zeta\|\mu(\|\zeta\|) \leq \frac{\varepsilon}{2} \|\zeta\|^2 h'(U(\zeta)) \quad \forall \zeta \in \mathbb{R}^n. \quad (3.17)$$

Combination of (3.16) and (3.17) yields

$$h'(U(\zeta))\|C\zeta\|^2 + 2\|\zeta\|\mu(\|\zeta\|) \leq \varepsilon\|\zeta\|^2 h'(U(\zeta)) + r\|C\zeta\|\alpha(\|C\zeta\|) \quad \forall \zeta \in \mathbb{R}^n,$$

which is equivalent to (3.10), completing the proof. □

The ISS property of the Lur’e system (A, B, B_e, C, f) , guaranteed by Theorem 3.2, means that there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that the ISS estimate (3.2) holds for all $(v, x) \in \mathcal{B}(A, B, B_e, C, f)$. As follows from ISS theory, the comparison functions ψ and φ depend only on the \mathcal{K}_∞ -functions μ, γ, η_1 and η_2 , see (3.12)–(3.14). These functions in turn depend only on A, B, B_e, C, K, r and α , but not on f . This means that, in the context of Theorem 3.2, there exist comparison functions $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for every f satisfying (3.3), the ISS estimate (3.2) holds. Furthermore, it can be shown that if α is linear, then we can choose ψ and φ as follows: $\psi(s, t) = M e^{-at} s$ and $\varphi(s) = bs$ for suitable constants $M \geq 1$ and $a, b > 0$.

As the following example shows, Theorem 3.2 does not remain true if the condition on α is relaxed to $\alpha \in \mathcal{K}$.

Example 3.3 Define $\alpha \in \mathcal{K} \setminus \mathcal{K}_\infty$ by $\alpha(s) := 1 - e^{-s}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(\xi) := \xi - \text{sgn}(\xi)\alpha(|\xi|)$. Consider the one-dimensional forced Lur’e system

$$\dot{x}(t) = -x(t) + f(x(t)) + v(t).$$

Obviously, $-1 + k$ is Hurwitz for all $k \in \mathbb{C}$ with $|k| < 1$ and

$$|f(\xi)| = |\xi| - \alpha(|\xi|) \quad \forall \xi \in \mathbb{R}.$$

Consequently, with the exception of the condition $\alpha(s) \rightarrow \infty$ as $s \rightarrow \infty$, the hypotheses of Theorem 3.2 are satisfied. Choosing $v(t) = 1 + \varepsilon$ for some positive ε , we have $\dot{x}(t) \geq \varepsilon$ for all $t \geq 0$ and hence the Lur’e system is not ISS. □

We note that, in the unforced case ($v = 0$), the equilibrium 0 in Example 3.3 is globally asymptotically stable. In fact, it can be shown that if $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$, then, for any locally Lipschitz $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$, satisfying $\|f(\xi) - K\xi\| < r \|\xi\|$ for all $\xi \in \mathbb{R}^p \setminus \{0\}$, the equilibrium 0 of the unforced Lur’e system

$$\dot{x} = Ax + Bf(Cx)$$

is globally asymptotically stable.

The following result identifies a class of Lur’e systems for which condition (3.3) does not hold and hence Theorem 3.2 does not apply. The result also shows that, under a mild additional assumption, these Lur’e systems are not ISS.

Proposition 3.4 *Let $(A, B, B_e, C) \in \Sigma_e$, $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be locally Lipschitz, $r > 0$ and $K \in \mathbb{R}^{m \times p}$. Assume that A is not Hurwitz, f is bounded and $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$. Then the following statements hold.*

- (a) *There does not exist $\alpha \in \mathcal{K}_{\infty}$ such that $\|f(\xi) - K\xi\| \leq r \|\xi\| - \alpha(\|\xi\|)$ for all $\xi \in \mathbb{R}^p$ (that is, condition (3.3) does not hold).*
- (b) *Under the additional assumption that $\text{im } B \subset \text{im } B_e$, the Lur’e system (A, B, B_e, C, f) is not ISS.*

Proof (a) Since A is not Hurwitz, it is clear that $r \leq \|K\|$. Moreover,

$$r \|\xi\| - \|f(\xi) - K\xi\| \leq r \|\xi\| - \|K\xi\| + \|f(\xi)\| \quad \forall \xi \in \mathbb{R}^p.$$

Let $\xi_0 \in \mathbb{R}^p$ be such that $\|\xi_0\| = 1$ and $\|K\xi_0\| = \|K\|$. Then, for all $a \geq 0$, we have

$$r \|a\xi_0\| - \|f(a\xi_0) - K(a\xi_0)\| \leq a(r - \|K\|) + \|f(a\xi_0)\| \leq \sup_{\xi \in \mathbb{R}^p} \|f(\xi)\| < \infty,$$

yielding the claim.

(b) We first prove the claim under the assumption that (A, B) is controllable. Let $z(\cdot; w)$ denote the solution of the initial value problem

$$\dot{z} = Az + Bw, \quad z(0) = 0.$$

Then there exists $w \in L^{\infty}(\mathbb{R}_+, \mathbb{R}^m)$ such that $x := z(\cdot; w)$ is unbounded (because otherwise the linear system (A, B, I) would be bounded-input–bounded-output stable, and therefore, by controllability and observability of (A, B, I) , A would be Hurwitz, which is not possible). By boundedness of f , we have that $w - f(Cx) \in L^{\infty}(\mathbb{R}_+, \mathbb{R}^m)$, and, since $\text{im } B \subset \text{im } B_e$, there exists $v \in L^{\infty}(\mathbb{R}_+, \mathbb{R}^{m_e})$ such that $B_e v = B(w - f(Cx))$. Thus,

$$\dot{x} = Ax + Bw = Ax + Bf(Cx) + B_e v.$$

Since v is bounded and x is unbounded, it follows that the Lur’e system is not ISS.

If (A, B) is not controllable, then combining an argument similar to that used above with Kalman’s controllability decomposition yields the claim. □

Results which are (vaguely) related to Proposition 3.4 can be found in [20], where it is shown that, under suitable assumptions, a “small” signal ISS property holds for Lur’e systems with nonlinearities of “saturation” type.

We now illustrate Theorem 3.2 by two examples.

Example 3.5 We consider a system modelling a sequence of linked chemical reactions inspired by [21]:

$$\left. \begin{aligned} \dot{z}_1 &= g(z_3) - a_1 z_1 + d_1, \\ \dot{z}_2 &= z_1 - a_2 z_2 + d_2, \\ \dot{z}_3 &= z_2 - a_3 z_3 + d_3, \end{aligned} \right\} \tag{3.18}$$

where z_1, z_2 and z_3 represent the concentrations of reagents, a_1, a_2 and a_3 are positive constants, d_1, d_2 and d_3 represent external disturbances and the locally Lipschitz nonlinearity $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ represents inhibition of creation of reagent z_1 depending on the concentration of reagent z_3 . The latter means that g is a decreasing function and hence g has negative derivative (provided that g is differentiable). The feedback loop corresponding to g , sometimes referred to as *negative feedback*, is common in metabolic control mechanisms, see Section 7.2 from [21]. Setting

$$A := \begin{pmatrix} -a_1 & 0 & 0 \\ 1 & -a_2 & 0 \\ 0 & 1 & -a_3 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C := (0 \ 0 \ 1),$$

the system (3.18) can be written in the form

$$\dot{z} = Az + Bg(Cz) + d, \tag{3.19}$$

where $z := (z_1, z_2, z_3)^*$ and $d := (d_1, d_2, d_3)^*$.

Note that z_1, z_2 and z_3 are naturally non-negative. Since A is a Metzler matrix (all off-diagonal entries are non-negative), B and C have non-negative entries and g maps \mathbb{R}_+ into \mathbb{R}_+ , it is well known that, for non-negative initial conditions and for non-negative disturbances, the corresponding trajectories of (3.19) are non-negative (here vectors are referred to as non-negative if each component is non-negative).

The matrix A is Hurwitz and thus, the transfer function G of the single-input single-output system (A, B, C) , given by $G(s) = C(sI - A)^{-1}B$, is bounded and holomorphic on \mathbb{C}_+ . From a routine argument, it follows that

$$\|G\|_{H^\infty} = G(0) = \frac{1}{a_1 a_2 a_3}.$$

Consequently, setting $r := a_1 a_2 a_3 > 0$, we have

$$\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C). \tag{3.20}$$

Since $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is decreasing (and excluding the trivial case $g(\xi) \equiv 0$), it is clear that there exists a unique number $\xi^\dagger > 0$ such that $g(\xi^\dagger) = r\xi^\dagger$. A straightforward calculation shows that the vector

$$z^\dagger := -A^{-1}br\xi^\dagger = (a_2a_3\xi^\dagger, a_3\xi^\dagger, \xi^\dagger)^* \neq 0$$

is the unique equilibrium of (3.19) with $d(t) \equiv 0$.

Before we can apply Theorem 3.2, we need to transform (3.19) in such a way that the equilibrium z^\dagger is moved to the origin. To this end, define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(\xi) = \begin{cases} g(\xi + \xi^\dagger) - g(\xi^\dagger) & \text{for } \xi \geq -\xi^\dagger \\ g(0) - g(\xi^\dagger) & \text{for } \xi < -\xi^\dagger. \end{cases}$$

Let $z(0)$ and d be non-negative and let z be the corresponding (non-negative) solution z of (3.19). Defining the function x by $x(t) = z(t) - z^\dagger$, it follows that

$$\dot{x} = Ax + Bf(Cx) + d. \tag{3.21}$$

We note that 0 is an equilibrium of (3.21) with $d(t) \equiv 0$. Furthermore, if (3.21) is ISS (with respect to the equilibrium 0), then (3.19) is ISS (with respect to the equilibrium z^\dagger) for non-negative disturbances d , that is, there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}_\infty$ such that, for all $z(0) \in \mathbb{R}_+^3$ and non-negative $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}_+^3)$,

$$\|z(t) - z^\dagger\| \leq \psi\left(\|z(0) - z^\dagger\|, t\right) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall t \geq 0. \tag{3.22}$$

Therefore, appealing to (3.20) and invoking Theorem 3.2, we may conclude that (3.19) is ISS, provided that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$|g(\xi + \xi^\dagger) - g(\xi^\dagger)| \leq r|\xi| - \alpha(|\xi|) \quad \forall \xi \geq -\xi^\dagger. \tag{3.23}$$

Let us consider a typical negative feedback nonlinearity g :

$$g(\xi) := \frac{1}{1 + \xi} \quad \forall \xi \geq 0. \tag{3.24}$$

It is easy to verify that, in this case,

$$|g(\xi + \xi^\dagger) - g(\xi^\dagger)| \leq \frac{|\xi|}{1 + \xi^\dagger} \quad \forall \xi \geq -\xi^\dagger. \tag{3.25}$$

If $r > 1/2$, then a routine calculation shows that $\xi^\dagger < 1$ and so,

$$\frac{1}{1 + \xi^\dagger} = g(\xi^\dagger) = r\xi^\dagger < r,$$

showing that (3.23) holds with α given by $\alpha(s) = r(1 - \xi^\dagger)s$. Consequently, if g is given by (3.24), then (3.19) is ISS, provided that $r = a_1a_2a_3 > 1/2$. We mention that this conclusion can also be obtained by writing (3.21) in component form

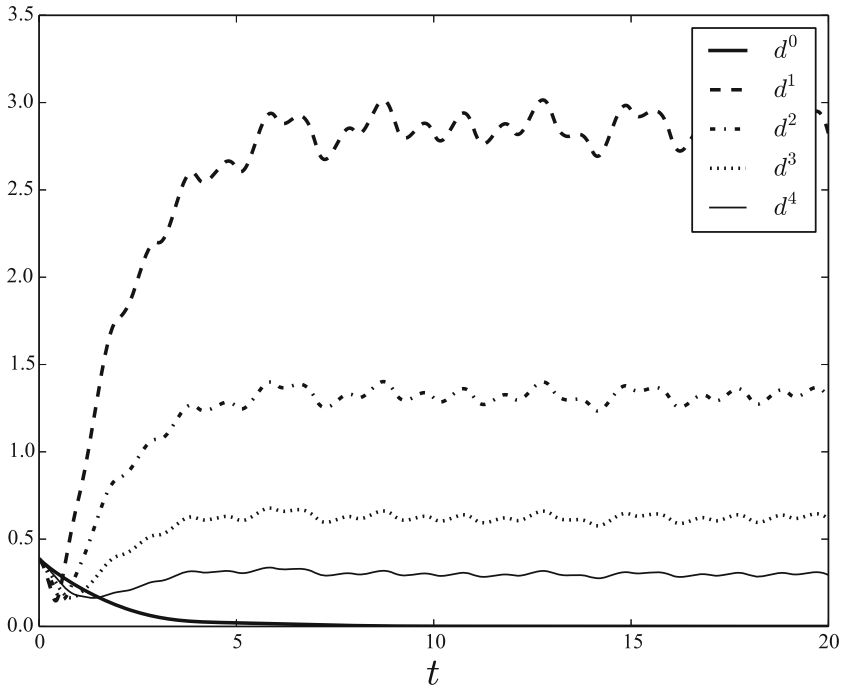


Fig. 1 $\|z(t) - z^\dagger\|_2$ for different disturbances: $d^0(t) = 0$, $d^1(t) = (|\sin(t)|, |\sin(\sqrt{2}t)|, |\sin(\pi t)|)^*$, $d^2(t) = \frac{1}{2}d^1(t)$, $d^3(t) = \frac{1}{4}d^1(t)$, $d^4(t) = \frac{1}{8}d^1(t)$

$$\left. \begin{aligned} \dot{x}_1 &= f(x_3) - a_1x_1 + d_1, \\ \dot{x}_2 &= x_1 - a_2x_2 + d_2, \\ \dot{x}_3 &= x_2 - a_3x_3 + d_3 \end{aligned} \right\} \tag{3.26}$$

and applying a suitable nonlinear small-gain ISS theorem for feedback interconnections of several subsystems, see [4, Theorem 11] or [5, Corollary 5.6].¹ We will make more systematic contact with small-gain ideas further below (see Corollary 3.8 and the paragraph after the proof of Corollary 3.8).

To consider a specific numerical example, let g is given by (3.24) and choose $a_1 = a_2 = 1$ and $a_3 = 3/5$. Then $r = a_1a_2a_3 = 3/5 > 1/2$ and hence (3.19) is ISS. Note that in this case $\xi^\dagger = (\sqrt{69}-3)/6$ and consequently $z^\dagger = ((\sqrt{69}-3)/10, (\sqrt{69}-3)/10, (\sqrt{69}-3)/6)^*$. Simulations with initial state $z(0) = (1/2, 1/2, 1/2)^*$ and a range of disturbances are shown in Fig. 1.

¹ For example, using the notation of [4], we have $\gamma_{11} = \gamma_{12} = \gamma_{22} = \gamma_{23} = \gamma_{31} = \gamma_{33} = 0$,

$$\gamma_{13}(s) = \frac{s}{a_1(1 + \xi^\dagger)}, \quad \gamma_{21}(s) = \frac{s}{a_2} \quad \text{and} \quad \gamma_{32}(s) = \frac{s}{a_3},$$

and defining $\alpha_i(s) = \varepsilon_i s$, where $\varepsilon_1, \varepsilon_2$ and ε_3 are positive numbers such that $(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) < r(1 + \xi^\dagger)$, it follows from [4, Theorem 11] that (3.26) is ISS, provided that $r > 1/2$.

Finally, to conclude the example, we mention that the above arguments establishing ISS also show that, if (3.22) holds, then, for all $z(0) \in \mathbb{R}_+^3$ and all disturbances $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^3)$, possibly negative-valued, such that

$$\psi(\|z(0) - z^\dagger\|, 0) + \varphi(\|d\|_{L^\infty(0,\infty)}) \leq \min\{z_j^\dagger : j = 1, 2, 3\} =: \mu,$$

where z_j^\dagger is the j -th component of z^\dagger ,

the solution z of (3.19) remains in the non-negative orthant for all times (or, equivalently, does not “escape” from the non-negative orthant in finite time). For example, if $\psi(\|z(0) - z^\dagger\|, 0) \leq \mu/2$, then the solution z of (3.19) stays in \mathbb{R}_+^3 for all times in the presence of componentwise negative disturbances d satisfying $\varphi(\|d\|_{L^\infty(0,\infty)}) \leq \mu/2$. □

Example 3.5 is a single-input single-output system in the sense that $m = p = 1$. In the following example, we consider a system with $m = 2$ and $p = 4$.

Example 3.6 Consider $(A, B, B_e, C) \in \Sigma_e$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $B_e \in \mathbb{R}^{4 \times m_e}$, $B_e \neq 0$, is arbitrary. It is obvious that A is not Hurwitz and thus, the transfer function G of the minimal triple (A, B, C) is not in $H^\infty(\mathbb{C}^{4 \times 2})$. A MATLAB calculation reveals that,

$$K := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 15 & -20/3 & 4/3 & 6 \end{pmatrix},$$

is a stabilizing output feedback matrix and we have $\|G^K\|_{H^\infty} = 3.8383$. Therefore, by Lemma 2.1, there exists $r > 1/4$ (for example, $r = 10/39$) such that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(G) = \mathbb{S}_{\mathbb{C}}(A, B, C)$. Invoking Theorem 3.2, we conclude that the Lur’e system (A, B, B_e, C, f) is ISS for every locally Lipschitz $f: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ such that

$$\|f(\xi) - K\xi\| \leq \frac{1}{4} \|\xi\| \quad \forall \xi \in \mathbb{R}^4. \tag{3.27}$$

To provide a specific example satisfying (3.27), consider the function $f: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ given by

$$f(\xi) = K\xi + \begin{pmatrix} \sin(\|\xi\|)/5 \\ 3g(\xi)/20 \end{pmatrix} \quad \forall \xi \in \mathbb{R}^4,$$

where $g: \mathbb{R}^4 \rightarrow \mathbb{R}$ is locally Lipschitz and such that $|g(\xi)| \leq \|\xi\|$ for all $\xi \in \mathbb{R}^4$. Then

$$\|f(\xi) - K\xi\| = \sqrt{\frac{1}{25} \sin^2(\|\xi\|) + \frac{9}{400} g^2(\xi)} < \frac{1}{4} \|\xi\| \quad \forall \xi \in \mathbb{R}^4, \quad \xi \neq 0,$$

implying that the Lur’e system (A, B, B_e, C, f) is ISS. □

Theorem 3.2 says, roughly speaking, that linear stability (namely, $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$) implies ISS for all nonlinearities $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ satisfying (3.3). In this sense, Theorem 3.2 is reminiscent of the Aizerman conjecture, see, for example [9, 10, 17, 27]. We emphasize though that stability of the linear feedback system $\dot{x} = (A + BFC)x$ has to hold for all complex output feedback matrices F satisfying $\|F - K\| < r$. It is easy to see that the ISS conclusion in Theorem 3.2 remains true for all complex nonlinearities $f : \mathbb{C}^p \rightarrow \mathbb{C}^m$ satisfying (3.3) for all $\xi \in \mathbb{C}^p$. We will now identify a special case wherein the complex condition $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$ can be replaced by its real counterpart $\mathbb{B}_{\mathbb{R}}(K, r) \subseteq \mathbb{S}_{\mathbb{R}}(A, B, C)$.

Recall that a square matrix $M \in \mathbb{R}^{n \times n}$ is said to be Metzler (or essentially non-negative or quasi positive) if all its off-diagonal entries are non-negative. It is well known (and straightforward to prove) that $M \in \mathbb{R}^{n \times n}$ is Metzler if, and only if, $e^{Mt}\zeta \in \mathbb{R}_+^n$ for all $\zeta \in \mathbb{R}_+^n$ and all $t \geq 0$. We say that a matrix with real entries is non-negative if all its entries are non-negative.

Corollary 3.7 *Let $(A, B, B_e, C) \in \Sigma_e$, $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be locally Lipschitz, $r > 0$ and $K \in \mathbb{R}^{m \times p}$. Assume that B and C are non-negative and $A + BKC$ is Metzler. If $\mathbb{B}_{\mathbb{R}}(K, r) \subseteq \mathbb{S}_{\mathbb{R}}(A, B, C)$ and there exists $\alpha \in \mathcal{K}_{\infty}$ such that*

$$\|f(\xi) - K\xi\| \leq r \|\xi\| - \alpha(\|\xi\|) \quad \forall \xi \in \mathbb{R}^p, \tag{3.28}$$

then the Lur’e system (A, B, B_e, C, f) is ISS.

Proof By hypothesis, B and C are non-negative and $A_K := A + BKC$ is Metzler. Since $\mathbb{B}_{\mathbb{R}}(K, r) \subseteq \mathbb{S}_{\mathbb{R}}(A, B, C)$, we have $\mathbb{B}_{\mathbb{R}}(0, r) \subseteq \mathbb{S}_{\mathbb{R}}(A_K, B, C)$, and thus, $r \leq r_{\mathbb{R}}(A_K; B, C)$. By a stability radius result for non-negative systems proved in [11], $r_{\mathbb{R}}(A_K; B, C) = r_{\mathbb{C}}(A_K; B, C)$, and hence, $\mathbb{B}_{\mathbb{C}}(0, r) \subseteq \mathbb{S}_{\mathbb{C}}(A_K, B, C)$, or, equivalently, $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$. The claim now follows from Theorem 3.2. □

The corollary below provides a “small-gain” interpretation of Theorem 3.2.

Corollary 3.8 *Let $(A, B, B_e, C) \in \Sigma_e$, $K \in \mathbb{S}_{\mathbb{R}}(A, B, C)$, let $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be locally Lipschitz and let G denote the transfer function of (A, B, C) . If there exists $\alpha \in \mathcal{K}_{\infty}$ such that*

$$\|G^K\|_{H^{\infty}} \frac{\|f(\xi) - K\xi\|}{\|\xi\|} \leq 1 - \frac{\alpha(\|\xi\|)}{\|\xi\|} \quad \forall \xi \in \mathbb{R}^p, \quad \xi \neq 0,$$

then the Lur’e system (A, B, B_e, C, f) is ISS.

Proof Setting $r := 1/\|G^K\|_{H^{\infty}}$, it follows that $\mathbb{B}_{\mathbb{C}}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$ and an application of Theorem 3.2 yields the claim. □

We note that Corollary 3.8 is not a special case of general nonlinear small-gain ISS results as can be found, for example, in [14,26]. The reason for this is that, in general, the H^∞ -gain $\|G^K\|_{H^\infty}$ and the ISS gain of the linear system $(A + BKC, B, C)$ do not coincide: the former is always less or equal to the latter and the difference between these two gains can be large.

Next we derive a version of Theorem 3.2 which is reminiscent of the well-known circle criterion (see [6,7,16,27]). To this end, let $\mathbb{R}(s)$ denote the field of real rational functions, and recall that $H \in \mathbb{R}(s)^{m \times m}$ is said to be positive real if for every $s \in \mathbb{C}_+$ which is not a pole of H , the matrix $H^*(s) + H(s)$ is positive semi-definite.

For convenience, we state the following well-known lemma.

Lemma 3.9 *Let $H \in \mathbb{R}(s)^{m \times m}$. If H is positive real, then H does not have any poles in \mathbb{C}_+ , -1 is not an eigenvalue of $H(s)$ for every $s \in \mathbb{C}_+$ and*

$$\left\| (I - H)(I + H)^{-1} \right\|_{H^\infty} \leq 1.$$

We are now in the position to state and prove a corollary of Theorem 3.2 which shows that, under conditions very similar to those of the circle criterion, the Lur’e system (A, B, B_e, C, f) is ISS.

Corollary 3.10 *Let $(A, B, B_e, C) \in \Sigma_e$, $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be locally Lipschitz, $K_1, K_2 \in \mathbb{R}^{m \times p}$ and let G denote the transfer function of (A, B, C) . Assume that (A, B, C) is stabilizable and detectable and that $(I - K_2G)(I - K_1G)^{-1}$ is positive real. If there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi \rangle \leq -\alpha(\|\xi\|) \|\xi\| \quad \forall \xi \in \mathbb{R}^p, \tag{3.29}$$

then the Lur’e system (A, B, B_e, C, f) is ISS.

Proof Setting

$$K := \frac{1}{2}(K_1 + K_2) \quad \text{and} \quad L := \frac{1}{2}(K_1 - K_2),$$

we rewrite the left-hand side of the sector condition (3.29) in terms of K and L :

$$\begin{aligned} \langle f(\xi) - K_1\xi, f(\xi) - K_2\xi \rangle &= \langle f(\xi) - (K + L)\xi, f(\xi) - (K - L)\xi \rangle \\ &= \|f(\xi) - K\xi\|^2 - \|L\xi\|^2 \quad \forall \xi \in \mathbb{R}^p. \end{aligned} \tag{3.30}$$

Note that in conjunction with (3.29) this implies $\ker L = \{0\}$. Thus L^*L is invertible and $L^\sharp := (L^*L)^{-1}L^* \in \mathbb{R}^{p \times m}$ is a left inverse of L . Furthermore,

$$(I - K_2G)(I - K_1G)^{-1} = (I - K_1G + 2LG)(I - K_1G)^{-1} = I + 2LG^{K_1},$$

showing that $I + 2LG^{K_1}$ is positive real. Thus, by Lemma 3.9,

$$\left\| LG^{K_1}(I + LG^{K_1})^{-1} \right\|_{H^\infty} \leq 1.$$

Trivially,

$$LG^{K_1}(I + LG^{K_1})^{-1} = LG^{K_1}(I - (-LL^\sharp)LG^{K_1})^{-1} = (LG^{K_1})^{-LL^\sharp},$$

and so, appealing to statement (d) of Lemma 2.1,

$$\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subseteq \mathbb{S}_{\mathbb{C}}(LG^{K_1}). \tag{3.31}$$

By stabilizability and detectability of (A, B, C) and left invertibility of L , it follows that (A_{K_1}, B, LC) is stabilizable and detectable, where $A_{K_1} := A + BK_1C$. The transfer function of (A_{K_1}, B, LC) is equal to LG^{K_1} and so (3.31) implies

$$\mathbb{B}_{\mathbb{C}}(-LL^\sharp, 1) \subseteq \mathbb{S}_{\mathbb{C}}(A_{K_1}, B, LC). \tag{3.32}$$

Defining $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by $g(\xi) := f(L^\sharp\xi) - K_1L^\sharp\xi$ for all $\xi \in \mathbb{R}^m$, it is straightforward to show that

$$\mathcal{B}(A, B, B_e, C, f) = \mathcal{B}(A_{K_1}, B, B_e, LC, g). \tag{3.33}$$

We claim that it is sufficient to prove that there exists $\beta \in \mathcal{K}_\infty$ such that

$$\|g(\xi) + LL^\sharp\xi\| \leq \|\xi\| - \beta(\|\xi\|) \quad \forall \xi \in \mathbb{R}^m. \tag{3.34}$$

Indeed, if (3.34) holds, then it follows from (3.32) and an application of Theorem 3.2 that (A_{K_1}, B, B_e, LC, g) is ISS, and consequently, by (3.33), the Lur'e system (A, B, B_e, C, f) is also ISS.

We proceed to establish the existence of a function $\beta \in \mathcal{K}_\infty$ such that (3.34) holds. To this end, note that

$$\|g(\xi) + LL^\sharp\xi\|^2 = \|f(L^\sharp\xi) - K_1L^\sharp\xi + LL^\sharp\xi\|^2 = \|f(L^\sharp\xi) - KL^\sharp\xi\|^2 \quad \forall \xi \in \mathbb{R}^m.$$

In conjunction with (3.29) and (3.30) this leads to

$$\|g(\xi) + LL^\sharp\xi\|^2 \leq \|LL^\sharp\xi\|^2 - \|L^\sharp\xi\|\alpha(\|L^\sharp\xi\|) \quad \forall \xi \in \mathbb{R}^m.$$

Let $\xi \in \mathbb{R}^m$ and decompose $\xi = \xi_1 + \xi_2$, where

$$\xi_1 \in \text{im } L = (\ker L^*)^\perp = (\ker L^\sharp)^\perp \quad \text{and} \quad \xi_2 \in (\text{im } L)^\perp = \ker L^* = \ker L^\sharp.$$

Then $\|LL^\sharp\xi\| = \|LL^\sharp\xi_1\| = \|\xi_1\|$. Moreover, there exists $c > 0$ such that

$$\|L^\sharp\xi\| \geq c\|\xi\| \quad \forall \xi \in (\ker L^\sharp)^\perp.$$

It follows that

$$\begin{aligned} \|g(\xi) + LL^\sharp \xi\|^2 &\leq \|\xi_1\|^2 - c\|\xi_1\|\alpha(c\|\xi_1\|) \\ &= \|\xi\|^2 - (c\|\xi_1\|\alpha(c\|\xi_1\|) + \|\xi_2\|^2) \quad \forall \xi \in \mathbb{R}^m. \end{aligned} \tag{3.35}$$

Defining $\beta \in \mathcal{K}_\infty$ by

$$\beta(s) := \frac{1}{4} \min\{c\alpha(cs/2), s/2\} \quad \forall s \geq 0,$$

we have that

$$4s\beta(2s) = \min\{cs\alpha(cs), s^2\} \quad \forall s \geq 0. \tag{3.36}$$

Now

$$\begin{aligned} \sqrt{s_1^2 + s_2^2} \beta\left(\sqrt{s_1^2 + s_2^2}\right) &\leq (s_1 + s_2)\beta(s_1 + s_2) \\ &\leq 2s_1\beta(2s_1) + 2s_2\beta(2s_2) \quad \forall s_1, s_2 \geq 0, \end{aligned}$$

and thus, by (3.36),

$$2\sqrt{s_1^2 + s_2^2} \beta\left(\sqrt{s_1^2 + s_2^2}\right) \leq cs_1\alpha(cs_1) + s_2^2 \quad \forall s_1, s_2 \geq 0.$$

This, in combination with (3.35), yields

$$\|g(\xi) + LL^\sharp \xi\|^2 \leq \|\xi\|^2 - 2\|\xi\|\beta(\|\xi\|) \leq (\|\xi\| - \beta(\|\xi\|))^2 \quad \forall \xi \in \mathbb{R}^m,$$

showing that (3.34) holds and completing the proof. □

We recall that $H \in \mathbb{R}(s)^{m \times m}$ is said to be strictly positive real if there exists $\varepsilon > 0$ such that the rational matrix function $s \mapsto H(s - \varepsilon)$ is positive real.

Corollary 3.11 *Let $(A, B, B_e, C) \in \Sigma_e$, $f : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be locally Lipschitz, let G denote the transfer function of (A, B, C) , and let $K_1, K_2 \in \mathbb{R}^{m \times p}$ be such that $\ker(K_1 - K_2) = \{0\}$. If (A, B, C) is stabilizable and detectable, $(I - K_2G)(I - K_1G)^{-1}$ is strictly positive real and*

$$\langle f(\xi) - K_1\xi, f(\xi) - K_2\xi \rangle \leq 0 \quad \forall \xi \in \mathbb{R}^p, \tag{3.37}$$

then the Lur’e system (A, B, B_e, C, f) is ISS.

Note that the assumptions in Corollary 3.11 are identical to those imposed in the “classical” circle criterion which guarantees global asymptotic stability, see, for example, [6, Theorem 5.1], [7, Corollary 5.8] and [16, Theorem 7.1].² Interestingly, Corollary 3.11 shows that the conditions of the circle criterion are actually sufficient for

² Whilst in these results it is assumed that the linear system (A, B, C) is controllable and observable, Corollary 3.11 requires only stabilizability and detectability.

ISS. Also note that if $\ker(K_1 - K_2)$ is non-trivial, then, in general, Corollary 3.11 does not hold: indeed, if $F \in \mathbb{R}^{m \times p}$ is such that $G(I - FG)^{-1} \notin H^\infty(\mathbb{C}^{p \times m})$ (that is, the feedback gain F is not stabilizing), $f(\xi) = F\xi$ and $K_1 = K_2 = F$, then $(I - K_2G)(I - K_1G)^{-1} = I$ is trivially strictly positive real and (3.37) is satisfied, but 0 is not an asymptotically stable equilibrium of the (uncontrolled) Lur’e system.

The following lemma will be useful in the proof of Corollary 3.11.

Lemma 3.12 *Let $H \in \mathbb{R}(s)^{m \times m}$ be proper and assume that $H(\infty) + H^*(\infty)$ is positive definite. Then H is strictly positive real if, and only if, $H \in H^\infty(\mathbb{C}^{m \times m})$ and $H(i\omega) + H^*(i\omega)$ is positive definite for all $\omega \in \mathbb{R}$.*

The above lemma is an immediate consequence of a standard characterization of the strict positive real property, see, for example [7, Theorem 5.17] or [16, Lemma 6.1].

Proof of Corollary 3.11 Set $M := K_2 - K_1$, let $\rho \geq 0$ and define

$$H_\rho := (I - (K_2 + \rho M)G) (I - (K_1 - \rho M)G)^{-1}.$$

By hypothesis, H_0 is strictly positive real. We claim that there exists $\hat{\rho} > 0$ such that H_ρ is strictly positive real for all $\rho \in [0, \hat{\rho}]$. To this end, note that

$$H_\rho = I - (1 + 2\rho)MG(I - (K_1 - \rho M)G)^{-1}. \tag{3.38}$$

Since H_0 is strictly positive real, Lemma 3.12 yields that $H_0 \in H^\infty(\mathbb{C}^{m \times m})$ and, furthermore, there exists $\delta > 0$ such that

$$H_0(i\omega) + H_0^*(i\omega) \geq \delta I \quad \forall \omega \in \mathbb{R}. \tag{3.39}$$

Since $\ker M = \{0\}$, the matrix M is left invertible, and it follows from (3.38) (with $\rho = 0$) that $G(I - K_1G)^{-1} \in H^\infty(\mathbb{C}^{p \times m})$. Consequently, there exists $\tilde{\rho} > 0$ such that $G(I - (K_1 - \rho M)G)^{-1} \in H^\infty(\mathbb{C}^{p \times m})$ for all $\rho \in [0, \tilde{\rho}]$ and the map

$$[0, \tilde{\rho}] \rightarrow H^\infty(\mathbb{C}^{m \times m}), \quad \rho \mapsto H_\rho$$

is continuous. Invoking (3.39), we conclude that there exists $\hat{\rho} \in (0, \tilde{\rho}]$ such that, for each $\rho \in [0, \hat{\rho}]$, $H_\rho(i\omega) + H_\rho^*(i\omega) \geq (\delta/2)I$ for all $\omega \in \mathbb{R}$. An application of Lemma 3.12 shows that, for all $\rho \in [0, \hat{\rho}]$, H_ρ is strictly positive real and, a fortiori, positive real.

The claim will follow from Corollary 3.10, provided we can show that, for $\rho \in (0, \hat{\rho}]$, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\langle f(\xi) - (K_1 - \rho M)\xi, f(\xi) - (K_2 + \rho M)\xi \rangle \leq -\alpha(\|\xi\|)\|\xi\| \quad \forall \xi \in \mathbb{R}^p. \tag{3.40}$$

Invoking (3.37), a straightforward calculation shows that

$$\langle f(\xi) - (K_1 - \rho M)\xi, f(\xi) - (K_2 + \rho M)\xi \rangle \leq -\rho(\rho + 1)\|M\xi\|^2 \quad \forall \xi \in \mathbb{R}^p.$$

By left invertibility of M , there exists $\mu > 0$ such that $\|M\xi\| \geq \mu\|\xi\|$ for all $\xi \in \mathbb{R}^p$, and so,

$$\langle f(\xi) - (K_1 - \rho M)\xi, f(\xi) - (K_2 + \rho M)\xi \rangle \leq -\mu\rho(\rho + 1)\|\xi\|^2 \quad \forall \xi \in \mathbb{R}^p,$$

showing that (3.40) holds with $\alpha(s) = \mu\rho(\rho + 1)s$. □

We now reformulate the sector condition (3.29) in the special case wherein (A, B, C) is a single-input single-output system (that is, $m = p = 1$). In the single-input single-output setting, this reformulation seems more natural than (3.29).

Corollary 3.13 *Let $(A, B, B_e, C) \in \Sigma_e$, where (A, B, C) is a single-input single-output system (that is, $m = p = 1$). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be locally Lipschitz, let $k_1 < k_2$ and let G denote the transfer function of (A, B, C) . Assume that (A, B, C) is stabilizable and detectable and that $(1 - k_2G)/(1 - k_1G)$ is positive real. If there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$k_1\xi^2 + \alpha(|\xi|)|\xi| \leq f(\xi)\xi \leq k_2\xi^2 - \alpha(|\xi|)|\xi| \quad \forall \xi \in \mathbb{R}, \tag{3.41}$$

then the Lur'e system (A, B, B_e, C, f) is ISS.

Note that there exists $\alpha \in \mathcal{K}_\infty$ such that (3.41) holds if, and only if,

$$k_1\xi^2 < f(\xi)\xi < k_2\xi^2 \quad \forall \xi \in \mathbb{R}, \quad \xi \neq 0$$

and

$$|f(\xi) - k_i\xi| \rightarrow \infty \quad \text{as } |\xi| \rightarrow \infty, \quad i = 1, 2.$$

Proof of Corollary 3.13 The result will follow from Corollary 3.10, provided we show that there exists $\beta \in \mathcal{K}_\infty$ such that

$$(f(\xi) - k_1\xi)(f(\xi) - k_2\xi) \leq -\beta(|\xi|)|\xi| \quad \forall \xi \in \mathbb{R}. \tag{3.42}$$

To this end, set

$$k := \frac{k_1 + k_2}{2} \quad \text{and} \quad r := \frac{k_2 - k_1}{2} > 0,$$

and note that, by (3.41),

$$-r\xi^2 + \alpha(|\xi|)|\xi| \leq f(\xi)\xi - k\xi^2 \leq r\xi^2 - \alpha(|\xi|)|\xi| \quad \forall \xi \in \mathbb{R},$$

or, equivalently,

$$|f(\xi) - k\xi| \leq r|\xi| - \alpha(|\xi|) \quad \forall \xi \in \mathbb{R}. \tag{3.43}$$

Hence,

$$(f(\xi) - k\xi)^2 - r^2\xi^2 \leq -2r|\xi|\alpha(|\xi|) + \alpha^2(|\xi|) \quad \forall \xi \in \mathbb{R}.$$

Since

$$(f(\xi) - k\xi)^2 = (f(\xi) - k_1\xi)(f(\xi) - k_2\xi) + k^2\xi^2 - k_1k_2\xi^2 \quad \forall \xi \in \mathbb{R}$$

and $k^2 - r^2 = k_1k_2$, it follows that

$$(f(\xi) - k_1\xi)(f(\xi) - k_2\xi) \leq -2r|\xi|\alpha(|\xi|) + \alpha^2(|\xi|) \quad \forall \xi \in \mathbb{R}.$$

Finally, by (3.43), $\alpha(s) \leq rs$ for all $s \geq 0$, implying that

$$(f(\xi) - k_1\xi)(f(\xi) - k_2\xi) \leq -2r|\xi|\alpha(|\xi|) + r|\xi|\alpha(|\xi|) = -r|\xi|\alpha(|\xi|) \quad \forall \xi \in \mathbb{R}.$$

Consequently, (3.42) holds with $\beta := r\alpha$. □

Example 3.14 Consider the one-dimensional linear system $\dot{x} = u + v$ with feedback $u = f(x)$, resulting in the Lur’e system

$$\dot{x}(t) = f(x(t)) + v(t). \tag{3.44}$$

Here we have $(A, B, B_e, C) = (0, 1, 1, 1)$ and $G(s) = 1/s$. Let $k_1 < 0$ and $k_2 = 0$. Note that, for every $k_1 < 0$,

$$\frac{1 - k_2G(s)}{1 - k_1G(s)} = \frac{s}{s - k_1}$$

is positive real (but not strictly positive real). Let f be given by

$$f(\xi) = \begin{cases} -\xi^3 & \text{for } |\xi| \leq 1 \\ -\text{sgn}(\xi)(\ln(|\xi|) + 1) & \text{for } |\xi| > 1. \end{cases} \tag{3.45}$$

It is clear that, for any $k_1 < -1$, $k_1\xi^2 < f(\xi)\xi < 0$ for all $\xi \neq 0$, and, as $|\xi| \rightarrow \infty$, we have that $|f(\xi) - k_1\xi| \rightarrow \infty$ and $|f(\xi)| \rightarrow \infty$. Consequently, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$k_1\xi^2 + \alpha(|\xi|)|\xi| \leq f(\xi)\xi \leq -\alpha(|\xi|)|\xi| \quad \forall \xi \in \mathbb{R}.$$

It follows now from Corollary 3.13 that the Lur’e system (3.44) is ISS. Note that the equilibrium 0 of the uncontrolled ($v = 0$) system (3.44) is not exponentially stable. Also note that if f is replaced by a saturating nonlinearity g , for example,

$$g(\xi) = \begin{cases} -\xi^3 & \text{for } |\xi| \leq 1 \\ -\text{sgn}(\xi) & \text{for } |\xi| > 1, \end{cases}$$

then, by Proposition 3.4, the resulting Lur’e system is not ISS. □

4 Conclusions

We have developed an ISS theory for a class of Lur'e systems. The main result of this paper (Theorem 3.2) is an ISS result which is reminiscent of the complexified Aizerman conjecture in the following sense: if every linear feedback gain F in the complex ball $\mathbb{B}_{\mathbb{C}}(K, r)$ stabilizes the system (A, B, C) , then the Lur'e system $\dot{x} = Ax + Bf(Cx) + B_e v$ is ISS for every locally Lipschitz nonlinearity f for which there exists $\alpha \in \mathcal{K}_{\infty}$ such that $\|f(\xi) - K\xi\| \leq r\|\xi\| - \alpha(\|\xi\|)$ for all ξ . As corollaries we have obtained a new nonlinear small-gain condition for ISS of Lur'e systems (Corollary 3.8) and several ISS versions of the classical circle criterion (Corollaries 3.10, 3.11 and 3.13).

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