Results. Math. 67 (2015), 333–344

1422-6383/15/030333-12
published online February 4, 2015
DOI 10.1007/s00025-015-0440-8

Results in Mathematics

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Popoviciu Type Equations on Cylinders

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Abstract. Using a correspondence between the Popoviciu type functional equations and the Fréchet equation we investigate the solutions of the Popoviciu type functional equations on cylinders.

Mathematics Subject Classification. 39B12, 39B22.

Keywords. Popoviciu inequality, Popoviciu type equation, Fréchet equation, cylinder.

1. Introduction

In 1965 Popoviciu [18] proved that if $I \subset \mathbb{R}$ is an interval then a continuous function $f : I \to \mathbb{R}$ is convex if and only if f satisfies the following inequality

$$3F\left(\frac{x+y+z}{3}\right) + F(x) + F(y) + F(z) \geq 2\left[F\left(\frac{x+y}{2}\right) + F\left(\frac{y+z}{2}\right) + F\left(\frac{z+x}{2}\right)\right] \quad \text{for } x, y, z \in I. \tag{1}$$

Inequality (1) is known as *the Popoviciu inequality* (cf. e.g. [17]). In [18] it has been also proved that the only continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the equation

$$3F\left(\frac{x+y+z}{3}\right) + F(x) + F(y) + F(z) = 2\left[F\left(\frac{x+y}{2}\right) + F\left(\frac{y+z}{2}\right) + F\left(\frac{z+x}{2}\right)\right] \tag{2}$$

satisfied for all $x, y, z \in \mathbb{R}$, are the affine functions. This result has been generalized by Trif [21], who has proved that if X and Y are real linear spaces, then a function $f : X \to Y$ satisfies Eq. (2) for all $x, y, z \in X$ if and only if there exist an additive function $A : X \to Y$ and a $B \in Y$ such that $f(x) = A(x) + B$

for $x \in X$. Stability of Eq. (2) has been investigated in [3]. Solutions and stability problem for the following generalizations of Eq. (2) have been studied, among others, by:

- Lee [15], Smajdor [20], Trif [22]:

$$9f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) = 4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]; \tag{3}$$

- Lee [16]:

$$m^2f\left(\frac{x+y+z}{m}\right)+f(x)+f(y)+f(z) = n^2\left[f\left(\frac{x+y}{n}\right)+f\left(\frac{y+z}{n}\right)+f\left(\frac{z+x}{n}\right)\right]; \tag{4}$$

- Lee and Lee [14], Smajdor [19]:

$$sf\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z) = t\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]. \tag{5}$$

Popoviciu type functional equations on groups have been investigated in [4] and [5]. In this paper we consider Eq. (2) on cylinders. More precisely, given two groups $(G,+)$ and $(H,+)$ and nonempty subsets A and B of G, we deal with the equation

$$MF\left(\frac{x+y+z}{m}\right)+F(x)+F(y)+F(z) = N\left[F\left(\frac{x+y}{n}\right)+F\left(\frac{y+z}{n}\right)+F\left(\frac{z+x}{n}\right)\right] \quad \text{for } (x,y,z)\in G\times A\times B. \tag{6}$$

The paper is organized as follows. In the next section we show that there is a natural correspondence between the solutions of (6) and the solution of the Fréchet equation on cylinders, that is equation of the form

$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(z+x) \quad \text{for } (x,y,z)\in G\times A\times B. \tag{7}$$

In Sects. 3 and 4 we deal with the solution of (7) and (6), respectively. In the last section we present some comments and remarks concerning the solutions of (7).

Our results are motivated by the recent papers [1] and [2], where the solutions of the d'Alembert functional equation on cylinders have been considered.

A similar problem for the Cauchy equation has been earlier studied in [6] and [7].

2. Correspondence Between Popoviciu Type Equations and Fréchet Equation

A crucial role in our considerations will play the following result describing a correspondence between the Popoviciu type functional equations and the Fréchet equation. In order to formulate the result let us recall that, given a positive integer k, a semigroup $(G,+)$ is said to be (*uniquely*) *divisible by* k provided for every $y \in G$ there exists a (unique) $x \in G$ such that $kx = y$.

Theorem 2.1. *Let m, n, M and N be positive integers. Assume that $(G,+)$ is a commutative semigroup with* 0*, uniquely divisible by m and n and $(H,+)$ is a commutative group. Let A and B be subsets of G containing* 0*. Then a function $F : G \to H$ satisfies Eq.* (6) *if and only if there exist a function $f : G \to H$ satisfying* (7)*, a function $a : G \to H$ and a $b \in H$ such that*

$$(M - 3N + 3)b = 0, \tag{8}$$

$$Mf\left(\frac{x}{m}\right) - f(x) = 2a(x) \quad \textit{for } x \in G, \tag{9}$$

$$Nf\left(\frac{x}{n}\right) - f(x) = a(x) \quad \textit{for } x \in G, \tag{10}$$

$$a(x+y) = a(x) + a(y) \quad \textit{for } (x,y) \in G \times (A \cup B), \tag{11}$$

and

$$F(x) = f(x) + b \quad \textit{for } x \in G. \tag{12}$$

Proof. Assume that $F : G \to H$ satisfies (6). Let $b := F(0)$. Then, applying (6) with $x = y = z = 0$, we get (8). Furthermore, setting in (6) $y = z = 0$, we obtain

$$MF\left(\frac{x}{m}\right) + F(x) = 2NF\left(\frac{x}{n}\right) + (N-2)b \quad \text{for } x \in G. \tag{13}$$

Hence, for every $x, y, z \in G$, we have

$$MF\left(\frac{x+y+z}{m}\right) = 2NF\left(\frac{x+y+z}{n}\right) + (N-2)b - F(x+y+z),$$

so making use of (6), we get

$$\begin{aligned} &2NF\left(\frac{x+y+z}{n}\right) + (N-2)b - F(x+y+z) + F(x) + F(y) + F(z) \\ &\quad = N\left[F\left(\frac{x+y}{n}\right) + F\left(\frac{y+z}{n}\right) + F\left(\frac{z+x}{n}\right)\right] \\ &\qquad \text{for } (x,y,z) \in G \times A \times B. \end{aligned}$$

Putting in the last equality $y=0$ and then $z=0$, we obtain

$$2NF\left(\frac{x+z}{n}\right)+(N-1)b-F(x+z)+F(x)+F(z)$$
$$=N\left[F\left(\frac{x}{n}\right)+F\left(\frac{z}{n}\right)+F\left(\frac{z+x}{n}\right)\right] \text{ for } (x,z)\in G\times B$$

and

$$2NF\left(\frac{x+y}{n}\right)+(N-1)b-F(x+y)+F(x)+F(y)$$
$$=N\left[F\left(\frac{x+y}{n}\right)+F\left(\frac{y}{n}\right)+F\left(\frac{x}{n}\right)\right] \text{ for } (x,y)\in G\times A,$$

respectively. Therefore

$$NF\left(\frac{x+z}{n}\right)-F(x+z)-(N-1)b$$
$$=NF\left(\frac{x}{n}\right)-F(x)-(N-1)b+NF\left(\frac{z}{n}\right)-F(z)-(N-1)b \text{ for } (x,z)\in G\times B \tag{14}$$

and

$$NF\left(\frac{x+y}{n}\right)-F(x+y)-(N-1)b$$
$$=NF\left(\frac{x}{n}\right)-F(x)-(N-1)b+NF\left(\frac{y}{n}\right)-F(y)-(N-1)b \text{ for } (x,y)\in G\times A. \tag{15}$$

Let $a:G\to H$ be given by

$$a(x)=NF\left(\frac{x}{n}\right)-F(x)-(N-1)b \quad \text{for} \quad x\in G. \tag{16}$$

Then from (14) and (15) we derive (11). Moreover, in view of (13) and (16), we obtain

$$2NF\left(\frac{x}{n}\right)=MF\left(\frac{x}{m}\right)+F(x)-(N-2)b \quad \text{for } x\in G$$

and

$$NF\left(\frac{x}{n}\right)=a(x)+F(x)+(N-1)b \quad \text{for} \quad x\in G, \tag{17}$$

respectively. Hence

$$MF\left(\frac{x}{m}\right)=2a(x)+F(x)+(3N-4)b \quad \text{for } x\in G. \tag{18}$$

Thus

$$MF\left(\frac{x+y+z}{m}\right)=2a(x+y+z)+F(x+y+z)+(3N-4)b$$
$$\text{for } x,y,z\in G$$

and so (6) becomes

$$\begin{aligned}&2a(x+y+z)+F(x+y+z)+(3N-4)b+F(x)+F(y)+F(z)\\&\quad=N\left[F\left(\frac{x+y}{n}\right)+F\left(\frac{y+z}{n}\right)+F\left(\frac{z+x}{n}\right)\right]\\&\qquad\text{for }\ (x,y,z)\in G\times A\times B.\end{aligned}$$

Therefore, making use of (17), we get

$$\begin{aligned}&2a(x+y+z)+F(x+y+z)+(3N-4)b+F(x)+F(y)+F(z)\\&\quad=a(x+y)+F(x+y)+(N-1)b+a(y+z)+F(y+z)+(N-1)b\\&\qquad+a(z+x)+F(z+x)+(N-1)b\ \text{ for }\ (x,y,z)\in G\times A\times B\end{aligned}$$

which, together with (11), gives

$$\begin{aligned}&F(x+y+z)+F(x)+F(y)+F(z)\\&\quad=F(x+y)+F(y+z)+F(z+x)+b\ \text{ for }\ (x,y,z)\in G\times A\times B.\end{aligned}$$

Consequently, taking $f:=F-b$, we obtain (7) and (12). Moreover, in view of (8), (12) and (18), we get

$$Mf\left(\frac{x}{m}\right)-f(x)=MF\left(\frac{x}{m}\right)-F(x)+(1-M)b=2a(x)\quad\text{for }\ x\in G.$$

Thus (9) holds. Note also that taking into account (8), (12) and (17), we obtain

$$Nf\left(\frac{x}{n}\right)-f(x)=NF\left(\frac{x}{n}\right)-F(x)+(1-N)b=a(x)\quad\text{for }\ x\in G,$$

which implies (10).

The converse is easy to check. □

3. Fréchet Equation on Cylinders

According to Theorem 2.1, the Popoviciu type functional equations are closely related to Eq. (7), being the Fréchet equation on a cylinder. It is remarkable that the Fréchet equation

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x), \qquad (19)$$

known also as the Deeba equation (cf. [8,11]), is strictly related to the problem of characterization of inner product spaces. It is well known that a normed space $(X,\|\|)$ is an inner product space if and only if the function $f:X\to[0,\infty)$ given by $f(x)=\|x\|^2$ for $x\in X$, satisfies Eq. (19) for all $x,y,z\in X$. The solutions of (19) in a more general setting have been considered in [11] (see also [12]). Various aspects of stability problem for (19) have been studied in [8–10].

The following result will play an important role in the considerations of this section.

Proposition 3.1. *Let $(G,+)$ be a commutative semigroup, $(H,+)$ be a commutative group. Assume that A and B are nonempty subsets of G and a function $f: G \to H$ satisfies Eq. (7). Then the following two sets*

$$P_1(B) := \{y \in G : f(x+y+z) + f(x) + f(y) + f(z) \\ = f(x+y) + f(y+z) + f(z+x) \ \text{for} \ (x,z) \in G \times B\}$$

and

$$P_2(A) := \{z \in G : f(x+y+z) + f(x) + f(y) + f(z) \\ = f(x+y) + f(y+z) + f(z+x) \ \text{for} \ (x,y) \in G \times A\}$$

are subsemigroups of $(G,+)$ containing A and B, respectively. Moreover, if $(G,+)$ is a group then the sets $P_1(B)$ and $P_2(A)$ are subgroups of $(G,+)$.

Proof. From (7) it follows that $A \subset P_1(B)$ and $B \subset P_2(A)$, so $P_1(B)$ and $P_2(A)$ are nonempty. Let $y_1, y_2 \in P_1(B)$. Then, applying (7), we get

$$\begin{aligned} & f(x+(y_1+y_2)+z) + f(x) + f(y_1+y_2) + f(z) \\ &\quad = f((x+y_1)+y_2+z) + f(x) + f(y_1+y_2) + f(z) \\ &\quad = f(x+y_1+y_2) + f(y_2+z) + f(z+x+y_1) - f(x+y_1) \\ &\qquad - f(y_2) - f(z) + f(x) + f(y_1+y_2) + f(z) \\ &\quad = f(x+y_1+y_2) + f(y_2+z) + f(x+y_1) + f(y_1+z) + f(z+x) \\ &\qquad - f(x) - f(y_1) - f(z) - f(x+y_1) - f(y_2) + f(x) + f(y_1+y_2) \\ &\quad = f(x+y_1+y_2) + f(y_2+z) + f(z+y_1) + f(y_1+y_2) - f(z) \\ &\qquad - f(y_1) - f(y_2) + f(z+x) \\ &\quad = f(x+y_1+y_2) + f(y_1+y_2+z) + f(z+x) \quad \text{for} \quad (x,z) \in G \times B \end{aligned}$$

whence $y_1 + y_2 \in P_1(B)$. Thus $P_1(B)$ is a subsemigroup of $(G,+)$. The same arguments show that $P_2(B)$ is a subsemigroup of $(G,+)$.

Now, assume that $(G,+)$ is a group. As previously, we present the proof only for $P_1(B)$. Since we have already proved that $P_1(B)$ is a subsemigroup of $(G,+)$, it is enough to show that $-y \in P_1(B)$ for every $y \in P_1(B)$. To this end fix a $y \in P_1(B)$. Note that taking in (7) $x = 0$, we get $f(0) = 0$. Therefore, setting in (7) $x = -y$, we obtain

$$f(z) + f(-y) + f(y) + f(z) = f(y+z) + f(z-y) \quad \text{for} \quad z \in B,$$

that is

$$f(y+z) = 2f(z) + f(-y) + f(y) - f(-y+z) \quad \text{for} \quad z \in B. \tag{20}$$

On the other hand, in view of (7), for every $(x,z) \in G \times B$, we have

$$\begin{aligned} f(x+z) &= f((x-y)+y+z) \\ &= f((x-y)+y) + f(y+z) + f(z+x-y) \\ &\quad - f(x-y) - f(y) - f(z) \end{aligned}$$

and so

$$f(x-y+z)=f(x+z)+f(x-y)+f(y)+f(z)-f(x)-f(y+z).$$

Thus, making use of (20), for every $(x,z)\in G\times B$, we get

$$f(x-y+z)+f(x)+f(-y)+f(z)=f(x-y)+f(-y+z)+f(z+x).$$

This means that $-y\in P_1(B)$. □

Corollary 3.2. *Let $(G,+)$ be a commutative (semi)group and $(H,+)$ be a commutative group. Assume that A and B are nonempty subsets of G and a function $f:G\to H$ satisfies Eq. (7). Then*

$$\begin{aligned}&f(x+y+z)+f(x)+f(y)+f(z)\\&\quad=f(x+y)+f(y+z)+f(z+x)\quad\textit{for}\ \ (x,y,z)\in G\times G(A)\times G(B),\end{aligned}\tag{21}$$

where $G(A)$ and $G(B)$ are sub(semi)groups of $(G,+)$ generated by A and B, respectively.

Proof. According to Proposition 3.1, $P_1(B)$ is a sub(semi)group of the (semi) group $(G,+)$ containing A. Hence $G(A)\subset P_1(B)$ and so

$$\begin{aligned}&f(x+y+z)+f(x)+f(y)+f(z)\\&\quad=f(x+y)+f(y+z)+f(z+x)\quad\text{for}\ \ (x,y,z)\in G\times G(A)\times B.\end{aligned}\tag{22}$$

Applying again Proposition 3.1, we obtain that $P_2(G(A))$ is a sub(semi)group of the (semi)group $(G,+)$ containing B. Thus $G(B)\subset P_2(G(A))$, which together with (22) gives (21). □

From Corollary 3.2 we derive the following result.

Theorem 3.3. *Let $(G,+)$ be a commutative (semi)group and $(H,+)$ be a commutative group. Assume that A and B are subsets of G such that $G(A)=G(B)=G$, where $G(A)$ and $G(B)$ are sub(semi)groups of $(G,+)$ generated by A and B, respectively. Then every function $f:G\to H$ satisfying Eq. (7) satisfies equation*

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x)\quad\textit{for}\ \ x,y,z\in G.\tag{23}$$

The following example shows that the assumption $G(A)=G(B)=G$ is essential in Theorem 3.3.

Example 3.4. Let $(G,+)=(H,+)=(\mathbb{Z},+)$, $B=2\mathbb{Z}$ and let $f:\mathbb{Z}\to\mathbb{Z}$ be of the form $f(x)=x\ (\mathrm{mod}\ 2)$ for $x\in\mathbb{Z}$. Then $f(x+y)=f(x)$ for $(x,y)\in\mathbb{Z}\times 2\mathbb{Z}$, which means that f satisfies (7) with an arbitrary nonempty set $A\subset\mathbb{Z}$. On the other hand, we have

$$f(1+1+1)+f(1)+f(1)+f(1)=4\neq 0=f(1+1)+f(1+1)+f(1+1),$$

so f does not satisfy (23).

4. Popoviciu Type Equations on Cylinders

Applying the results of the previous sections, we obtain the following two theorems concerning the solutions of the Popoviciu type equations on cylinders.

Theorem 4.1. *Let m, n, M and N be positive integers. Assume that $(G, +)$ is a commutative semigroup with* 0 *(a commutative group) uniquely divisible by m and n, and $(H, +)$ is a commutative group. Let A and B be subsets of G containing* 0*. If a function $F : G \to H$ satisfies Eq.* (6) *then*

$$\begin{aligned} &MF\left(\frac{x+y+z}{m}\right) + F(x) + F(y) + F(z) \\ &\quad = N\left[F\left(\frac{x+y}{n}\right) + F\left(\frac{y+z}{n}\right) + F\left(\frac{z+x}{n}\right)\right] \\ &\quad \textit{for } (x, y, z) \in G \times G(A) \times G(B) \end{aligned} \tag{24}$$

where, as previously, $G(A)$ and $G(B)$ are subsemigroups (subgroups) of $(G, +)$ generated by A and B, respectively.

Proof. Assume that F satisfies (6). Then, according to Theorem 2.1, there exist functions $a, f : G \to H$ and a $b \in H$ such that f satisfies (7), (8)-(11) hold and F is of the form (12). Applying Corollary 3.2, we get (21). Moreover (cf. [13, Lemma 18.5.1, p. 552]) from (11) it follows that

$$a(x + y) = a(x) + a(y) \quad \text{for} \quad (x, y) \in G \times G(A \cup B),$$

where $G(A \cup B)$ denotes the subsemigroup (subgroup) of $(G, +)$ generated by $A \cup B$. Since $G(A) \cup G(B) \subset G(A \cup B)$, from the last equality we derive that

$$a(x + y) = a(x) + a(y) \quad \text{for} \quad (x, y) \in G \times [G(A) \cup G(B)].$$

Hence, making use of (8)–(10) and (21), according to Theorem 2.1, we obtain (24). □

From Theorem 4.1 we derive the following result.

Theorem 4.2. *Let m, n, M and N be positive integers, $(G, +)$ be a commutative semigroup with 0 (a commutative group) uniquely divisible by m and n, and $(H, +)$ be a commutative group. Assume that A and B are subsets of G containing* 0 *such that $G(A) = G(B) = G$, where $G(A)$ and $G(B)$ are subsemigroups (subgroups) of $(G, +)$ generated by A and B, respectively. If a function $F : G \to H$ satisfies* (6) *then*

$$\begin{aligned} &MF\left(\frac{x+y+z}{m}\right) + F(x) + F(y) + F(z) \\ &\quad = N\left[F\left(\frac{x+y}{n}\right) + F\left(\frac{y+z}{n}\right) + F\left(\frac{z+x}{n}\right)\right] \quad \textit{for} \quad x, y, z \in G. \end{aligned}$$

5. Final Remarks

The problem of determining the general solution of (6) as well as (7) is open. Nevertheless, we are going to present some remarks concerning the solutions of (7). In the whole section, given a group $(G,+)$ and a nonempty set $D \subset G$, by $G(D)$ we denote a subgroup of $(G,+)$ generated by D.

Remark 5.1. Let $(G,+)$ and $(H,+)$ be commutative groups such that $(H,+)$ contains at least two elements. Assume that A and B are nonempty subsets of G such that at least one of the groups $G/G(A)$ and $G/G(B)$, say $G/G(A)$, contains an element of order greater than 3. Then there is an $x_0 \in G$ such that $[-x_0] \neq [2x_0]$, where $[x]$ denotes the equivalence class of x. Let $\phi : G/G(A) \to H$ be such that $\phi([0]) = \phi([x_0]) = 0$ and $\phi([-x_0]) \neq \phi([2x_0])$. Furthermore, let $f : G \to H$ be given by $f(x) = \phi([x])$ for $x \in G$. Then, for every $(x,y,z) \in G \times A \times B$, we get $f(x+y+z) = f(x+z)$, $f(x+y) = f(x)$ and $f(y+z) = f(z)$ whence f satisfies (7). On the other hand, we have

$$f(x_0 + x_0 - x_0) + f(x_0) + f(x_0) + f(-x_0) = 3\phi([x_0]) + \phi([-x_0]) = \phi([-x_0])$$

and

$$f(x_0 + x_0) + f(x_0 - x_0) + f(-x_0 + x_0) = \phi([2x_0]) + 2\phi([0]) = \phi([2x_0]).$$

Since $\phi([-x_0]) \neq \phi([2x_0])$, this means that f does not satisfy (23).

Remark 5.2. Let $(G,+)$ and $(H,+)$ be commutative groups such that $(H,+)$ contains at least one non-zero element a of order different from 3. Assume that A and B are nonempty subsets of G such that $G(A) \neq G$ and every non-zero element of the quotient group $G/G(A)$ has the order 3. Fix an $x_0 \in G\backslash G(A)$. Let $\phi : G/G(A) \to H$ be such that $\phi([0]) = 0$ and $\phi([x_0]) = a$. Furthermore, let $f : G \to H$ be given by $f(x) = \phi([x])$ for $x \in G$. Then, arguing as in Remark 5.1, one can easily check that f satisfies (7). On the other hand, as $[-x_0] = [2x_0]$, we have

$$\begin{aligned} &f(x_0 + x_0 - x_0) + f(x_0) + f(x_0) + f(-x_0) \\ &\quad = 3\phi([x_0]) + \phi([-x_0]) = 3a + \phi([2x_0]) \end{aligned}$$

and

$$f(x_0 + x_0) + f(x_0 - x_0) + f(-x_0 + x_0) = \phi([2x_0]) + 2\phi([0]) = \phi([2x_0]).$$

Since the order of a is different from 3, this means f does not satisfy (23).

The same arguments work also in the case where $G(B) \neq G$ and every non-zero element of the quotient group $G/G(B)$ has the order 3.

Remark 5.3. Let $(G,+)$ and $(H,+)$ be commutative groups such that $(H,+)$ contains at least one non-zero element a of order different from 2 and 4. Assume that A and B are nonempty subsets of G such that $G(A) \neq G$ and every non-zero element of the quotient group $G/G(A)$ has the order 2. Let $f : G \to H$ be of the form

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in G(A), \\ a & \text{for} \quad x \in G\setminus G(A). \end{cases} \tag{25}$$

Then, as previously, f satisfies (7). On the other hand, taking $x = y = z \in G\setminus G(A)$, we have $x+y$, $y+z$, $z+x \in G(A)$ and $x+y+z \notin G(A)$, so

$$f(x+y+z) + f(x) + f(y) + f(z) = 4a \neq 0 = f(x+y) + f(y+z) + f(z+x).$$

Thus f does not satisfy (23).

The same arguments show that if $G(B) \neq G$ and every non-zero element of the quotient group $G/G(B)$ has the order 2, then the function f given by (25) satisfies (7), but it does not satisfy (23).

If $(G, +)$ and $(H, +)$ are commutative groups such that $(H, +)$ contains at least two elements and A, B are nonempty subsets of G such that every solution $f : G \to H$ of (7) satisfies (23) then, according to Remarks 5.1–5.3, one of the following two conditions holds:

(i) every non-zero element of the groups $G/G(A)$, $G/G(B)$ and $(H, +)$ has the order 3;

(ii) every non-zero element of the groups $G/G(A)$ and $G/G(B)$ has the order 2 and every non-zero element of $(H, +)$ has either the order 2 or 4.

The next two examples show that, in general, neither (i) nor (ii) implies that every solution $f : G \to H$ of (7) satisfies (23).

Example 5.4. Let $(G, +) = (\mathbb{Z}_9, +)$, $(H, +) = (\mathbb{Z}_3, +)$ and $A = B = \{3\}$. Then $G(A) = G(B) = \{0, 3, 6\}$ and so (i) is valid. Let $f : G \to H$ be of the form

$$f(x) = \begin{cases} 0 & \text{for} \quad x = 0, \\ 2 & \text{for} \quad x = 6, \\ 1 & \text{otherwise.} \end{cases}$$

Then a straightforward calculation shows that f satisfies (7). However, we have

$$f(1+1+1) + f(1) + f(1) + f(1) = 1 \neq 0 = f(1+1) + f(1+1) + f(1+1).$$

Therefore, f does not satisfy (23).

Example 5.5. Let $(G, +) = (H, +) = (\mathbb{Z}_4, +)$ and $A = B = \{2\}$. Then $G(A) = G(B) = \{0, 2\}$ whence (ii) holds. Let $f : G \to H$ be given by

$$f(x) = \begin{cases} 0 & \text{for} \quad x = 0, \\ 2 & \text{otherwise.} \end{cases}$$

Then f satisfies (7). On the other hand, we have

$$f(1+3+3) + f(1) + f(3) + f(3) = 0 \neq 2 = f(1+3) + f(3+3) + f(3+1),$$

which means that f does not satisfy (23).

We conclude the paper with a one more example.

Example 5.6. Let $(G,+)=(\mathbb{Z}_6,+)$, $(H,+)=(\mathbb{Z}_3,+)$ and $A=B=\{3\}$. Then $G(A)=G(B)=\{0,3\}$, so the order of every non-zero element of the groups $G/G(A)$, $G/G(B)$ and $(H,+)$ is equal to 3. Suppose that $f:\mathbb{Z}_6\to\mathbb{Z}_3$ satisfies Eq. (7), that is

$$f(x{+}3{+}3)+f(x)+f(3)+f(3)=f(x{+}3)+f(3{+}3)+f(x{+}3)\quad\text{for}\quad x\in\mathbb{Z}_6. \tag{26}$$

Applying (26) with $x=0$ and then with $x=3$, we obtain $f(0)=f(3)=0$. Therefore (26) becomes $2f(x)=2f(x+3)$ for $x\in\mathbb{Z}_6$ whence $f(x)=f(x+3)$ for $x\in\mathbb{Z}_6$. We show that f satisfies (23). To this end, fix $x,y,z\in\mathbb{Z}_6$. The case where $\{x,y,z\}\cap\{0,3\}\neq\emptyset$ is obvious. Furthermore, if $x,y,z\in\{1,2,4,5\}$, then both sides of (23) are equal 0 whenever $x+y+z\in\{0,3\}$; $f(2)$ whenever $x+y+z\in\{1,4\}$ and $f(1)$ whenever $x+y+z\in\{2,5\}$. Therefore f satisfies (23). So, every solution of (26) satisfies (23).

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Received: October 23, 2014.
Accepted: January 22, 2015.