Kim et al. *Journal of Inequalities and Applications* (2017) 2017:8
DOI 10.1186/s13660-016-1282-y

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fourier series of higher-order Bernoulli functions and their applications

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Abstract

In this paper, we study the Fourier series related to higher-order Bernoulli functions and give new identities for higher-order Bernoulli functions which are derived from the Fourier series of them.

MSC: 11B68; 42A16

Keywords: Fourier series; Bernoulli polynomials; Bernoulli functions

1 Introduction

As is well known, Bernoulli polynomials are defined by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\text{(see [1–23]).}\tag{1.1}$$

When $x=0$, $B_n=B_n(0)$ are called Bernoulli numbers. From (1.1), we note that

$$B_n(x)=\sum_{l=0}^{n}\binom{n}{l}B_lx^{n-l}\in\mathbb{Q}[x]\quad(n\geq 0),\tag{1.2}$$

with $\deg B_n(x)=n$ (see [9–11]). By (1.1), we easily get

$$(B+1)^n-B_n=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases}\quad\text{and}\quad B_0=1,\tag{1.3}$$

with the usual convention about replacing B^n by B_n (see [9, 10]). From (1.2), we note that

$$\begin{aligned}\frac{dB_n(x)}{dx}&=\frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}=\sum_{k=0}^{n-1}\binom{n}{k}B_k(n-k)x^{n-k-1}\\&=n\sum_{k=0}^{n-1}\frac{(n-1)!}{(n-k-1)!k!}B_kx^{n-k-1}=n\sum_{k=0}^{n-1}\binom{n-1}{k}B_kx^{n-1-k}\\&=nB_{n-1}(x)\quad(n\geq 1)\text{ (see [9–18]).}\end{aligned}\tag{1.4}$$

Thus, by (1.4), we get

$$\int_0^x B_n(x)\,dx = \frac{1}{n+1}\big(B_{n+1}(x) - B_{n+1}(0)\big) \quad (n \geq 0). \tag{1.5}$$

For any real number x, we define

$$\langle x\rangle = x - [x] \in [0,1), \tag{1.6}$$

where $[x]$ is the integral part of x. Then $B_n(\langle x\rangle)$ are functions defined on $(-\infty,\infty)$ and periodic with period 1, which are called Bernoulli functions. The Fourier series for $B_m(\langle x\rangle)$ is given by

$$B_m\big(\langle x\rangle\big) = -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.7}$$

where $m \geq 1$ and $x \notin \mathbb{Z}$ (see [1, 2, 8, 14, 22]). For a positive integer N, we have

$$\begin{aligned}
\sum_{k=0}^{N-1} B_m\left(\left\langle \frac{x+k}{N}\right\rangle\right) &= -m! \sum_{k=0}^{N-1} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi in(\frac{x+k}{N})}}{(2\pi in)^m} \\
&= -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi in\frac{x}{N}}}{(2\pi in)^m} \sum_{k=0}^{N-1} e^{2\pi in\frac{k}{N}} \\
&= -m! N^{1-m} \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{e^{2\pi ilx}}{(2\pi il)^m} = N^{1-m} B_m\big(\langle x\rangle\big) \quad (x \notin \mathbb{Z}).
\end{aligned}$$

For $r \in \mathbb{N}$, the higher-order Bernoulli polynomials are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1, 10, 11, 22])}. \tag{1.8}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called Bernoulli numbers of order r (see [1, 22]). Then $B_n^{(r)}(\langle x\rangle)$ are functions defined on $(-\infty,\infty)$ and periodic with period 1, which are called Bernoulli functions of order r. In this paper, we study the Fourier series related to higher-order Bernoulli functions and give some new identities for the higher-order Bernoulli functions which are derived from the Fourier series of them.

2 Fourier series of higher-order Bernoulli functions and their applications

From (1.8), we note that

$$B_m^{(r)}(x+1) = B_m^{(r)}(x) + mB_{m-1}^{(r-1)}(x) \quad (m \geq 0). \tag{2.1}$$

Indeed,

$$\begin{aligned}\sum_{m=0}^{\infty} B_m^{(r)}(x+1)\frac{t^m}{m!} &= \left(\frac{t}{e^t-1}\right)^r e^{(x+1)t} = \left(\frac{t}{e^t-1}\right)^r e^{xt}\left(e^t-1+1\right) \\ &= \left(\frac{t}{e^t-1}\right)^{r-1} te^{xt} + \left(\frac{t}{e^t-1}\right)^r e^{xt} \\ &= \sum_{m=0}^{\infty} B_m^{(r-1)}(x)\frac{t^{m+1}}{m!} + \sum_{m=0}^{\infty} B_m^{(r)}(x)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(mB_{m-1}^{(r-1)}(x) + B_m^{(r)}(x)\right)\frac{t^m}{m!}. \end{aligned} \tag{2.2}$$

Let $x = 0$ in (2.1). Then we have

$$B_m^{(r)}(1) = B_m^{(r)}(0) + mB_{m-1}^{(r-1)}(0) \quad (m \geq 0). \tag{2.3}$$

Now, we assume that $m \geq 1$, $r \geq 2$. $B_m^{(r)}(\langle x\rangle)$ is piecewise C^∞. Further, in view of (2.3), $B_m^{(r)}(\langle x\rangle)$ is continuous for those (r,m) with $B_{m-1}^{(r-1)}(0) = 0$, and is discontinuous with jump discontinuities at integers for those (r,m) with $B_{m-1}^{(r-1)}(0) \neq 0$. The Fourier series of $B_m^{(r)}(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi inx} \quad (i = \sqrt{-1}), \tag{2.4}$$

where

$$\begin{aligned} C_n^{(r,m)} &= \int_0^1 B_m^{(r)}\left(\langle x\rangle\right)e^{-2\pi inx}\,dx = \int_0^1 B_m^{(r)}(x)e^{-2\pi inx}\,dx \\ &= \left[\frac{1}{m+1}B_{m+1}^{(r)}(x)e^{-2\pi inx}\right]_0^1 + \frac{2\pi in}{m+1}\int_0^1 B_{m+1}^{(r)}(x)e^{-2\pi inx}\,dx \\ &= \frac{1}{m+1}\left(B_{m+1}^{(r)}(1) - B_{m+1}^{(r)}(0)\right) + \frac{2\pi in}{m+1}C_n^{(r,m+1)} \\ &= B_m^{(r-1)}(0) + \frac{2\pi in}{m+1}C_n^{(r,m+1)}. \end{aligned} \tag{2.5}$$

Replacing m by $m-1$ in (2.5), we get

$$C_n^{(r,m-1)} = B_{m-1}^{(r-1)}(0) + \frac{2\pi in}{m}C_n^{(r,m)}. \tag{2.6}$$

Case 1. Let $n \neq 0$. Then we have

$$\begin{aligned} C_n^{(r,m)} &= \frac{m}{2\pi in}C_n^{(r,m-1)} - \frac{m}{2\pi in}B_{m-1}^{(r-1)} \\ &= \frac{m}{2\pi in}\left(\frac{m-1}{2\pi in}C_n^{(r,m-2)} - \frac{m-1}{2\pi in}B_{m-2}^{(r-1)}\right) - \frac{m}{2\pi in}B_{m-1}^{(r-1)} \\ &= \frac{m(m-1)}{(2\pi in)^2}C_n^{(r,m-2)} - \frac{m(m-1)}{(2\pi in)^2}B_{m-2}^{(r-1)} - \frac{m}{2\pi in}B_{m-1}^{(r-1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{m(m-1)}{(2\pi in)^2}\left\{\frac{m-2}{2\pi in}C_n^{(r,m-3)} - \frac{m-2}{2\pi in}B_{m-3}^{(r-1)}\right\} - \frac{m(m-1)}{(2\pi in)^2}B_{m-2}^{(r-1)} - \frac{m}{2\pi in}B_{m-1}^{(r-1)} \\
&= \frac{m(m-1)(m-2)}{(2\pi in)^3}C_n^{(r,m-3)} - \frac{m(m-1)(m-2)}{(2\pi in)^3}B_{m-3}^{(r-1)} \\
&\quad - \frac{m(m-1)}{(2\pi in)^2}B_{m-2}^{(r-1)} - \frac{m}{2\pi in}B_{m-1}^{(r-1)} \\
&= \cdots \\
&= \frac{m(m-1)(m-2)\cdots 2}{(2\pi in)^{m-1}}C_n^{(r,1)} - \sum_{k=1}^{m-1}\frac{(m)_k}{(2\pi in)^k}B_{m-k}^{(r-1)},
\end{aligned} \tag{2.7}
$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, for $n \geq 1$, and $(x)_0 = 1$. Now, we observe that

$$
\begin{aligned}
C_n^{(r,1)} &= \int_0^1 B_1^{(r)}(x)e^{-2\pi inx}\,dx = \int_0^1 \left(x + B_1^{(r)}\right)e^{-2\pi inx}\,dx \\
&= \int_0^1 xe^{-2\pi inx}\,dx + B_1^{(r)}\int_0^1 e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in}\left[xe^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in}.
\end{aligned} \tag{2.8}
$$

From (2.7) and (2.8), we can derive equation (2.9):

$$
\begin{aligned}
C_n^{(r,m)} &= -\frac{m!}{(2\pi in)^m} - \sum_{k=1}^{m-1}\frac{(m)_k}{(2\pi in)^k}B_{m-k}^{(r-1)} \\
&= -\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}B_{m-k}^{(r-1)}.
\end{aligned} \tag{2.9}
$$

Case 2. Let $n = 0$. Then we have

$$
\begin{aligned}
C_0^{(r,m)} &= \int_0^1 B_m^{(r)}\left(\langle x\rangle\right)dx = \int_0^1 B_m^{(r)}(x)\,dx = \frac{1}{m+1}\left[B_{m+1}^{(r)}(x)\right]_0^1 \\
&= \frac{1}{m+1}\left(B_{m+1}^{(r)}(1) - B_{m+1}^{(r)}(0)\right) = B_m^{(r-1)}.
\end{aligned} \tag{2.10}
$$

Before proceeding, we recall the following equations:

$$
B_m\left(\langle x\rangle\right) = -m!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m}, \quad (m \geq 2) \text{ (see [1])}, \tag{2.11}
$$

and

$$
-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}, \text{ (see [1, 22])}. \end{cases} \tag{2.12}
$$

The series in (2.11) converges uniformly, while that in (2.12) converges pointwise. Assume first that $B_{m-1}^{(r-1)}(0) = 0$. Then we have $B_m^{(r)}(1) = B_m^{(r)}(0)$, and $m \geq 2$. As $B_m^{(r)}(\langle x \rangle)$ is piecewise C^∞ and continuous, the Fourier series of $B_m^{(r)}(\langle x \rangle)$ converges uniformly to $B_m^{(r)}(\langle x \rangle)$, and

$$\begin{aligned} B_m^{(r)}\big(\langle x \rangle\big) &= \sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi inx} \\ &= B_m^{(r-1)} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} B_{m-k}^{(r-1)} \right) e^{2\pi inx} \\ &= B_m^{(r-1)} + \sum_{k=1}^{m} \frac{(m)_k}{k!} B_{m-k}^{(r-1)} \cdot \left(-k! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^k} \right) \\ &= B_m^{(r-1)} + \sum_{k=2}^{m} \binom{m}{k} B_{m-k}^{(r-1)} B_k\big(\langle x \rangle\big) \\ &\quad + \binom{m}{1} B_{m-1}^{(r-1)} \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\ &= \begin{cases} \sum_{k=0}^{m} \binom{m}{k} B_{m-k}^{(r-1)} B_k(\langle x \rangle) & \text{for } x \notin \mathbb{Z}, \\ \sum_{\substack{k=0 \\ k\neq 1}}^{m} \binom{m}{k} B_{m-k}^{(r-1)} B_k(\langle x \rangle) & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

Note that (2.13) holds whether $B_{m-1}^{(r-1)}(0) = 0$ or not. However, if $B_{m-1}^{(r-1)}(0) = 0$, then

$$B_m^{(r)}\big(\langle x \rangle\big) = \sum_{\substack{k=0 \\ k\neq 1}}^{m} \binom{m}{k} B_{m-k}^{(r-1)} B_k\big(\langle x \rangle\big), \quad \text{for all } x \in (-\infty, \infty).$$

Therefore, we obtain the following theorem.

Theorem 2.1 *Let $m \geq 2$, $r \geq 2$. Assume that $B_{m-1}^{(r-1)}(0) = 0$.*

(a) *$B_m^{(r)}(\langle x \rangle)$ has the Fourier series expansion*

$$B_m^{(r)}\big(\langle x \rangle\big) = B_m^{(r-1)}(0) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\sum_{k=1}^{m} \frac{(m)_k}{(2\pi in)^k} B_{m-k}^{(r-1)} \right) e^{2\pi inx},$$

for $x \in (-\infty, \infty)$. Here the convergence is uniform.

(b) *$B_m^{(r)}(\langle x \rangle) = \sum_{\substack{k=0 \\ k\neq 1}}^{m} \binom{m}{k} B_{m-k}^{(r-1)} B_k(\langle x \rangle)$, for all $x \in (-\infty, \infty)$, where $B_k(\langle x \rangle)$ is the Bernoulli function.*

Assume next that $B_{m-1}^{(r-1)}(0) \neq 0$. Then $B_m^{(r)}(1) \neq B_m^{(r)}(0)$, and hence $B_m^{(r)}(\langle x \rangle)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers Thus the Fourier series of $B_m^{(r)}(\langle x \rangle)$ converges pointwise to $B_m^{(r)}(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to $\frac{1}{2}(B_m^{(r)}(0) + B_m^{(r)}(1)) = B_m^{(r)}(0) + \frac{m}{2} B_{m-1}^{(r-1)}(0)$, for $x \in \mathbb{Z}$. Thus we obtain the following theorem.

Theorem 2.2 *Let $m \geq 1$, $r \geq 2$, Assume that $B_{m-1}^{(r-1)}(0) \neq 0$.*

$$\text{(a)}\quad B_m^{(r-1)}(0) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\sum_{k=1}^{m}\frac{(m)_k}{(2\pi in)^k}B_{m-k}^{(r-1)}\right)e^{2\pi inx} = \begin{cases} B_m^{(r)}(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ B_m^{(r)} + \frac{m}{2}B_{m-1}^{(r-1)}, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Here the convergence is pointwise,

$$\text{(b)}\quad \sum_{k=0}^{m}\binom{m}{k}B_{m-k}^{(r-1)}B_k\bigl(\langle x\rangle\bigr) = B_m^{(r)}(x), \quad \text{for } x \notin \mathbb{Z},$$

and

$$\sum_{\substack{k=0 \\ k\neq 1}}^{m}\binom{m}{k}B_{m-k}^{(r-1)}B_k\bigl(\langle x\rangle\bigr) = B_m^{(r)} + \frac{m}{2}B_{m-1}^{(r-1)}, \quad \text{for } x \in \mathbb{Z},$$

where $B_k(\langle x\rangle)$ is the Bernoulli function.

Remark Let $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, $(\mathrm{Re}(s) > 1)$. From (1.7), we note that, for $m \geq 1$,

$$\begin{aligned} B_{2m} &= -(2m)!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{(-1)^m}{(2\pi n)^{2m}} \\ &= -\frac{(2m)!}{(2\pi)^{2m}}2\sum_{n=1}^{\infty}\frac{(-1)^m}{n^{2m}} = (-1)^{m+1}\frac{2(2m)!}{(2\pi)^{2m}}\zeta(2m). \end{aligned}$$

3 Results and discussion

In this paper, we studied the Fourier series expansion of the higher-order Bernoulli functions $B_m^{(r)}(\langle x\rangle)$ which are obtained by extending by periodicity of period 1 the higher-order Bernoulli polynomials $B_m^{(r)}(x)$ on $[0,1)$. As it turns out, the Fourier series of $B_m^{(r)}(\langle x\rangle)$ converges uniformly to $B_m^{(r)}(\langle x\rangle)$, if $B_{m-1}^{(r-1)}(0) = 0$, and converges pointwise to $B_m^{(r)}(\langle x\rangle)$ for $x \notin \mathbb{Z}$ and converges to $B_m^{(r)} + \frac{m}{2}B_{m-1}^{(r-1)}$ for $x \in \mathbb{Z}$, if $B_{m-1}^{(r-1)}(0) \neq 0$. Here the Fourier series of the higher-order Bernoulli functions $B_m^{(r)}(\langle x\rangle)$ are explicitly determined. In addition, in each case the Fourier series of the higher-order Bernoulli functions $B_m^{(r)}(\langle x\rangle)$ are expressed in terms of Bernoulli functions which are obtained by extending by periodicity of period 1 the ordinary Bernoulli polynomials $B_m(x)$ on $[0,1)$. The Fourier series expansion of the Bernoulli functions are useful in computing the special values of the Dirichlet L-functions. For details, one is referred to [24].

It is expected that the Fourier series of the higher-order Bernoulli functions will find some applications in connections with a certain generalization of Dirichlet L-functions and higher-order generalized Bernoulli numbers.

4 Conclusion

In this paper, we considered the Fourier series expansion of the higher-order Bernoulli functions $B_m^{(r)}(\langle x\rangle)$ which are obtained by extending by periodicity of period 1 the higher-order Bernoulli polynomials $B_m^{(r)}(x)$ on $[0,1)$. The Fourier series are explicitly determined. Depending on whether $B_{m-1}^{(r-1)}(0)$ is zero or not, the Fourier series of $B_m^{(r)}(\langle x\rangle)$ converges uniformly to $B_m^{(r)}(\langle x\rangle)$ or converges pointwise to $B_m^{(r)}(\langle x\rangle)$ for $x \notin \mathbb{Z}$ and converges to $B_m^{(r)} + \frac{m}{2}B_{m-1}^{(r-1)}$ for $x \in \mathbb{Z}$. In addition, the Fourier series of the higher-order Bernoulli functions $B_m^{(r)}(\langle x\rangle)$ are expressed in terms of Bernoulli functions $B_k(\langle x\rangle)$. Thus we established the relations between higher-order Bernoulli functions and Bernoulli functions. Just as the Fourier series expansion of the Bernoulli functions are useful in computing the special values of Dirichlet L-functions, we would like to see some applications to a certain generalization of Dirichlet L-functions and higher-order generalized Bernoulli numbers in near future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Acknowledgements
This paper is supported by grant NO 14-11-00022 of Russian Scientific Fund.

Received: 3 November 2016 Accepted: 22 December 2016 Published online: 05 January 2017

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