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Aequationes Mathematicae

Alienation of the logarithmic and exponential functional equations

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Dedicated to Professor Roman Ger on the occasion of his 70-th birthday

Abstract. We study the logarithmic–exponential functional equation and check whether the alienation phenomenon takes place.

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1. Introduction

Let $E_1 = 0$ and $E_2 = 0$ be arbitrary functional the equations. Consider equation of the form $E_1 = E_2$. We ask whether or not this equation is equivalent to the system of functional equations $E_1 = 0$ and $E_2 = 0$. If the answer is "yes" then we say that equations $E_1 = 0$ and $E_2 = 0$ are alien. The phenomenon of alienation was first discussed by Dhombres [2]. Later on, several papers and lectures have appeared on this subject (see [3–12]).

The aim of the present paper is to solve the functional equation

$$f(xy) - f(x) - f(y) = g(x + y) - g(x)g(y),$$

which is strictly connected with the problem of alienation of logarithmic and exponential Cauchy functional equations for real functions of a real variable. We solve this equation both in the case where we consider all real variables, and—taking into account the nature of a logarithmic function—for non-zero variables. In the latter case the problem turns out to be much more complicated. The crucial point in this case is a very technical Lemma 2. Generally, we do not assume any regularity conditions on f and g. But if $g(1) = 1$ and $f(1) = 0$, unfortunately, the method of the proof which we use forces us to assume the continuity of g at the origin.

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2. Functions defined on the whole real line

This section we devote to solving our equation considered for functions of all real variables, i.e., we solve

$$f(xy) - f(x) - f(y) = g(x+y) - g(x)g(y), \quad x, y \in \mathbb{R}. \tag{1}$$

We start the section with presenting a result which is an easy consequence of Dhombres [2, Theorem 11] or Ger and Reich [8, Proposition 3] in the case of real functions (see also [1, Proposition 5.6]).

Proposition 1. *Given three nonzero real constants a, b, c, if a map $f\colon \mathbb{R} \to \mathbb{R}$ is a solution of the equation*

$$af(xy) + bf(x)f(y) + c\big(f(x+y) - f(x) - f(y)\big) = 0, \quad x, y \in \mathbb{R}$$

then

(i) $f(x) \equiv 0$, *or*
(ii) $f(x) \equiv (c-a)b^{-1}$, *or*
(iii) $f(x) = -ab^{-1}x$ *for all* $x \in \mathbb{R}$.

Now we are able to proceed with our main result of this section.

Theorem 1. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be functions satisfying (1). Then*

$$f(x) \equiv 0 \quad \textit{and} \quad g \textit{ is an arbitrary exponential function},$$

or there exists a nonzero real constant α such that

$$f(x) \equiv \alpha(\alpha+1), \qquad g(x) \equiv \alpha + 1,$$

or

$$f(x) = -\alpha x^2 + \alpha(\alpha+1), \quad g(x) = -\alpha x + \alpha + 1, \quad x \in \mathbb{R}.$$

Conversely, each of the above pairs of functions is a solution of (1) with any $\alpha \in \mathbb{R}$.

Proof. Put $y = 0$ in (1). Then

$$f(x) = \big(g(0) - 1\big)g(x), \quad x \in \mathbb{R}.$$

Let

$$\alpha := g(0) - 1.$$

If $\alpha = 0$, i.e., $g(0) = 1$, then $f(x) = 0$ for all $x \in \mathbb{R}$, and g is a nonzero exponential function.

Assume that $\alpha \neq 0$. We may write (1) in the form

$$\alpha g(xy) - \alpha g(x) - \alpha g(y) = g(x+y) - g(x)g(y), \quad x, y \in \mathbb{R}.$$

Setting $G(x) := g(x) - g(0),\ x \in \mathbb{R}$, we obtain $G(0) = 0$ and

$$\alpha G(xy) + G(x)G(y) - G(x+y) + G(x) + G(y) = 0, \quad x, y \in \mathbb{R}.$$

By Proposition 1 it follows that $G(x) \equiv 0$, or $G(x) \equiv -1-\alpha$, or $G(x) = -\alpha x$ for all $x \in \mathbb{R}$, which yields $g(x) \equiv \alpha + 1$, or $g(x) \equiv 0$ or $g(x) = -\alpha x + \alpha + 1$ for all $x \in \mathbb{R}$ and $f(x) \equiv \alpha(\alpha+1)$, or $f(x) \equiv 0$, or $f(x) = -\alpha^2 x + \alpha^2 + \alpha$ for all $x \in \mathbb{R}$, respectively.

The converse we obtain by a direct checking. □

3. Equation on a restricted domain

The situation becomes more complicated if we assume that f is not defined on the whole real line but (which is natural for a logarithmic function) on $\mathbb{R}\backslash\{0\}$.

In what follows let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions satisfying

$$f(xy) - f(x) - f(y) = g(x+y) - g(x)g(y), \quad x, y \in \mathbb{R} \setminus \{0\}. \tag{2}$$

For the rest of the paper we denote

$$c := g(1), \quad d := -f(1).$$

We begin our considerations from a simple case where $c = 0$.

Lemma 1. *Let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ satisfy (2) and let $c = 0$. Then*

$$g = 0 \quad \textit{and} \quad f(xy) = f(x) + f(y), \quad x, y \neq 0.$$

Proof. Put $y = 1$ into (2). Then, by the assumption that $c = 0$, we have $g(x+1) = d$ for all $x \neq 0$, whence

$$g(x) = d, \quad x \neq 1. \tag{3}$$

Substituting first $x = y = 2$, and then $x = y = -2$ in (2), and comparing the so obtained results, by (3) we conclude that

$$f(2) = f(-2). \tag{4}$$

Put now in turn, $x = 2$, $y = -1$ and $x = -2$, $y = -1$ in (2). Comparing the results, by (3) and (4) we get $-d^2 = d - d^2$, whence $d = 0$. This finishes the proof. □

From now on we will assume that $c \neq 0$ and we start with a technical lemma.

Lemma 2. *Let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ satisfy (2) and assume that $c \neq 0$. Then the following cases are the only possible ones*

(i) *$c \neq 0$ arbitrary, $d = 0$, $g(0) = 1$,*
(ii) *$c = 1$, d arbitrary, $g(0) = 1 - d$,*
(iii) *$c = -1$, d arbitrary, $g(0) = 1 + d$,*
(iv) *$c \neq 0$ arbitrary, $d = c - c^2$, $g(0) = c$.*

Proof. Substituting $y = 1$ into (2), we get

$$g(x+1) = cg(x) + d, \quad x \in \mathbb{R}\backslash\{0\}.$$

Hence,

$$g(n+1) = c^{n+1} + \sum_{k=0}^{n-1} c^k d, \quad g(-n) = \frac{g(0)}{c^n} - \sum_{k=1}^{n} \frac{d}{c^k}, \quad n \in \mathbb{N}. \tag{5}$$

Put now $x = y = -2$ and $x = y = 2$ into (2) and subtract the so obtained results side by side to obtain

$$2f(2) - 2f(-2) = g(-4) - g(-2)^2 - g(4) + g(2)^2. \tag{6}$$

Set now $x = -1$ and $y = 2$ in (2):

$$f(-2) - f(-1) - f(2) = g(1) - g(-1)g(2). \tag{7}$$

Multiplying (7) by 2 and adding it to (6) we obtain

$$-2f(-1) = g(-4) - g(-2)^2 - g(4) + g(2)^2 + 2g(1) - 2g(-1)g(2). \tag{8}$$

Substitute $x = y = -1$ in (2):

$$f(1) - 2f(-1) = g(-2) - g(-1)^2. \tag{9}$$

By (8) and (9) it follows that

$$f(1) + g(-4) - g(-2)^2 - g(4) + g(2)^2 + 2g(1) - 2g(-1)g(2) = g(-2) - g(-1)^2. \tag{10}$$

Set $x = -1$ and $y = -2$ in (2):

$$f(2) - f(-1) - f(-2) = g(-3) - g(-1)g(-2). \tag{11}$$

Adding (7) and (11) side by side, we get

$$-2f(-1) = g(1) - g(-1)g(2) + g(-3) - g(-1)g(-2),$$

which put into (9), gives

$$f(1) + g(1) - g(-1)g(2) + g(-3) - g(-1)g(-2) = g(-2) - g(-1)^2. \tag{12}$$

Finally, put $x = -1$, $y = 3$ and $x = -1$, $y = -3$ in (2),

$$f(-3) - f(-1) - f(3) = g(2) - g(-1)g(3),$$
$$f(3) - f(-1) - f(-3) = g(-4) - g(-1)g(-3).$$

Adding the above equalities side by side yields

$$-2f(-1) = g(2) - g(-1)g(3) + g(-4) - g(-1)g(-3),$$

whence, on account of (8) we have

$$-g(-2)^2 - g(4) + g(2)^2 + 2g(1) - 2g(-1)g(2) = g(2) - g(-1)g(3) - g(-1)g(-3). \tag{13}$$

Taking into account our notations and formula (5), by (10), (12) and (13) we obtain the system of equations with unknown variables c, d and $g(0)$:

$$\begin{cases} c^4+c^2d+cd+2d-\dfrac{g(0)}{c^4}+\dfrac{d}{c^4}+\dfrac{d}{c^3}-\left(c^2+d\right)^2+\left(\dfrac{g(0)}{c^2}-\dfrac{d}{c^2}-\dfrac{d}{c}\right)^2+\dfrac{g(0)}{c^2} \\ \qquad -\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)^2+2\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)\left(c^2+d\right)-2c=0 \\ \dfrac{g(0)}{c^2}-\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)^2+\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)\left(c^2+d\right)-\dfrac{g(0)}{c^3}+\dfrac{d}{c^3} \\ \qquad +\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)\left(\dfrac{g(0)}{c^2}-\dfrac{d}{c^2}-\dfrac{d}{c}\right)-c+d=0 \\ c^4+c^2d+cd+2d-\left(c^2+d\right)^2+\left(\dfrac{g(0)}{c^2}-\dfrac{d}{c^2}-\dfrac{d}{c}\right)^2+c^2+2\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)\left(c^2+d\right) \\ \qquad -\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)\left(c^3+cd+d\right)-\left(\dfrac{g(0)}{c}-\dfrac{d}{c}\right)\left(\dfrac{g(0)}{c^3}-\dfrac{d}{c^3}-\dfrac{d}{c^2}-\dfrac{d}{c}\right)-2c=0, \end{cases}$$

whence,

$$\begin{cases} c^6d+c^5d-2c^5g(0)+2c^5+c^4d^2-2c^4d+2c^3d^2-2c^3dg(0)+c^2g(0)^2-c^2g(0) \\ \qquad -2c^2dg(0)+2cdg(0)-2cd^2-cd-d^2+2dg(0)-d-g(0)^2+g(0)=0 \\ c^4+c^4d-c^4g(0)-c^3d+c^2d^2-c^2dg(0)+cg(0)^2-cdg(0)-cg(0)-d^2 \\ \qquad +2dg(0)-d-g(0)^2+g(0)=0 \\ -c^5+c^5g(0)+2c^4+c^4d-2c^4g(0)-2c^3d+c^3dg(0)+c^2d^2-c^2dg(0) \\ \qquad -cdg(0)+dg(0)-d^2=0. \end{cases} \tag{14}$$

From the last equation we have

$$\left(c^5-2c^4+c^3d-c^2d-cd+d\right)g(0)=c^5-2c^4-c^4d+2c^3d-c^2d^2+d^2. \tag{15}$$

Denote $w := c^5-2c^4+c^3d-c^2d-cd+d$ and assume first that $w=0$, that is,

$$d\left(c^3-c^2-c+1\right)=2c^4-c^5,$$
$$d\left(c-1\right)^2\left(c+1\right)=c^4(2-c).$$

For $c=1$ or $c=-1$ we would get $c=0$ or $c=2$, which is impossible. Consequently, in this case, it must be $c\neq\pm1$ and then

$$d=\frac{c^4(2-c)}{(c-1)^2(c+1)}.$$

Substituting this into the right-hand side of (15), which equals zero, we get (after simplifying)

$$\frac{c^4(c-2)(2c-1)}{(c-1)^3(c+1)}=0,$$

whence, $c = \frac{1}{2}$ or $c = 2$. Then, $d = \frac{1}{4}$ or $d = 0$, respectively. Substituting the pair $(c, d) = \left(\frac{1}{2}, \frac{1}{4}\right)$ into the first two equations of (14) we obtain

$$\begin{cases} 12g(0)^2 - 20g(0) + 7 = 0 \\ 4g(0)^2 - 6g(0) + 2 = 0, \end{cases}$$

which yields $g(0) = \frac{1}{2}$.

Putting $(c, d) = (2, 0)$ into the first two equations in (14) gives

$$\begin{cases} 3g(0)^2 - 67g(0) + 64 = 0 \\ g(0)^2 - 17g(0) + 16 = 0, \end{cases}$$

and $g(0) = 1$.

Assume now that $w \neq 0$. Then

$$d\,(1-c)^2\,(1+c) \neq c^4(2-c)$$

and

$$g(0) = \frac{c^5 - 2c^4 - c^4 d + 2c^3 d - c^2 d^2 + d^2}{c^5 - 2c^4 + c^3 d - c^2 d - cd + d}.$$

Substituting this into the first two equations in (14), we obtain (after simplification)

$$\begin{cases} dc^3(c-1)(c+1)\left(d - c + c^2\right)\left(c^9 - 2c^8 + 2c^7 d - c^7 - 2c^6 d + c^5 + c^5 d^2 - 4c^5 d \right. \\ \qquad \left. + 4c^4 - 2c^3 d^2 + 3c^3 + 2c^2 d + 2c^2 + c + cd^2 + 2cd - 2\right) = 0 \\ dc^3(c-1)(c+1)\left(c^3 - 2c^2 + cd + c - d - 2\right)\left(d - c + c^2\right) \\ \qquad \times\left(c^4 - c^3 + c^2 d - c^2 - c - d + 1\right) = 0. \end{cases} \tag{16}$$

Solving (16), we get the following cases

- $d = 0$ and $c \neq 2$ arbitrary; then $g(0) = 1$. This, together with the earlier obtained triple $(c, d, g(0)) = (2, 0, 1)$, gives assertion (i).
- $c = 1$ and d arbitrary; then $g(0) = 1 - d$ and we get (ii).
- $c = -1$ and d arbitrary; then $g(0) = 1 + d$, which gives (iii).
- $c^3 - 2c^2 + cd + c - d - 2 = 0$, so $d(c-1) = (1+c^2)(2-c)$. Hence, $c \neq 1$ and

  $$d = \frac{(1+c^2)(2-c)}{c-1}.$$

  Then for $c \neq 2$ we have $w \neq 0$ and substituting the above d into the first equation of (16), we obtain

  $$\frac{2c^3(c+1)(c-2)^2(c^2+1)}{c-1} = 0,$$

  whence, $c = -1$. But then $d = -3$ and $g(0) = -2$, and such a triple is already included in (iii).

- $d = c - c^2$ with an arbitrary c such that $w \neq 0$, which means $c \neq \frac{1}{2}$. Then $g(0) = c$. This case with the earlier obtained triple $(c, d, g(0)) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)$ gives (iv).
- $c^4 - c^3 + c^2 d - c^2 - c - d + 1 = 0$, i.e., $d(1 - c^2) = c^4 - c^3 - c^2 - c + 1$, whence, $c \neq \pm 1$ and
$$d = \frac{c^4 - c^3 - c^2 - c + 1}{1 - c^2}.$$
Then for $c \neq 2$ we have $w \neq 0$ and substituting such d into the first equation of (16), we obtain
$$\frac{2c^3(c^4 - c^3 - c^2 - c + 1)(2c - 1)^2}{c^2 - 1} = 0,$$
whence, $c = \frac{1}{2}$ (then $d = \frac{1}{4}$ and $g(0) = \frac{1}{2}$) or $c^4 - c^3 - c^2 - c + 1 = 0$. In the latter case we have $d = 0$, which even with an arbitrary value of c gives $g(0) = 1$.

This finishes the proof. □

Corollary 1. *Let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ satisfy (2) and $c \neq 0$. Then*
$$g(x + 1) = cg(x) + d, \quad x \in \mathbb{R}. \tag{17}$$

Lemma 3. *Let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ satisfy (2). If $c \neq 0$ and $d = 0$, then*
$$f(rx) = f(r) + f(x), \quad r \in \mathbb{Q}\backslash\{0\}, \quad x \neq 0 \tag{18}$$
and
$$g(r + x) = g(r)g(x), \quad r \in \mathbb{Q}\backslash\{0\}, \quad x \neq 0. \tag{19}$$

Proof. By Lemma 2 and (17), we have $g(0) = 1$ and $g(-1) = \frac{1}{c}$. Hence, $g(x + p) = c^p g(x)$ for all $p \in \mathbb{Z}$ and $x \in \mathbb{R}$. Therefore,
$$g(x + p) = g(x)g(p), \quad p \in \mathbb{Z}, \quad x \in \mathbb{R}.$$
On account of (2),
$$f(px) - f(p) - f(x) = 0, \quad p \in \mathbb{Z}\backslash\{0\}, \quad x \neq 0.$$
Substituting $x = \frac{1}{p}$, $p \in \mathbb{Z}\backslash\{0\}$ in the above equality gives
$$f(p) + f\left(\frac{1}{p}\right) = 0, \quad p \in \mathbb{Z}\backslash\{0\}.$$
Hence, for $p, q \in \mathbb{Z}\backslash\{0\}$ and $x \neq 0$,
$$f\left(\frac{p}{q}x\right) = f(p) + f\left(\frac{1}{q}x\right) + f(q) + f\left(\frac{1}{q}\right) = f\left(\frac{p}{q}\right) + f(x),$$
so, we have (18), and consequently, by (2), we obtain (19). □

Corollary 2. *Let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ and $d = 0$. If $c < 0$ then (2) does not have solutions.*

Proof. Putting $x = r = \frac{1}{2}$ in (19) we get

$$c = g(1) = g\left(\tfrac{1}{2}\right)^2 \geq 0,$$

which is a contradiction. □

Under the assumptions of the following lemma we get the "full" alienation of logarithmic and exponential Cauchy functional equations.

Lemma 4. *Let* $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ *satisfy* (2). *If* $c > 0$, $c \neq 1$ *and* $d = 0$, *then*

$$g(x+y) = g(x)g(y), \quad x, y \in \mathbb{R} \quad \textit{and} \quad f(xy) = f(x) + f(y), \quad x, y \neq 0.$$

Proof. On account of (2),

$$f(2xy) - f(2x) - f(y) = g(2x+y) - g(2x)g(y), \quad x, y \neq 0, \tag{20}$$

and by (18),

$$f(2xy) = f(2) + f(xy), \quad x, y \neq 0,$$

and

$$f(2x) = f(2) + f(x), \quad x \neq 0.$$

Therefore, from (20),

$$f(xy) - f(x) - f(y) = g(2x+y) - g(2x)g(y), \quad x, y \neq 0.$$

By (2) and Lemma 2 we have $g(0) = 1$ and

$$g(x+y) - g(x)g(y) = g(2x+y) - g(2x)g(y), \quad x, y \in \mathbb{R}. \tag{21}$$

Putting $x+1$ in the place of x and recalling that $g(x+1) = cg(x)$ for all $x \in \mathbb{R}$, we get

$$c\big(g(x+y) - g(x)g(y)\big) = c^2\big(g(2x+y) - g(2x)g(y)\big), \quad x, y \in \mathbb{R}.$$

This together with (21) and the assumption that $c \neq 1$ yields

$$g(x+y) = g(x)g(y), \quad x, y \in \mathbb{R}$$

and, as a consequence,

$$f(xy) = f(x) + f(y), \quad x, y \neq 0,$$

which was to be proved. □

In the next lemma we obtain a particular case of alienation of logarithmic and exponential Cauchy functional equations (with $g(x) \equiv 1$), but we assume the continuity of g at the origin. Without this assumption we are not able to get such type of results. But on the other side, we also do not have an example showing that in such a case the phenomenon of alienation does not occur.

Lemma 5. *Let* $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ *satisfy* (2). *If* $c = 1$, $d = 0$, *and* g *is continuous at the origin, then*

$$g(x) \equiv 1 \quad \textit{and} \quad f(xy) = f(x) + f(y), \quad x, y \in \mathbb{R}.$$

Proof. By (18),

$$f(-x) = f(x) + f(-1), \quad x \in \mathbb{R}\backslash\{0\}$$

and

$$0 = f(1) = 2f(-1),$$

we get

$$f(x) = f(-x), \quad x \neq 0.$$

By (17)

$$g(x+1) = g(x), \quad x \in \mathbb{R},$$

and consequently,

$$g(p) = 1, \quad p \in \mathbb{Z}.$$

On account of (19),

$$g\left(n \cdot \frac{p}{q}\right) = g\left(\frac{p}{q}\right)^n, \quad n, q \in \mathbb{N}, \quad p \in \mathbb{Z},$$

whence

$$1 = g\left(q \cdot \frac{p}{q}\right) = g\left(\frac{p}{q}\right)^q, \quad p \in \mathbb{Z},\ q \in \mathbb{N},$$

thus

$$g(r) = 1, \quad r \in \mathbb{Q}.$$

Therefore $g(x+r) = g(x)$ for all $x \in \mathbb{R}$ and $r \in \mathbb{Q}$.

We shall show that $g(x) \equiv 1$. Suppose to the contrary that $g(x_0) \neq 1$ for some $x_0 \in \mathbb{R}$. Put $\varepsilon := \frac{1}{2}|1 - g(x_0)|$. By the continuity of g at the origin it follows that there exists $\delta > 0$ such that if $|x| < \delta$ then $|g(x) - 1| < \varepsilon$. Observe that we can take $x = x_0 + w$ with a suitably chosen rational number w. Hence,

$$2\varepsilon = |g(x_0) - 1| = |g(x_0 + w) - 1| < \varepsilon.$$

This contradiction shows that $g(x) \equiv 1$, which was our assertion. □

In the remaining cases with $d \neq 0$ the phenomenon of alienation does not occur, although we are not very far from it. We may observe that functions f and g are logarithmic and exponential, respectively, up to linear functions.

Lemma 6. *Let* $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ *satisfy* (2). *If* $c = -1$ *and* $d \neq 0$, *then* $g(x) \equiv -1$ *and* $f(x) = F(x) + 2$ *for all* $x \in \mathbb{R}\backslash\{0\}$, *where* $F\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *satisfies*

$$F(xy) = F(x) + F(y), \quad x, y \neq 0.$$

Proof. By Lemma 2, $g(0) = d + 1$ and on account of (17),

$$g(x+1) = -g(x) + d, \quad x \in \mathbb{R}, \tag{22}$$

whence, $g(-1) = -1$ and

$$g(x+2) = g(x), \quad x \in \mathbb{R}. \tag{23}$$

Substituting $y = -1$ into (2) we obtain

$$f(-x) - f(x) - f(-1) = g(x-1) - g(x)g(-1), \quad x, y \neq 0,$$

whence, by use of (22) we get

$$f(-x) - f(x) = d + f(-1), \quad x \neq 0.$$

Interchanging the roles of x and $-x$ we see that $f(-1) = -d$, and, consequently,

$$f(-x) = f(x), \quad x \neq 0. \tag{24}$$

By (2) and (23),

$$f(2x) - f(x) - f(2) = g(x+2) - g(x)g(2) = -dg(x), \quad x \neq 0,$$

whence, on account of (24),

$$g(x) = g(-x), \quad x \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} g(2x) - g(x)^2 &= f(x^2) - 2f(x) = f(-x^2) - f(x) - f(-x) \\ &= g(0) - g(x)g(-x) = 1 + d - g(x)^2 \end{aligned}$$

for all $x \neq 0$, and

$$g(x) = 1 + d, \quad x \in \mathbb{R}.$$

Since $g(1) = -1$, we have $g(x) = -1$ for all $x \in \mathbb{R}$, and

$$f(xy) - f(x) - f(y) = -2, \quad x, y \neq 0.$$

Finally, if we define $F(x) := f(x) - 2$ for all $x \neq 0$, then

$$F(xy) = F(x) + F(y), \quad x, y \neq 0,$$

and the proof is finished. □

In the proof of Lemmas 8 and 9 we will use the following result.

Lemma 7. *Let* $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ *satisfy* (2). *If* $c \notin \{0, -1\}$ *and* $d \neq 0$, *then*

$$2g(x+y) = \frac{Kc^2 + d(1+c)}{c^2}\big[g(x)+g(y)\big] + K - K^2 - \frac{(1+c)d(c-d-cd+c^3)}{c^4}, \tag{25}$$

for all $x, y \neq 0$, *where* $K = \dfrac{c+c^3-cd-d}{c^2}$.

Proof. By (17), we have

$$cg(0) = c-d, \quad c^2g(-1) = c-d-cd, \quad cg(x-1) = g(x)-d, \quad x \in \mathbb{R}. \tag{26}$$

On account of (2) and (26),

$$c^2\big[f(-x) - f(x) - f(-1)\big] = d(1+c)g(x) - cd, \quad x \neq 0. \tag{27}$$

Substituting $-x$ in the place of x in the above equation and adding these two equations we obtain

$$-2c^2 f(-1) = d(1+c)\big[g(x)+g(-x)\big] - 2cd, \quad x \neq 0. \tag{28}$$

Set $x = 1$ in (28) in order to obtain

$$f(-1) = \frac{d}{2c^4}\big[(1-c)c^3 + (1+c)(cd+d-c)\big]. \tag{29}$$

Putting it back to (28) we get

$$(1+c)\big[g(x)+g(-x)\big] = \frac{1+c}{c^2}\big[c(1+c^2) - d(1+c)\big], \quad x \in \mathbb{R},$$

which, by the assumption $c \neq -1$, gives

$$g(x) + g(-x) = \frac{c+c^3-cd-d}{c^2} =: K, \quad x \neq 0. \tag{30}$$

On the other side, substituting (29) into (27), yields

$$f(-x) - f(x) = \frac{d(1+c)}{c^2}\big[g(x) - \tfrac{1}{2}K\big], \quad x \neq 0. \tag{31}$$

Using again (2), on account of (30), we obtain

$$\begin{aligned} f(xy) - f(-x) - f(-y) &= K - g(x+y) - (K-g(x))(K-g(y)) \\ &= K - K^2 - g(x+y) + Kg(x) + Kg(y) \\ &\quad -g(x)g(y), \quad x, y \neq 0. \end{aligned}$$

Subtracting (2) from the above equality, we get

$$\begin{aligned} &f(x) + f(y) - f(-x) - f(-y) \\ &\quad = -2g(x+y) + K - K^2 + Kg(x) + Kg(y), \quad x, y \neq 0. \end{aligned}$$

By (31) we may write

$$-\frac{d(1+c)}{c^2}g(x)-\frac{d(1+c)}{c^2}g(y)+\frac{(1+c)d(c-d-cd+c^3)}{c^4}$$
$$=-2g(x+y)+K-K^2+Kg(x)+Kg(y),\quad x,y\neq 0,$$

which is equivalent to our assertion (25). □

Lemma 8. *Let* $f\colon \mathbb{R}\backslash\{0\}\to\mathbb{R}$, $g\colon\mathbb{R}\to\mathbb{R}$ *satisfy* (2)*. If* $c=1$ *and* $d\neq 0$*, then*

$$g(x)=dx+(1-d),\quad x\in\mathbb{R}$$

and there exists a function $F:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *satisfying*

$$F(xy)=F(x)+F(y),\quad x,y\neq 0,$$

such that

$$f(x)=F(x)-d^2x-d(1-d),\quad x\neq 0.$$

Proof. We will use Lemma 7. Setting $c=1$ we get $K=2-2d$ and (25) takes the form

$$g(x+y)=g(x)+g(y)+d-1,\quad x,y\neq 0.$$

Since by Lemma 2, $g(0)=1-d$, we have in fact,

$$g(x+y)=g(x)+g(y)+d-1,\quad x,y\in\mathbb{R}.$$

Therefore, the function $G\colon\mathbb{R}\to\mathbb{R}$ defined by $G(x):=g(x)+d-1$ for all $x\in\mathbb{R}$ is additive.

By the additivity of G it follows that $G(2x)=2G(x)$ for all $x\in\mathbb{R}$, whence, by the definition of G we get

$$\begin{aligned}g(x+y)-g(x)g(y)&=G(x+y)+1-d-(G(x)+(1-d))(G(y)+(1-d))\\&=G(x+y)+1-d-G(x)G(y)-(1-d)(G(x)\\&\quad+G(y))-(1-d)^2\\&=dG(x)+dG(y)-G(x)G(y)+(1-d)d.\end{aligned}$$

Consequently, (2) takes the form

$$f(xy)-f(x)-f(y)=dG(x)+dG(y)-G(x)G(y)+d(1-d),\quad x,y\neq 0.\quad (32)$$

Substituting $y=2$ in the above equality, we get

$$f(2x)-f(x)-f(2)=-dG(x)+d^2+d,\quad x\neq 0.$$

Therefore, according to (32), for all $x, y \neq 0$ we have

$$\begin{aligned} dG(xy) &= d^2 + d - [f(2xy) - f(xy) - f(2)] \\ &= d^2 + d - [f(2xy) - f(2x) - f(y) + f(2x) - f(2) - f(x) \\ &\quad + f(x) + f(y) - f(xy)] \\ &= d^2 + d - [dG(2x) + dG(y) - G(2x)G(y) + d(1-d) \\ &\quad + dG(x) + dG(2) - G(x)G(2) \\ &\quad + d(1-d) - dG(x) - dG(y) \quad + G(x)G(y) - d(1-d)] \\ &= G(x)G(y). \end{aligned}$$

The function $\frac{1}{d}G$, being additive, is of the form $\frac{1}{d}G(x) = x$ for all $x \in \mathbb{R}$ (see [1, Proposition 5.6]), whence,

$$G(x) = dx, \quad x \in \mathbb{R}.$$

Consequently, we may rewrite (32) as

$$f(xy) - f(x) - f(y) = d^2x + d^2y - d^2xy + d(1-d), \quad x, y \neq 0.$$

Defining

$$F(x) := f(x) + d^2x + d(1-d), \quad x \neq 0,$$

we observe that

$$F(xy) = F(x) + F(y), \quad x, y \neq 0.$$

Hence,

$$g(x) = dx + (1-d),\ x \in \mathbb{R} \quad \text{and} \quad f(x) = F(x) - d^2x - d(1-d), \quad x \neq 0,$$

and the proof is finished. □

Lemma 9. *Let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ satisfy (2). If $c \notin \{0, 1, -1\}$ and $d \neq 0$, then $g(x) \equiv c$ and there exists $F : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ satisfying*

$$F(xy) = F(x) + F(y), \quad x, y \neq 0,$$

so that $f(x) = F(x) + c^2 - c$ for all $x \neq 0$.

Proof. We will apply Lemma 7. Put $y = 1$ in (25). By (17), for all $x \neq 0$ we have

$$\begin{aligned} 2(cg(x) + d) = \frac{Kc^2 + d(1+c)}{c^2} g(x) + \frac{Kc^2 + d(1+c)}{c} \\ + K - K^2 - \frac{(1+c)d(c - d - cd + c^3)}{c^4}, \end{aligned}$$

whence,

$$\left(2c - \frac{Kc^2 + d(1+c)}{c^2}\right) g(x) = -2d + \frac{Kc^2 + d(1+c)}{c} + K - K^2 - \frac{(1+c)d(c-d-cd+c^3)}{c^4},$$

for all $x \neq 0$. This together with the definition of K yields

$$c(c^2-1)g(x) = A, \quad x \neq 0,$$

where

$$A := -2dc^2 + Kc^3 + dc(1+c) + Kc^2 - K^2c^2 - \frac{(1+c)d(c-d-cd+c^3)}{c^2}.$$

Therefore,

$$g(x) = \frac{A}{c(c^2-1)}, \quad x \neq 0.$$

Hence, $g(x) = g(1) = c$ for all $x \neq 0$. By Lemma 2, $g(0) = c$. Consequently, $g(x) = c$ for all $x \in \mathbb{R}$. By (2), the function $F(x) := f(x) + c - c^2,\ x \neq 0$, satisfies

$$F(xy) = F(x) + F(y), \quad x, y \neq 0,$$

so,

$$f(x) = F(x) - c + c^2, \quad x \neq 0,$$

with $c \notin \{0, 1, -1\}$. □

Consequently, considering all the above results, we may formulate the final theorem.

Theorem 2. *Assume that* $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *satisfy* (2).

If $g(1) \neq 1$ *or* $f(1) \neq 0$, *then*

$$g(x+y) = g(x)g(y), \quad x, y \in \mathbb{R} \quad \text{and} \quad f(xy) = f(x) + f(y), \quad x, y \in \mathbb{R}\backslash\{0\}, \tag{33}$$

or there exist $\alpha \in \mathbb{R}\backslash\{0\}$ *and a function* $F : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *satisfying* $F(xy) = F(x) + F(y)$ *for all* $x, y \in \mathbb{R}\backslash\{0\}$ *such that*

$$g(x) = \alpha x + (1-\alpha), \quad x \in \mathbb{R} \quad \text{and} \quad f(x) = F(x) - \alpha^2 x - \alpha(1-\alpha), \quad x \in \mathbb{R}\backslash\{0\}, \tag{34}$$

or there exist $\beta \in \mathbb{R}\backslash\{1\}$ *and a function* $F : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *satisfying* $F(xy) = F(x) + F(y)$ *for all* $x, y \in \mathbb{R}\backslash\{0\}$ *such that*

$$g(x) = \beta, \quad x \in \mathbb{R} \quad \text{and} \quad f(x) = F(x) + \beta^2 - \beta, \quad x \in \mathbb{R}\backslash\{0\}. \tag{35}$$

If $g(1) = 1$, $f(1) = 0$ *and* g *is continuous at the origin, then*

$$g(x) = 1, \quad x \in \mathbb{R} \quad \text{and} \quad f(xy) = f(x) + f(y), \quad x, y \in \mathbb{R}\backslash\{0\}.$$

Conversely, each pair of functions described by (33)*,* (34) *and* (35) *is a solution of* (2) *with any real* α *and* β*.*

We terminate our paper with the following open problem.

Problem 1. Assume $g(1) = 1$ and $f(1) = 0$. Find all solutions of (2) without assuming the continuity of g at the origin (cf., Lemma 5).

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