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Valence Double Parton Distributions of the Nucleon in a Simple Model

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Abstract Valence double parton distribution functions of the nucleon are evaluated in the framework of a simple model, where the conservation of the longitudinal momentum is taken into account. The leading-order DGLAP QCD evolution from the low quark-model scale to higher renormalization scales is carried out via the Mellin moments of the distributions. Results of the valence quark correlation function show that in general the double distributions cannot be approximated as a product of the single-particle distributions.

Theoretical interest in double parton distributions (dPDFs) in hadrons has been recently renewed (see the reviews [1,2] and the references therein). The activity is largely triggered by the experimental program at the LHC, where the double parton scattering (DPS) may bring substantial contribution in certain production processes [3,4]. Phenomenological aspects of the inclusion of the multi-parton effects have been widely pursued since the early days of the parton model [5], while many formal features of the multiparton evolution [6,7] follow from a pioneering study on fragmentation functions [8]. Elements of the QCD formulation are presented in [9–12], while a ${}_2$ GPD interpretation is provided in [13]. Positivity bounds for dPDFs were recently discussed in [14].

Dynamic modeling of dPDFs has been up to now very little explored. In fact, the only calculations which nonperturbatively predict the valence dPDFs of the nucleon have been carried out in the framework of the MIT bag model [15] and in the constituent quark model [16]. In both calculations, the approximate implementation of relativity has the side effect of producing distributions with unphysical support, i.e., extending outside the interval $0 \leq x_1, x_2, x_1 + x_2 \leq 1$. An important aspect of the dPDFs is that they allow to detect tight diquark correlations, in which case the dPDF becomes a function of the sum $x_1 + x_2$. In the (x_1, x_2) distribution plot the fingerprints are the straight-line structures inclined by 45° .

The purpose of this talk is to explore the valence dPDF of the proton in a very simple model which includes the exact momentum-conservation constraints and complies to the Lorentz invariance. Satisfaction of these requirements results in proper theoretical features, such as the correct support in the x -variables or the quark-number and momentum sum rules [17]. In the considered model, the correlations between partons are due

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solely to the longitudinal momentum conservation. Importantly, we carry out the LO DGLAP evolution [6,7] of the obtained valence dPDF, confirming its crucial effects for the corresponding quark correlation function as the evolution scale is moved up to the experimentally-accessible values. We find that with the increasing renormalization scale, the correlation between the valence quarks becomes large and positive along the edges where x_1 or x_2 is small, while it is negative in the region where x_1 and x_2 are similar and relatively large. This shows in explicit terms that approximating dPDFs with a product of two single-parton distributions (sPDF) is not justified.

Similarly to sPDF, which is interpreted as the probability distribution for the probed parton to carry the fraction x of the total p^+ momentum of the hadron, the dPDF has the interpretation of the joint probabilistic distribution that the j_1 and j_2 partons carry the momentum fractions x_1 and x_2 of the hadron, respectively. Denoting the sPDF as $D_j(x)$ and dPDF as $D_{j_1 j_2}(x_1, x_2)$, one may define the parton correlation function [15,16]

$$\rho_{j_1 j_2}(x_1, x_2) = \frac{D_{j_1 j_2}(x_1, x_2)}{D_{j_1}(x_1)D_{j_2}(x_2)} - 1. \quad (1)$$

As mentioned, a dynamical nonperturbative calculation for the proton has been made in the MIT bag model [15], where the correlations stem from the conservation of the linear momentum (implemented in terms of the Peierls-Yoccoz projection). As is well known, projecting onto good linear momentum coincides with boosting from the rest frame only for the exact eigenstates of relativistic systems [18]. A constituent quark model study has been carried out in [16]. Here we consider a much simpler model which in our view grasps the essential features of the problem and has the proper support. The role of transverse degrees of freedom is discussed at the end.

Let us introduce the probability that the parton carries the momentum fraction x as the square of its wave function, $\phi(x) = |\psi(x)|^2$. Then the *three-particle* probability distribution is proposed in the form

$$D_3(x_1, x_2, x_3) = \phi(x_1)\phi(x_2)\phi(x_3)\delta(1 - x_1 - x_2 - x_3), \quad (2)$$

where the delta function enforces the longitudinal momentum conservation. Its marginal projections define the dPDF and sPDF:

$$D_2(x_1, x_2) = \int_0^1 dx_3 D_3(x_1, x_2, x_3) = \phi(x_1)\phi(x_2)\phi(1 - x_1 - x_2), \quad (3)$$

$$D_1(x_1) = \int_0^1 dx_2 D_2(x_1, x_2) = \int_0^1 \theta(1 - x_1 - x_2) dx_2 \phi(x_1)\phi(x_2)\phi(1 - x_1 - x_2). \quad (4)$$

Note that Eq. (4) is equivalent to the quark number sum rule of Gaunt and Stirling (GS) [17]. The momentum GS sum rule is also satisfied, as it is straightforward to show that

$$\int_0^{1-x_1} dx_2 x_2 D_2(x_1, x_2) + \int_0^{1-x_1} dx_3 x_3 D_2(x_1, x_3) = (1 - x_1)D_1(x_1). \quad (5)$$

As a matter of fact, the only feature used in these derivations is the symmetric form of Eq. (2). In addition, due to the symmetry, the extremum of $D_3(x_1, x_2, x_3)$ is located at the symmetric point $x_1 = x_2 = x_3 = 1/3$ regardless of its particular shape.

Early multiparton models [5] take the simple Regge-motivated parametrization $\psi(x) \sim x^a$ with $a = 1 - \alpha(0)$ and the intercept $\alpha(0) = 1/2$, while the constant phase-space model of Ref. [19] uses $\psi(x) = 1$. In this presentation we investigate two simple classes of models of the form

$$\phi(x) \equiv |\psi(x)|^2 = Ax^\alpha \text{ (model I)}, \quad \phi(x) \equiv |\psi(x)|^2 = A(1-x)^a \text{ (model II)}, \quad (6)$$

where A is a normalization constant such that $\int_0^1 dx \phi(x) = 1$. We consider the cases $\alpha = 0, 1/2, 1$ and $a = 1, 2$. The first class corresponds to the so-called *valon* model [20,21] for the proton, which assumes

$G_{uud/p}(x_1, x_2, x_3) = (x_1 x_2)^\alpha x_3^\beta \delta(1 - x_1 - x_2 - x_3)$ for $\alpha = \beta$. For instance, for the case of the second class with $a = 2$ in Eq. (6) we get

$$\begin{aligned} D_2(x_1, x_2) &= \frac{1008}{29} (1 - x_1)^2 (1 - x_2)^2 (x_1 + x_2)^2, \\ D_1(x) &= \frac{168}{145} (1 - x)^3 (1 + 6x + 16x^2 + 6x^3 + x^4) \end{aligned} \quad (7)$$

We note that the behavior of $D_1(x)$ near $x \rightarrow 1$ conforms to the QCD counting rules [22]. This is also the case for $\alpha = 1$ in model I of Eq. (6). As we see, at small x 's we can trace a diquark correlation.

The above forms hold by construction at the *quark-model scale*, where the only degrees of freedom are the valence quarks. To relate to results at experimental scales, the appropriate QCD evolution is necessary [23], which generates radiatively the sea quarks and gluons. The matching prescription based on the requirement that at $\mu = 2$ GeV the valence quarks carry the experimental momentum fraction of 41.6% [24] leads to a very low quark model scale, $\mu_0 = 285$ MeV (where the valence quarks carry all the momentum). This low value is compatible to the case of the pion in chiral quark models [25, 26].

The QCD evolution equations for multi-parton distributions have been derived long ago [6, 7]. The DGLAP evolution preserves the GS sum rules [17], hence their satisfaction at the quark-model scale implies satisfaction at higher scales. A simple and practical method of solving the evolution equations numerically is based on the Mellin moments, similarly to the case of sPDFs. One introduces the corresponding moments of the sPDFs and dPDFs,

$$\begin{aligned} M_j^n &= \int_0^1 dx x^n D_j(x), \\ M_{j_1 j_2}^{n_1 n_2} &= \int_0^1 dx_1 \int_0^1 dx_2 \theta(1 - x_1 - x_2) x_1^{n_1} x_2^{n_2} D_{j_1 j_2}(x_1, x_2), \end{aligned} \quad (8)$$

and the moments of the QCD splitting functions

$$\begin{aligned} P_{i \rightarrow j}^n &= \int_0^1 dx x^n P_{i \rightarrow j}(x), \\ P_{i \rightarrow j_1 j_2}^{n_1 n_2} &= \int_0^1 dx x^{n_1} (1 - x)^{n_2} P_{i \rightarrow j_1 j_2}(x), \\ \tilde{P}_{i \rightarrow j_1 j_2}^{n_1 n_2} &= \delta_{j_1 j_2} P_{i \rightarrow j_1}^{n_1 + n_2} - \delta_{i j_1} P_{j_1 \rightarrow j_2}^{n_2} - \delta_{i j_2} P_{j_2 \rightarrow j_1}^{n_1}. \end{aligned} \quad (9)$$

Then the LO dDGLAP evolution equations read [6, 7]

$$\frac{d}{dt} M_{j_1 j_2}^{n_1 n_2} = \sum_i P_{i \rightarrow j_1}^{n_1} M_{i j_2}^{n_1 n_2} + \sum_i P_{i \rightarrow j_2}^{n_2} M_{j_1 i}^{n_1 n_2} + \sum_i \left(P_{i \rightarrow j_1 j_2}^{n_1 n_2} + \tilde{P}_{i \rightarrow j_1 j_2}^{n_1 n_2} \right) M_i^{n_1 + n_2}, \quad (10)$$

where the evolution variable is

$$t = \frac{1}{2\pi\beta} \log \left[1 + \alpha_s(\mu) \beta \log(\Lambda_{\text{QCD}}/\mu) \right], \quad \beta = \frac{11N_c - 2N_f}{12\pi}.$$

Partons i , j_1 , and j_2 may in general represent the valence quarks, the sea quarks, or the gluons, whose distributions are coupled. Moreover, the last term in Eq. (10) couples dPDFs to sPDFs. In this talk we consider only the evolution of the valence quarks which leads to a technical simplification, as in this case there are no partons i decaying into a pair of valence quarks and $P_{i \rightarrow j_1 j_2} = 0$. This means that the inhomogeneous term in Eq. (10) vanishes and our evolution equation simply reads

$$\frac{d}{dt} M_{j_1 j_2}^{n_1 n_2}(t) = \left(P_{j_1 \rightarrow j_1}^{n_1} + P_{j_2 \rightarrow j_2}^{n_2} \right) M_{j_1, j_2}^{n_1 n_2}(t). \quad (11)$$

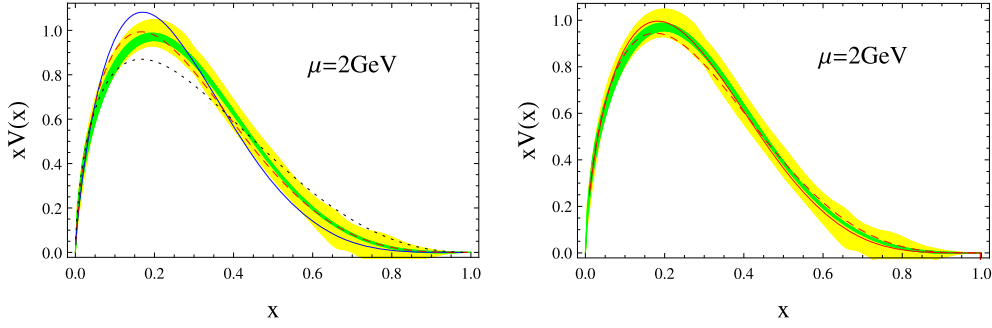


Fig. 1 The valence sPDF of the nucleon (multiplied by x) at the scale of $\mu = 2$ GeV, plotted as a function of the Bjorken variable x . The darker band corresponds to the NNPDF2.3 fit with no LHC data, while the broader light band is the the NNPDF2.3 fit with the collider data only [24]. We show sPDF evolved with the dDGLAP equations from the quark-model scale μ_0 to $\mu = 2$ GeV. *Left* The *solid*, *dashed*, and *dotted* lines correspond to the valon model $|\psi(x)|^2 = Ax^\alpha$ with $\alpha = 1, 1/2$, and 0 , respectively. *Right* The *solid* and *dashed* lines correspond to the model $|\psi(x)|^2 = A(1-x)^a$ with $a = 2$ and $a = 1$, respectively

Recall that for sPDFs the corresponding evolution equation is

$$\frac{d}{dt} M_j^n(t) = P_{j \rightarrow j}^n M_j^n(t). \quad (12)$$

Note that due to the presence of correlations we have $M_{j_1 j_2}^{n_1 n_2}(t) \neq M_{j_1}^{n_1}(t) M_{j_2}^{n_2}(t)$ and the system is not separable. The solutions of Eqs. (11, 12) are

$$M_j^n(t) = e^{P_{j \rightarrow j}^n(t-t_0)} M_j^n(t_0), \quad (13)$$

$$M_{j_1 j_2}^{n_1 n_2}(t) = e^{(P_{j_1 \rightarrow j_1}^{n_1} + P_{j_2 \rightarrow j_2}^{n_2})(t-t_0)} M_{j_1 j_2}^{n_1 n_2}(t_0). \quad (14)$$

The inverse Mellin transform brings us to the evolved solution in the x -space, namely

$$D_j(x; t) = \int_C \frac{dn}{2\pi i} x^{-n-1} M_j^n(t), \quad (15)$$

$$D_{j_1 j_2}(x_1, x_2; t) = \int_C \frac{dn_1}{2\pi i} x_1^{-n_1-1} \int_{C'} \frac{dn_2}{2\pi i} x_2^{-n_2-1} M_{j_1, j_2}^{n_1, n_2}(t), \quad (16)$$

where n and n' are treated as complex variables and the contours C and C' , lying right to all singularities of M , are bended by 45° (see e.g. Ref. [27]). The bending of the contours helps with the pace of numerical convergence. Schwarz's reflection principle $M_{n^*} = M_n^*$ is satisfied since the anomalous dimensions are real for any integer n and thus one segment of the bended contour is needed in the evaluation. The method is most practical when the moments M are analytic functions of their arguments; if they were not (as could be the case when some numerical fits to the data are made), then they should be approximated with analytic functions, e.g., with sums of the Euler Beta functions, whence the procedure may be carried out with no difficulty.

Next, we proceed to presenting our results of the evolution in the considered model. Before showing the main findings for the valence dPDF, we explore the valence sPDF, where the results can be straightforwardly compared to the existing data parameterizations. Figure 1 presents the valence quark distribution, customarily multiplied by x , evolved to the benchmark scale of $\mu = 2$ GeV and compared to the NNPDF2.3 fits [24], indicated with bands (the darker narrow band is the fit excluding the LHC data, and the broad band uses the collider data only). The reasonable reproduction of the valence sPDF gives us confidence that the simple model of Eqs. (3,4) grasps the most essential features of the valence parton distributions of the nucleon, particularly for the case $a = 2$, $\psi(x) = \sqrt{3}(1-x)$. We may thus move towards the double distributions. The effect of the evolution on the valence dPDFs is presented in Fig. 2, where we show the contour maps of $x_1 x_2 D_2(x_1, x_2)$ at three scales: the quark-model scale μ_0 , and the higher scales $\mu = 2$ GeV and $\mu = 1$ TeV. Of course, the support of the function is the region $x_1 + x_2 \leq 1$, $x_i \geq 0$. We observe how the dDGLAP equations cause the drift of the distributions towards low values of x_1 and x_2 . This drift to low values of x is well-known for the case of the DGLAP evolution of sPDFs; it has also been observed for the dPDFs [17]. We note the lack

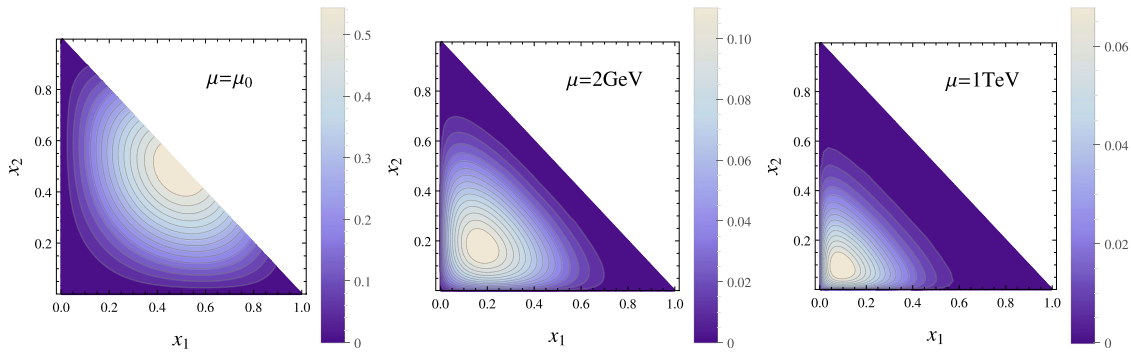


Fig. 2 Contour plots of the valence dPDF of the nucleon multiplied by $x_1 x_2$, i.e., the quantity $x_1 x_2 D_2(x_1, x_2)$, at the scale μ_0 and evolved to $\mu = 2 \text{ GeV}$ and $\mu = 1 \text{ TeV}$ with the LO dDGLAP equations. Model with $a = 2$ corresponding to $\psi(x) = \sqrt{3}(1-x)$

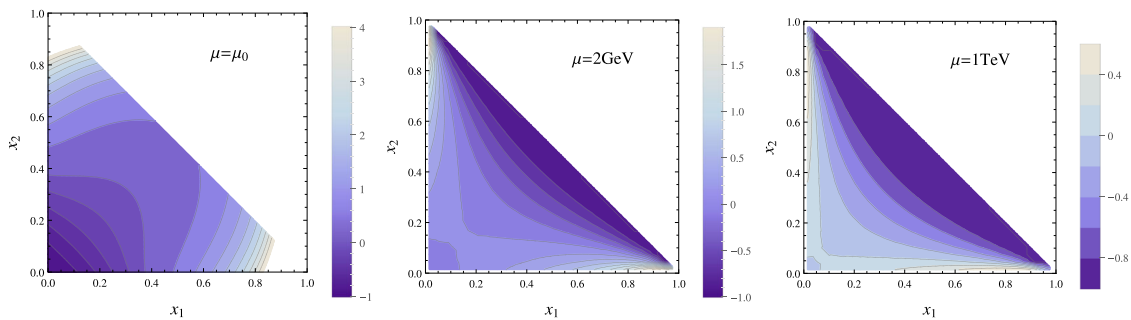


Fig. 3 Contour plots of the valence quark correlation function $\rho(x_1, x_2) = D_2(x_1, x_2)/[D_1(x_1)D_1(x_2)] - 1$, at the scales μ_0 , $\mu = 2 \text{ GeV}$, and $\mu = 1 \text{ TeV}$. Model with $a = 2$ corresponding to $\psi(x) = \sqrt{3}(1-x)$

of factorization in the x_1 and x_2 variables within the DGLAP approach, as emphasized in Ref. [28,29]. The behavior at the end-lines [30] shows that for a fast variable and a slow variable, i.e., $x_1 \rightarrow 1$ and $x_2 \rightarrow 0$, the correlation becomes small, since

$$D_2(x_1, x_2; t) \rightarrow H(t)(1-x_1)^{k_{i,j_1,j_2}}(1-x_1-x_2)^{2C_{Ft}+h_{i,j_1,j_2}}, \quad (17)$$

a feature confirmed with our numerical analysis.

For a probability interpretation to hold, it is necessary that the evolution equations preserve the positivity of the initial condition. Amusingly, it has been found that the upward evolution for sPDFs produces positive distributions [31], unlike the downward evolution where a lower bound for the evolution ratio is found on this basis [27] (cf. also the irreversible evolution of Ref. [32]). Within the considered model, the upward evolution preserves the positivity of the valence dPDF, as can be seen from Fig. 2.

In Fig. 3 we plot the double-parton correlation function as defined in Eq. (1). As the scale increases, the correlation becomes large and positive along the lines $x_i = 0$, while in the central region it is negative. At the line of the kinematic constraint $x_1 + x_2 = 1$ we find $\rho(x_1, x_2) = -1$. The results of Fig. 3 show that the valence dPDF cannot be approximated with the uncorrelated product of the sPDFs. We note that our plots bare qualitative similarity to the results obtained in the MIT bag model [15].

We end with some remarks concerning the longitudinal and transverse degrees of freedom. Our calculation is essentially 1-dimensional, as the transverse dynamics plays no active role. This dimensional reduction could be possible if the relative momentum between the two quarks is assumed to be small. To analyze this assumption from a non-perturbative point of view, let us remind ourselves that the transverse lattice approach [33] allows to effectively freeze the transverse degrees of freedom when the transverse spacing, \mathbf{a}_\perp , becomes larger than the characteristic resolution of the parton momentum in the hadron, $\sim 1/\Lambda_{\text{QCD}}$. This also limits the multiparton Fock state by powers of a_\perp . For instance, for the nucleon $N = q^3 + a_\perp^3 q^4 \bar{q} + a_\perp^4 q^3 G + \dots$. However, as soon as a_\perp starts decreasing, gluons and sea quarks are radiated and it is amazing that the DGLAP evolution equations roughly reproduce the observed behavior for sPDF [26].

A simple way to mimic the transverse lattice dynamics is by using a 2-dimensional harmonic oscillator to trap the partons in the hadron [34] in the transverse plane, with a mass squared operator given by

$$M^2 = \sum_i \left[\frac{\mathbf{k}_{i\perp}^2 + m_i^2}{x_i} + \sigma^2 x_i \mathbf{b}_i^2 \right]. \quad (18)$$

Here, x_i are the longitudinal momentum fractions fulfilling $\sum_i x_i = 1$, \mathbf{b}_i are the impact-parameter variables, $\mathbf{k}_{\perp,i}$ their corresponding Fourier-conjugate variables, and σ is the oscillator constant. If $\psi_{n_\perp}(x, \mathbf{b})$ is the single particle two dimensional HO wave function normalized as $\int d^2b |\psi_{n_\perp}(x, \mathbf{b})|^2 = 1$ with quantum numbers $\mathbf{n}_\perp = (n_x, n_y)$ and mass squared $M^2 = 2\sigma(n_x + n_y + 1)$, then the general solution is a symmetrized product of factorized states

$$\Psi_{\mathbf{n}_\perp}(x_1, \mathbf{b}_1, \dots, x_N, \mathbf{b}_N) = \varphi(x_1, \dots, x_N) \prod_{i=1}^N \psi_{n_{i,\perp}}(x_i, \mathbf{b}_i) \chi_{\text{spin-flavor}}. \quad (19)$$

Due to the fact that the single-particle wave functions have an x -independent normalization, one clearly has

$$D_N(x_1, \dots, x_N) = \prod_i \int d^2b_i |\Psi_{\mathbf{n}_\perp}(x_1, \mathbf{b}_1, \dots, x_N, \mathbf{b}_N)|^2 = |\varphi(x_1, \dots, x_N)|^2. \quad (20)$$

Note that here the variables x_i are fixed, i.e., they are not dynamical, and we can still multiply by an arbitrary function $\varphi(x_1, \dots, x_N)$ which would be generated after requantization by some unspecified 1-dimensional dynamics, say $M^2 = M_{\text{HO}}^2 + M_{1+1}^2$. Our model in Eq. (2) corresponds to the product $\varphi(x_1, \dots, x_N) = \prod_i \psi(x_i)$. This is similar to the Born-Oppenheimer approximation where the longitudinal variables obey classical dynamics. In this extreme case the mass terms give after minimization $x_i = m_i/m$, with $m = \sum_i m_i$ yielding a contribution m^2 to M^2 and $D_N(x_1, \dots, x_N) \rightarrow \prod_i \delta(x_i - m_i/m)$, which for equal masses corresponds to locating the distribution in our Eq. (2) at the maximum, $x_i = 1/N$. That model breaks the transverse translational invariance, which can be restored by projecting onto zero total transverse momentum using the Peierls-Yoccoz method [18], such that for $m_i = 0$ one finds $M_N^2 = 2N\sigma$ for the symmetric state. A straightforward calculation with the rest frame wave function yields the following result for the unintegrated dPDF:

$$D_2(x_1, x_2, \mathbf{k}_\perp) = \int d^2b d^2B |\Psi_0\left(x_1, \mathbf{B} + \frac{1}{2}\mathbf{b}, x_2, \mathbf{B} - \frac{1}{2}\mathbf{b}\right)|^2 e^{i\mathbf{b}\cdot\mathbf{k}_\perp} = D_2(x_1, x_2) e^{-\frac{\mathbf{k}_\perp^2}{4\sigma} \frac{x_1+x_2}{x_1x_2}}. \quad (21)$$

Thus, there exists a transverse dynamics given by Eq. (18) where the effective 1-dimensional treatment used here holds, but it does not support the often assumed transverse-longitudinal factorization.

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