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A coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions

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Abstract

In this paper, we investigate the existence and the uniqueness of solutions for coupled and uncoupled systems of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions. The existence and the uniqueness of the solutions are established by using the Banach contraction principle, while the existence of solutions is derived by applying Leray-Schauder's alternative. Examples illustrating our results are also presented.

MSC: 34A08; 34A12; 34B15; 93A10

Keywords: fractional q-difference equations; existence; uniqueness; fixed point theorems

1 Introduction

In this paper, we investigate a coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions given by

$$\begin{cases} D_q^{\alpha} x(t) = f(t, x(t), I_r^{\delta} y(t)), & t \in [0,T], 1 < \alpha \le 2, \\ D_p^{\beta} y(t) = g(t, y(t), I_z^{\varepsilon} x(t)), & t \in [0,T], 1 < \beta \le 2, \\ x(0) = 0, \quad \lambda_1 I_m^{\gamma} x(\eta) = I_n^{\kappa} y(\xi), \\ y(0) = 0, \quad \lambda_2 I_h^{\mu} y(\theta) = I_k^{\nu} x(\tau), \end{cases} \tag{1.1}$$

where $0 < p, q, r, z, m, n, h, k < 1$ are quantum numbers, $\eta, \xi, \theta, \tau \in (0, T)$ are fixed points, $\delta, \varepsilon, \gamma, \kappa, \mu, \nu > 0$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ are given constants, D_ω^ρ is the fractional ω-derivative of Riemann-Liouville type of order ρ, when $\rho \in \{\alpha, \beta\}$ and $\omega \in \{p, q\}$, I_ϕ^ψ is the fractional ϕ-integral of order ψ with $\phi \in \{r, z, m, n, h, k\}$ and $\psi \in \{\delta, \varepsilon, \gamma, \kappa, \mu, \nu\}$ and $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The early work on q-difference calculus or *quantum calculus* dates back to Jackson's paper [1]. Basic definitions and properties of quantum calculus can be found in the book [2]. The fractional q-difference calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. Motivated by recent interest in the study of fractional-order differential equations, the topic of q-fractional equations has attracted the attention of many researchers. The details of some recent development of the subject can be found in [5–18], and the references cited therein, whereas the background material on q-fractional calculus can be found in a recent book [19].

Recently in [20], we have studied the existence and the uniqueness of solutions of a class of boundary value problems for fractional q-integro-difference equations with nonlocal fractional q-integral conditions which have different quantum numbers. Here we extend the results of [20] to a coupled system of fractional q-integro-difference equations with nonlocal fractional q-integral boundary conditions.

The paper is organized as follows: In Section 2 we will present some useful preliminaries and lemmas. Some auxiliary lemmas are presented in Section 3. In Section 4, we establish an existence and a uniqueness result via the Banach contraction principle, and an existence result by applying Leray-Schauder's alternative. Results on the uncoupled integral boundary conditions case are contained in Section 5. Examples illustrating our results are also presented.

2 Preliminaries

To make this paper self-contained, below we recall some well-known facts on fractional q-calculus. The presentation here can be found in, for example, [6, 19].

For $q \in (0,1)$, define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}. \tag{2.1}$$

The q-analog of the power function $(a-b)^k$ with $k \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(k)} = \prod_{i=0}^{k-1}\bigl(a-bq^i\bigr), \quad k \in \mathbb{N}, a, b \in \mathbb{R}. \tag{2.2}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(a-b)^{(\gamma)} = a^\gamma \prod_{i=0}^{\infty} \frac{1-(b/a)q^i}{1-(b/a)q^{\gamma+i}}, \quad a \neq 0. \tag{2.3}$$

Note if $b = 0$, then $a^{(\gamma)} = a^\gamma$. We also use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, \quad t \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}. \tag{2.4}$$

Obviously, $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$.

The q-derivative of a function h is defined by

$$(D_q h)(t) = \frac{h(t)-h(qt)}{(1-q)t} \quad \text{for } t \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{t\to 0}(D_q h)(t), \tag{2.5}$$

and q-derivatives of higher order are given by

$$\bigl(D_q^0 h\bigr)(t) = h(t) \quad \text{and} \quad \bigl(D_q^k h\bigr)(t) = D_q\bigl(D_q^{k-1} h\bigr)(t), \quad k \in \mathbb{N}. \tag{2.6}$$

The q-integral of a function h defined on the interval $[0,b]$ is given by

$$(I_q h)(t) = \int_0^t h(s)\,d_q s = t(1-q)\sum_{i=0}^{\infty} h\bigl(tq^i\bigr)q^i, \quad t \in [0,b]. \tag{2.7}$$

If $a \in [0,b]$ and h is defined in the interval $[0,b]$, then its integral from a to b is defined by

$$\int_a^b h(s)\,d_qs = \int_0^b h(s)\,d_qs - \int_0^a h(s)\,d_qs. \tag{2.8}$$

Similar to derivatives, an operator I_q^k is given by

$$\big(I_q^0 h\big)(t) = h(t) \quad \text{and} \quad \big(I_q^k h\big)(t) = I_q\big(I_q^{k-1}h\big)(t), \quad k \in \mathbb{N}. \tag{2.9}$$

The fundamental theorem of calculus applies to these operators D_q and I_q, *i.e.*,

$$(D_qI_qh)(t) = h(t), \tag{2.10}$$

and if h is continuous at $t = 0$, then

$$(I_qD_qh)(t) = h(t) - h(0). \tag{2.11}$$

Definition 2.1 Let $\nu \geq 0$ and h be a function defined on $[0,T]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0h)(t) = h(t)$ and

$$\big(I_q^\nu h\big)(t) = \frac{1}{\Gamma_q(\nu)}\int_0^t (t-qs)^{(\nu-1)}h(s)\,d_qs, \quad \nu > 0, t \in [0,T]. \tag{2.12}$$

Definition 2.2 The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0h)(t) = h(t)$ and

$$\big(D_q^\nu h\big)(t) = \big(D_q^l I_q^{l-\nu}h\big)(t), \quad \nu > 0, \tag{2.13}$$

where l is the smallest integer greater than or equal to ν.

Definition 2.3 For any $t, s > 0$,

$$B_q(t,s) = \int_0^1 u^{(t-1)}(1-qu)^{(s-1)}\,d_qu \tag{2.14}$$

is called the q-beta function.

The expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

Lemma 2.4 [4] *Let* $\alpha, \beta \geq 0$ *and* f *be a function defined in* $[0,T]$. *Then the following formulas hold*:

(1) $(I_q^\beta I_q^\alpha f)(t) = (I_q^{\alpha+\beta}f)(t)$,

(2) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 2.5 [6] *Let $\alpha > 0$ and n be a positive integer. Then the following equality holds*:

$$\big(I_q^\alpha D_q^n f\big)(t) = \big(D_q^n I_q^\alpha f\big)(t) - \sum_{i=0}^{n-1} \frac{t^{\alpha-n+i}}{\Gamma_q(\alpha+i-n+1)} \big(D_q^i f\big)(0). \tag{2.15}$$

3 Some auxiliary lemmas

The following formulas have been modified from Lemmas 3.2 and 7 in [21] and [20], respectively.

Lemma 3.1 *Let $x, y, z > 0$ and $0 < u, v, w < 1$. Then, for $\phi \in \mathbb{R}_+$, we have*

(i) $I_u^x I_v^y (1)(\phi) = \frac{\Gamma_u(y+1)}{\Gamma_u(x+y+1)\Gamma_v(y+1)} \phi^{x+y}$;

(ii) $I_u^x I_v^y I_w^z (1)(\phi) = \frac{\Gamma_u(y+z+1)\Gamma_v(z+1)}{\Gamma_u(x+y+z+1)\Gamma_v(y+z+1)\Gamma_w(z+1)} \phi^{x+y+z}$.

Lemma 3.2 *Given $u, v \in C([0,T], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_q^\alpha x(t) = u(t), & t \in [0,T], 1 < \alpha \le 2, \\ D_p^\beta y(t) = v(t), & t \in [0,T], 1 < \beta \le 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa y(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\nu x(\tau), \end{cases} \tag{3.1}$$

is

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s)\, d_q s + \frac{\lambda_2 \Omega_1}{\Omega} t^{\alpha-1} I_h^\mu I_p^\beta v(\theta) \\ & - \frac{\Omega_1}{\Omega} t^{\alpha-1} I_k^\nu I_q^\alpha u(\tau) + \frac{\lambda_1\lambda_2\Omega_4}{\Omega} t^{\alpha-1} I_m^\gamma I_q^\alpha u(\eta) \\ & - \frac{\lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_n^\kappa I_p^\beta v(\xi) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma_p(\beta)} \int_0^t (t-ps)^{(\beta-1)} v(s)\, d_p s + \frac{\lambda_1 \Omega_2}{\Omega} t^{\beta-1} I_m^\gamma I_q^\alpha u(\eta) \\ & - \frac{\Omega_2}{\Omega} t^{\beta-1} I_n^\kappa I_p^\beta v(\xi) + \frac{\lambda_1\lambda_2\Omega_3}{\Omega} t^{\beta-1} I_h^\mu I_p^\beta v(\theta) \\ & - \frac{\lambda_1 \Omega_3}{\Omega} t^{\beta-1} I_k^\nu I_q^\alpha u(\tau), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \Omega_1 &= \frac{\Gamma_n(\beta)}{\Gamma_n(\beta+\kappa)} \xi^{\beta+\kappa-1}, \\ \Omega_2 &= \frac{\Gamma_k(\alpha)}{\Gamma_k(\alpha+\nu)} \tau^{\alpha+\nu-1}, \\ \Omega_3 &= \frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha+\gamma)} \eta^{\alpha+\gamma-1}, \\ \Omega_4 &= \frac{\Gamma_h(\beta)}{\Gamma_h(\beta+\mu)} \theta^{\beta+\mu-1}, \\ \Omega &= \Omega_1\Omega_2 - \lambda_1\lambda_2\Omega_3\Omega_4 \neq 0. \end{aligned}$$

Proof From $1 < \alpha \leq 2$, we let $n = 2$. Applying Lemma 2.5, the equations in (3.1) can be expressed as equivalent integral equations

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} u(s)\, d_q s, \tag{3.4}$$

$$y(t) = d_1 t^{\beta-1} + d_2 t^{\beta-2} + \frac{1}{\Gamma_p(\beta)} \int_0^t (t - ps)^{(\beta-1)} v(s)\, d_p s \tag{3.5}$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0$ and $y(0) = 0$ imply that $c_2 = 0$ and $d_2 = 0$, respectively. Taking the Riemann-Liouville fractional ϕ-integral of order $\psi > 0$ for (3.4) and (3.5), we have the system

$$\begin{aligned} I_\phi^\psi x(t) = {}& c_1 \frac{\Gamma_\phi(\alpha)}{\Gamma_\phi(\alpha+\psi)} t^{\alpha+\psi-1} \\ &+ \frac{1}{\Gamma_\phi(\psi)\Gamma_q(\alpha)} \int_0^t \int_0^s (t - \phi s)^{(\psi-1)} (s - qw)^{(\alpha-1)} u(w)\, d_q w\, d_\phi s, \end{aligned} \tag{3.6}$$

$$\begin{aligned} I_\phi^\psi y(t) = {}& d_1 \frac{\Gamma_\phi(\beta)}{\Gamma_\phi(\beta+\psi)} t^{\beta+\psi-1} \\ &+ \frac{1}{\Gamma_\phi(\psi)\Gamma_p(\beta)} \int_0^t \int_0^s (t - \phi s)^{(\psi-1)} (s - pw)^{(\beta-1)} v(w)\, d_p w\, d_\phi s. \end{aligned} \tag{3.7}$$

Substituting (ψ, ϕ, t) by (γ, m, η), (ν, k, τ) in (3.6), and (κ, n, ξ), (μ, h, θ) in (3.7) and using Lemma 2.4 with nonlocal conditions in (3.1), we have

$$\begin{aligned} c_1 = {}& \frac{\lambda_2 \Omega_1}{\Omega} I_h^\mu I_p^\beta v(\theta) - \frac{\Omega_1}{\Omega} I_k^\nu I_q^\alpha u(\tau) \\ &+ \frac{\lambda_1 \lambda_2 \Omega_4}{\Omega} I_m^\gamma I_q^\alpha u(\eta) - \frac{\lambda_2 \Omega_4}{\Omega} I_n^\kappa I_p^\beta v(\xi) \end{aligned}$$

and

$$\begin{aligned} d_1 = {}& \frac{\lambda_1 \Omega_2}{\Omega} I_m^\gamma I_q^\alpha u(\eta) - \frac{\Omega_2}{\Omega} I_n^\kappa I_p^\beta v(\xi) \\ &+ \frac{\lambda_1 \lambda_2 \Omega_3}{\Omega} I_h^\mu I_p^\beta v(\theta) - \frac{\lambda_1 \Omega_3}{\Omega} I_k^\nu I_q^\alpha u(\tau). \end{aligned}$$

Substituting the values of c_1, c_2, d_1, and d_2 in (3.4) and (3.5), we obtain the solutions (3.2) and (3.3) as required. □

4 Main results

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$. Let us introduce the space $X = \{x(t) | x(t) \in C([0,T],\mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t) | y(t) \in C([0,T],\mathbb{R})\}$ be endowed with the norm $\|y\| = \sup\{|y(t)|, t \in [0,T]\}$. Obviously the product space $(X \times Y, \|(x,y)\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 3.2, we define an operator $\mathcal{K} : X \times Y \to X \times Y$ by

$$\mathcal{K}(x,y)(t) = \begin{pmatrix} \mathcal{K}_1(x,y)(t) \\ \mathcal{K}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned}\mathcal{K}_1(x,y)(t) &= I_q^{\alpha} f\big(s,x(s),I_r^{\delta}y(s)\big)(t) + \frac{\lambda_2\Omega_1}{\Omega}t^{\alpha-1}I_h^{\mu}I_p^{\beta}g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(\theta)\\ &\quad - \frac{\Omega_1}{\Omega}t^{\alpha-1}I_k^{\nu}I_q^{\alpha}f\big(s,x(s),I_r^{\delta}y(s)\big)(\tau)\\ &\quad + \frac{\lambda_1\lambda_2\Omega_4}{\Omega}t^{\alpha-1}I_m^{\gamma}I_q^{\alpha}f\big(s,x(s),I_r^{\delta}y(s)\big)(\eta)\\ &\quad - \frac{\lambda_2\Omega_4}{\Omega}t^{\alpha-1}I_n^{\kappa}I_p^{\beta}g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(\xi)\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\mathcal{K}_2(x,y)(t) &= I_p^{\beta} g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(t) + \frac{\lambda_1\Omega_2}{\Omega}t^{\beta-1}I_m^{\gamma}I_q^{\alpha}f\big(s,x(s),I_r^{\delta}y(s)\big)(\eta)\\ &\quad - \frac{\Omega_2}{\Omega}t^{\beta-1}I_n^{\kappa}I_p^{\beta}g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(\xi)\\ &\quad + \frac{\lambda_1\lambda_2\Omega_3}{\Omega}t^{\beta-1}I_h^{\mu}I_p^{\beta}g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(\theta)\\ &\quad - \frac{\lambda_1\Omega_3}{\Omega}t^{\beta-1}I_k^{\nu}I_q^{\alpha}f\big(s,x(s),I_r^{\delta}y(s)\big)(\tau).\end{aligned}\tag{4.2}$$

For the sake of convenience, we set

$$\begin{aligned}
&A_1 = \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}, \qquad A_2 = \frac{T^{\beta}}{\Gamma_p(\beta+1)},\\
&A_3 = \frac{\Gamma_q(\delta+1)T^{\alpha+\delta}}{\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)}, \qquad A_4 = \frac{\Gamma_p(\varepsilon+1)T^{\beta+\varepsilon}}{\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},\\
&A_5 = \frac{\Gamma_m(\alpha+1)\eta^{\gamma+\alpha}}{\Gamma_m(\gamma+\alpha+1)\Gamma_q(\alpha+1)}, \qquad A_6 = \frac{\Gamma_n(\beta+1)\xi^{\kappa+\beta}}{\Gamma_n(\kappa+\beta+1)\Gamma_p(\beta+1)},\\
&A_7 = \frac{\Gamma_h(\beta+1)\theta^{\mu+\beta}}{\Gamma_h(\mu+\beta+1)\Gamma_p(\beta+1)}, \qquad A_8 = \frac{\Gamma_k(\alpha+1)\tau^{\nu+\alpha}}{\Gamma_k(\nu+\alpha+1)\Gamma_q(\alpha+1)},\\
&A_9 = \frac{\Gamma_m(\alpha+\delta+1)\Gamma_q(\delta+1)\eta^{\gamma+\alpha+\delta}}{\Gamma_m(\gamma+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)},\\
&A_{10} = \frac{\Gamma_n(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\xi^{\kappa+\beta+\varepsilon}}{\Gamma_n(\kappa+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},\\
&A_{11} = \frac{\Gamma_h(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\theta^{\mu+\beta+\varepsilon}}{\Gamma_h(\mu+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},\\
&A_{12} = \frac{\Gamma_k(\alpha+\delta+1)\Gamma_q(\delta+1)\tau^{\nu+\alpha+\delta}}{\Gamma_k(\nu+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)}.
\end{aligned}$$

Theorem 4.1 *Assume that $f,g:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions and there exist positive constants M_i, N_i, $i=1,2$, such that for all $t\in[0,T]$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$,*

$$\big|f(t,u_1,u_2)-f(t,v_1,v_2)\big| \le M_1|u_1-v_1| + M_2|u_2-v_2|$$

and

$$\big|g(t,u_1,u_2)-g(t,v_1,v_2)\big| \le N_1|u_1-v_1| + N_2|u_2-v_2|.$$

In addition, we suppose that

$$B_1 + B_2 + C_1 + C_2 < 1,$$

where

$$B_1 = M_1A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\big(\Omega_1N_2A_{11} + |\lambda_1|\Omega_4M_1A_5 + \Omega_4N_2A_{10}\big) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}M_1A_8,$$

$$B_2 = M_2A_3 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\big(\Omega_1N_1A_7 + |\lambda_1|\Omega_4M_2A_9 + \Omega_4N_1A_6\big) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}M_2A_{12},$$

$$C_1 = N_2A_4 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\big(\Omega_2M_1A_5 + |\lambda_2|\Omega_3N_2A_{11} + \Omega_3M_1A_8\big) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}N_2A_{10},$$

$$C_2 = N_1A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\big(\Omega_2M_2A_9 + |\lambda_2|\Omega_3N_1A_7 + \Omega_3M_2A_{12}\big) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}N_1A_6.$$

Then the system (1.1) *has a unique solution on* $[0,T]$.

Proof Firstly, we define $\sup_{t\in[0,T]}|f(t,0,0)| = G_1 < \infty$ and $\sup_{t\in[0,T]}|g(t,0,0)| = G_2 < \infty$ such that

$$r \geq \max\left\{\frac{B_3}{1-(B_1+B_2)}, \frac{C_3}{1-(C_1+C_2)}\right\},$$

where

$$B_3 = G_1A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\big(\Omega_1G_2A_7 + |\lambda_1|\Omega_4G_1A_5 + \Omega_4G_2A_6\big) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}G_1A_8,$$

$$C_3 = G_2A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\big(\Omega_2G_1A_5 + |\lambda_2|\Omega_3G_2A_7 + \Omega_3G_1A_8\big) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}G_2A_6.$$

We will show that $\mathcal{K}B_r \subset B_r$, where $B_r = \{(x,y) \in X \times Y : \|(x,y)\| \leq r\}$.

For $(x,y) \in B_r$, taking into account Lemma 3.1, we have

$$\begin{aligned}
&\big|\mathcal{K}_1(x,y)(t)\big| \\
&\quad\leq \sup_{t\in T}\bigg\{I_q^{\alpha}\big|f\big(s,x(s),I_r^{\delta}y(s)\big)\big|(t) + \frac{|\lambda_2|\Omega_1}{|\Omega|}t^{\alpha-1}I_h^{\mu}I_p^{\beta}\big|g\big(s,y(s),I_z^{\varepsilon}x(s)\big)\big|(\theta) \\
&\qquad + \frac{\Omega_1}{|\Omega|}t^{\alpha-1}I_k^{\nu}I_q^{\alpha}\big|f\big(s,x(s),I_r^{\delta}y(s)\big)\big|(\tau) + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|}t^{\alpha-1}I_m^{\gamma}I_q^{\alpha}\big|f\big(s,x(s),I_r^{\delta}y(s)\big)\big|(\eta) \\
&\qquad + \frac{|\lambda_2|\Omega_4}{|\Omega|}t^{\alpha-1}I_n^{\kappa}I_p^{\beta}\big|g\big(s,y(s),I_z^{\varepsilon}x(s)\big)\big|(\xi)\bigg\} \\
&\quad\leq I_q^{\alpha}\big(\big|f\big(s,x(s),I_r^{\delta}y(s)\big) - f(s,0,0)\big| + \big|f(s,0,0)\big|\big)(t) \\
&\qquad + \frac{|\lambda_2|\Omega_1}{|\Omega|}T^{\alpha-1}I_h^{\mu}I_p^{\beta}\big(\big|g\big(s,y(s),I_z^{\varepsilon}x(s)\big) - g(s,0,0)\big| + \big|g(s,0,0)\big|\big)(\theta) \\
&\qquad + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}I_k^{\nu}I_q^{\alpha}\big(\big|f\big(s,x(s),I_r^{\delta}y(s)\big) - f(s,0,0)\big| + \big|f(s,0,0)\big|\big)(\tau) \\
&\qquad + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|}T^{\alpha-1}I_m^{\gamma}I_q^{\alpha}\big(\big|f\big(s,x(s),I_r^{\delta}y(s)\big) - f(s,0,0)\big| + \big|f(s,0,0)\big|\big)(\eta)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_n^{\kappa} I_p^{\beta}\big(\big|g\big(s,y(s),I_z^{\varepsilon}x(s)\big)-g(t,0,0)\big| + \big|g(t,0,0)\big|\big)(\xi) \\
&\leq M_1\|x\|A_1 + M_2\|y\|A_3 + G_1A_1 \\
&\quad + \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1}\big(N_1\|y\|A_7 + N_2\|x\|A_{11} + G_2A_7\big) \\
&\quad + \frac{\Omega_1}{|\Omega|} T^{\alpha-1}\big(M_1\|x\|A_8 + M_2\|y\|A_{12} + G_1A_8\big) \\
&\quad + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1}\big(M_1\|x\|A_5 + M_2\|y\|A_9 + G_1A_5\big) \\
&\quad + \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1}\big(N_1\|y\|A_6 + N_2\|x\|A_{10} + G_2A_6\big) \\
&= B_1\|x\| + B_2\|y\| + B_3 \\
&\leq (B_1+B_2)r + B_3 \leq r.
\end{aligned}
$$

In a similar way, we get

$$
\begin{aligned}
&\big|\mathcal{K}_2(x,y)(t)\big| \\
&\leq I_p^{\beta}\big(\big|g\big(s,y(s),I_z^{\varepsilon}x(s)\big)-g(s,0,0)\big| + \big|g(s,0,0)\big|\big)(t) \\
&\quad + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} I_m^{\gamma} I_q^{\alpha}\big(\big|f\big(s,x(s),I_r^{\delta}y(s)\big)-f(s,0,0)\big| + \big|f(s,0,0)\big|\big)(\eta) \\
&\quad + \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^{\kappa} I_p^{\beta}\big(\big|g\big(s,y(s),I_z^{\varepsilon}x(s)\big)-g(s,0,0)\big| + \big|g(s,0,0)\big|\big)(\xi) \\
&\quad + \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} I_h^{\mu} I_p^{\beta}\big(\big|g\big(s,y(s),I_z^{\varepsilon}x(s)\big)-g(s,0,0)\big| + \big|g(s,0,0)\big|\big)(\theta) \\
&\quad + \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} I_k^{\nu} I_q^{\alpha}\big(\big|f\big(s,x(s),I_r^{\delta}y(s)\big)-f(s,0,0)\big| + \big|f(s,0,0)\big|\big)(\tau) \\
&\leq N_1\|y\|A_2 + N_2\|x\|A_4 + G_2A_2 \\
&\quad + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1}\big(M_1\|x\|A_5 + M_2\|y\|A_9 + G_1A_5\big) \\
&\quad + \frac{\Omega_2}{|\Omega|} T^{\beta-1}\big(N_1\|y\|A_6 + N_2\|x\|A_{10} + G_2A_6\big) \\
&\quad + \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1}\big(N_1\|y\|A_7 + N_2\|x\|A_{11} + G_2A_7\big) \\
&\quad + \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1}\big(M_1\|x\|A_8 + M_2\|y\|A_{12} + G_1A_8\big) \\
&= C_1\|x\| + C_2\|y\| + C_3 \\
&\leq (C_1+C_2)r + B_3 \leq r.
\end{aligned}
$$

Consequently, $\|\mathcal{K}(x,y)(t)\| \leq r$.

Next, for $(x_2,y_2),(x_1,y_1) \in X \times Y$, and for any $t \in [0,T]$, we have

$$
\begin{aligned}
&\big|\mathcal{K}_1(x_2,y_2)(t) - \mathcal{K}_1(x_1,y_1)(t)\big| \\
&\leq I_q^{\alpha}\big(\big|f\big(s,x_2(s),I_r^{\delta}y_2(s)\big) - f\big(s,x_1(s),I_r^{\delta}y_1(s)\big)\big|\big)(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} I_h^{\mu} I_p^{\beta} \bigl(\bigl|g\bigl(s, y_2(s), I_z^{\varepsilon} x_2(s)\bigr) - g\bigl(s, y_1(s), I_z^{\varepsilon} x_1(s)\bigr)\bigr|\bigr)(\theta) \\
&+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^{\nu} I_q^{\alpha} \bigl(\bigl|f\bigl(s, x_2(s), I_r^{\delta} y_2(s)\bigr) - f\bigl(s, x_1(s), I_r^{\delta} y_1(s)\bigr)\bigr|\bigr)(\tau) \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_m^{\gamma} I_q^{\alpha} \bigl(\bigl|f\bigl(s, x_2(s), I_r^{\delta} y_2(s)\bigr) - f\bigl(s, x_1(s), I_r^{\delta} y_1(s)\bigr)\bigr|\bigr)(\eta) \\
&+ \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_n^{\kappa} I_p^{\beta} \bigl(\bigl|g\bigl(s, y_2(s), I_z^{\varepsilon} x_2(s)\bigr) - g\bigl(s, y_1(s), I_z^{\varepsilon} x_1(s)\bigr)\bigr|\bigr)(\xi) \\
\le{}& M_1 \|x_2 - x_1\| I_q^{\alpha}(1)(T) + M_2 \|y_2 - y_1\| I_q^{\alpha} I_r^{\delta}(1)(T) \\
&+ \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} \bigl(N_1 \|y_2 - y_1\| I_h^{\mu} I_p^{\beta}(1)(\theta) + N_2 \|x_2 - x_1\| I_h^{\mu} I_p^{\beta} I_z^{\varepsilon}(1)(\theta)\bigr) \\
&+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} \bigl(M_1 \|x_2 - x_1\| I_k^{\nu} I_q^{\alpha}(1)(\tau) + M_2 \|y_2 - y_1\| I_k^{\nu} I_q^{\alpha} I_r^{\delta}(1)(\tau)\bigr) \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} \bigl(M_1 \|x_2 - x_1\| I_m^{\gamma} I_q^{\alpha}(1)(\eta) + M_2 \|y_2 - y_1\| I_m^{\gamma} I_q^{\alpha} I_r^{\delta}(1)(\eta)\bigr) \\
&+ \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} \bigl(N_1 \|y_2 - y_1\| I_n^{\kappa} I_p^{\beta}(1)(\xi) + N_2 \|x_2 - x_1\| I_n^{\kappa} I_p^{\beta} I_z^{\varepsilon}(1)(\xi)\bigr) \\
={}& B_1 \|x_2 - x_1\| + B_2 \|y_2 - y_1\|.
\end{aligned}
$$

Therefore, we have

$$
\bigl\|\mathcal{K}_1(x_2, y_2)(t) - \mathcal{K}_1(x_1, y_1)(t)\bigr\| \le (B_1 + B_2)\bigl(\|x_2 - x_1\| + \|y_2 - y_1\|\bigr). \tag{4.3}
$$

In the same way, we have

$$
\begin{aligned}
&\bigl|\mathcal{K}_2(x_2, y_2)(t) - \mathcal{K}_2(x_1, y_1)(t)\bigr| \\
&\quad\le I_p^{\beta} \bigl(\bigl|g\bigl(s, y_2(s), I_z^{\varepsilon} x_2(s)\bigr) - g\bigl(s, y_1(s), I_z^{\varepsilon} x_1(s)\bigr)\bigr|\bigr)(t) \\
&\qquad+ \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} I_m^{\gamma} I_q^{\alpha} \bigl(\bigl|f\bigl(s, x_2(s), I_r^{\delta} y_2(s)\bigr) - f\bigl(s, x_1(s), I_r^{\delta} y_1(s)\bigr)\bigr|\bigr)(\eta) \\
&\qquad+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^{\kappa} I_p^{\beta} \bigl(\bigl|g\bigl(s, y_2(s), I_z^{\varepsilon} x_2(s)\bigr) - g\bigl(s, y_1(s), I_z^{\varepsilon} x_1(s)\bigr)\bigr|\bigr)(\xi) \\
&\qquad+ \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} I_h^{\mu} I_p^{\beta} \bigl(\bigl|g\bigl(s, y_2(s), I_z^{\varepsilon} x_2(s)\bigr) - g\bigl(s, y_1(s), I_z^{\varepsilon} x_1(s)\bigr)\bigr|\bigr)(\theta) \\
&\qquad+ \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} I_k^{\nu} I_q^{\alpha} \bigl(\bigl|f\bigl(s, x_2(s), I_r^{\delta} y_2(s)\bigr) - f\bigl(s, x_1(s), I_r^{\delta} y_1(s)\bigr)\bigr|\bigr)(\tau) \\
&\quad\le N_1 \|y_2 - y_1\| A_2 + N_2 \|x_2 - x_1\| A_4 \\
&\qquad+ \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} \bigl(M_1 \|x_2 - x_1\| A_5 + M_2 \|y_2 - y_1\| A_9\bigr) \\
&\qquad+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} \bigl(N_1 \|y_2 - y_1\| A_6 + N_2 \|x_2 - x_1\| A_{10}\bigr) \\
&\qquad+ \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} \bigl(N_1 \|y_2 - y_1\| A_7 + N_2 \|x_2 - x_1\| A_{11}\bigr) \\
&\qquad+ \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} \bigl(M_1 \|x_2 - x_1\| A_8 + M_2 \|y_2 - y_1\| A_{12}\bigr) \\
&\quad= C_1 \|x_2 - x_1\| + C_2 \|y_2 - y_1\|,
\end{aligned}
$$

which implies

$$\left\|\mathcal{K}_2(x_2,y_2)(t)-\mathcal{K}_2(x_1,y_1)(t)\right\| \leq (C_1+C_2)\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr). \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\left\|\mathcal{K}(x_2,y_2)(t)-\mathcal{K}(x_1,y_1)(t)\right\| \leq (B_1+B_2+C_1+C_2)\bigl(\|x_2-x_1\|+\|y_2-y_1\|\bigr).$$

Since $B_1 + B_2 + C_1 + C_2 < 1$, therefore, $\mathcal{K}$ is a contraction operator. So, by Banach's fixed point theorem, the operator $\mathcal{K}$ has a unique fixed point, which is the unique solution of problem (1.1). The proof is completed. □

In the next result, we prove the existence of solutions for the problem (1.1) by applying the Leray-Schauder alternative.

Lemma 4.2 (Leray-Schauder alternative, see [22], p.4) *Let $F : E \to E$ be a completely continuous operator* (*i.e.*, *a map that restricted to any bounded set in E is compact*). *Let*

$$\mathcal{E}(F)=\bigl\{x\in E : x=\lambda F(x) \text{ for some } 0<\lambda<1\bigr\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, *or F has at least one fixed point*.

For convenience, we set constants

$$\begin{aligned}
E_0 &= P_0A_1+\frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1Q_0A_7+|\lambda_1|\Omega_4P_0A_5+\Omega_4Q_0A_6\bigr)+\frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_0A_8,\\
E_1 &= P_1A_1+\frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1Q_2A_{11}+|\lambda_1|\Omega_4P_1A_5+\Omega_4Q_2A_{10}\bigr)+\frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_1A_8,\\
E_3 &= P_2A_3+\frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1Q_1A_7+|\lambda_1|\Omega_4P_2A_9+\Omega_4Q_1A_6\bigr)+\frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_2A_{12},\\
F_0 &= Q_0A_2+\frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\bigl(\Omega_2P_0A_5+|\lambda_2|\Omega_3Q_0A_7+\Omega_3P_0A_8\bigr)+\frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_0A_6,\\
F_1 &= Q_2A_4+\frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\bigl(\Omega_2P_1A_5+|\lambda_2|\Omega_3Q_2A_{11}+\Omega_3P_1A_8\bigr)+\frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_2A_{10},\\
F_2 &= Q_1A_2+\frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\bigl(\Omega_2P_2A_9+|\lambda_2|\Omega_3Q_1A_7+\Omega_3P_2A_{12}\bigr)+\frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_1A_6
\end{aligned}$$

and

$$G^*=\max\bigl\{1-(E_1+F_1),1-(E_2+F_2)\bigr\}.$$

Theorem 4.3 *Assume that there exist real constants $P_i, Q_i \geq 0$ ($i = 1,2$), and $P_0 > 0$, $Q_0 > 0$ such that for all $u_i, v_i \in \mathbb{R}$ ($i = 1,2$) we have*

$$\begin{aligned}
\bigl|f(t,u_1,u_2)\bigr| &\leq P_0+P_1|u_1|+P_2|u_2|,\\
\bigl|g(t,v_1,v_2)\bigr| &\leq Q_0+Q_1|v_1|+Q_2|v_2|.
\end{aligned}$$

In addition it is assumed that

$$E_1 + F_1 < 1 \quad \textit{and} \quad E_2 + F_2 < 1.$$

Then there exists at least one solution for the system (1.1).

Proof We first prove that the operator $\mathcal{K} : X \times Y \to X \times Y$ is completely continuous. The continuity of functions f and g imply that the operator $\mathcal{K}$ is continuous. Let $\Phi \subset X \times Y$ be a bounded set. Then there exist positive constants D_1 and D_2 such that

$$\left|f\left(t, u_1(t), u_2(t)\right)\right| \le D_1, \qquad \left|g\left(t, v_1(t), v_2(t)\right)\right| \le D_2, \quad \forall (u_1, u_2), (v_1, v_2) \in \Phi.$$

Then for any $(u_1, u_2), (v_1, v_2) \in \Phi$, and using Lemma 3.1, we have

$$\begin{aligned}
\left\|\mathcal{K}_1(x, y)\right\| \le{}& I_q^{\alpha}\left|f\left(s, x(s), I_r^{\delta} y(s)\right)\right|(t) + \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} I_h^{\mu} I_p^{\beta}\left|g\left(s, y(s), I_z^{\varepsilon} x(s)\right)\right|(\theta) \\
&+ \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^{\nu} I_q^{\alpha}\left|f\left(s, x(s), I_r^{\delta} y(s)\right)\right|(\tau) \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_m^{\gamma} I_q^{\alpha}\left|f\left(s, x(s), I_r^{\delta} y(s)\right)\right|(\eta) \\
&+ \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_n^{\kappa} I_p^{\beta}\left|g\left(s, y(s), I_z^{\varepsilon} x(s)\right)\right|(\xi) \\
\le{}& D_1 A_1 + \frac{D_2|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} A_7 + \frac{D_1\Omega_1}{|\Omega|} T^{\alpha-1} A_8 \\
&+ \frac{D_1|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} A_5 + \frac{D_2|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} A_6.
\end{aligned}$$

In the same way, we deduce that

$$\begin{aligned}
\left\|\mathcal{K}_2(x, y)\right\| \le{}& I_p^{\beta}\left(\left|g\left(s, y(s), I_z^{\varepsilon} x(s)\right)\right|\right)(t) + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} I_m^{\gamma} I_q^{\alpha}\left(\left|f\left(s, x(s), I_r^{\delta} y(s)\right)\right|\right)(\eta) \\
&+ \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^{\kappa} I_p^{\beta}\left(\left|g\left(s, y(s), I_z^{\varepsilon} x(s)\right)\right|\right)(\xi) \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} I_h^{\mu} I_p^{\beta}\left(\left|g\left(s, y(s), I_z^{\varepsilon} x(s)\right)\right|\right)(\theta) \\
&+ \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} I_k^{\nu} I_q^{\alpha}\left(\left|f\left(s, x(s), I_r^{\delta} y(s)\right)\right|\right)(\tau) \\
\le{}& D_2 A_2 + \frac{D_1|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} A_5 + \frac{D_2\Omega_2}{|\Omega|} T^{\beta-1} A_6 \\
&+ \frac{D_2|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} A_7 + \frac{D_1|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} A_8.
\end{aligned}$$

Thus, it follows from the above inequalities that the operator $\mathcal{K}$ is uniformly bounded.

Next, we show that $\mathcal{K}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\left|\mathcal{K}_1(x, y)(t_2) - \mathcal{K}_1(x, y)(t_1)\right| \\
&\quad \le \left|I_q^{\alpha} f\left(s, x(s), I_r^{\delta} y(s)\right)(t_2) - I_q^{\alpha} f\left(s, x(s), I_r^{\delta} y(s)\right)(t_1)\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|\lambda_2|\Omega_1}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| I_h^{\mu} I_p^{\beta}\left|g\left(s, y_2(s), I_z^{\varepsilon} x_2(s)\right)\right|(\theta) \\
&+ \frac{\Omega_1}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| I_k^{\nu} I_q^{\alpha}\left|f\left(s, x(s), I_r^{\delta} y(s)\right)\right|(\tau) \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| I_m^{\gamma} I_q^{\alpha}\left|f\left(s, x(s), I_r^{\delta} y(s)\right)\right|(\eta) \\
&+ \frac{|\lambda_2|\Omega_4}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| I_n^{\kappa} I_p^{\beta}\left|g\left(s, y(s), I_z^{\varepsilon} x(s)\right)\right|(\xi) \\
\leq{}& \frac{D_1}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}\right] d_q s \\
&+ \frac{D_1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)}\, d_q s \\
&+ \frac{|\lambda_2|\Omega_1 D_2}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| A_7 + \frac{\Omega_1 D_1}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| A_8 \\
&+ \frac{|\lambda_1||\lambda_2|\Omega_4 D_1}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| A_5 + \frac{|\lambda_2|\Omega_4 D_2}{|\Omega|}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| A_6.
\end{aligned}
$$

Analogously, we can get

$$
\begin{aligned}
&\left|\mathcal{K}_2(x,y)(t_2) - \mathcal{K}_2(x,y)(t_1)\right| \\
&\quad\leq \left|I_p^{\beta} g\left(s, y(s), I_z^{\varepsilon} x(s)\right)(t_2) - I_p^{\beta} g\left(s, y(s), I_z^{\varepsilon} x(s)\right)(t_1)\right| \\
&\qquad+ \frac{|\lambda_1|\Omega_2}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| I_m^{\gamma} I_q^{\alpha}\left|f\left(s, x_2(s), I_r^{\delta} y_2(s)\right)\right|(\eta) \\
&\qquad+ \frac{\Omega_2}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| I_n^{\kappa} I_p^{\beta}\left|g\left(s, y_2(s), I_z^{\varepsilon} x_2(s)\right)\right|(\xi) \\
&\qquad+ \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| I_h^{\mu} I_p^{\beta}\left|g\left(s, y_2(s), I_z^{\varepsilon} x_2(s)\right)\right|(\theta) \\
&\qquad+ \frac{|\lambda_1|\Omega_3}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| I_k^{\nu} I_q^{\alpha}\left|f\left(s, x_2(s), I_r^{\delta} y_2(s)\right)\right|(\tau) \\
&\quad\leq \frac{D_2}{\Gamma_p(\beta)} \int_0^{t_1} \left[(t_2 - ps)^{(\beta-1)} - (t_1 - ps)^{(\beta-1)}\right] d_p s \\
&\qquad+ \frac{D_2}{\Gamma_p(\beta)} \int_{t_1}^{t_2} (t_2 - ps)^{(\beta-1)}\, d_p s \\
&\qquad+ \frac{|\lambda_1|\Omega_2 D_1}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| A_5 + \frac{\Omega_2 D_2}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| A_6 \\
&\qquad+ \frac{|\lambda_1||\lambda_2|\Omega_3 D_2}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| A_7 + \frac{|\lambda_1|\Omega_3 D_1}{|\Omega|}\left|t_2^{\beta-1} - t_1^{\beta-1}\right| A_8.
\end{aligned}
$$

Therefore, the operator $\mathcal{K}(x,y)$ is equicontinuous, and thus the operator $\mathcal{K}(x,y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(x,y) \in X \times Y : (x,y) = \lambda\mathcal{K}(x,y), 0 \leq \lambda \leq 1\}$ is bounded. Let $(x,y) \in \mathcal{E}$, then $(x,y) = \lambda\mathcal{K}(x,y)$. For any $t \in [0,T]$, we have

$$
x(t) = \lambda\mathcal{K}_1(x,y)(t), \qquad y(t) = \lambda\mathcal{K}_2(x,y)(t).
$$

Then we have

$$
\begin{aligned}
|x(t)| &\le P_0 I_q^{\alpha}(1)(t) + P_1\|x\| I_q^{\alpha}(1)(t) + P_2\|y\| I_q^{\alpha} I_r^{\delta}(1)(t) \\
&\quad + \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1}\big(Q_0 I_h^{\mu} I_p^{\beta}(1)(\theta) + Q_1\|y\| I_h^{\mu} I_p^{\beta}(1)(\theta) + Q_2\|x\| I_h^{\mu} I_p^{\beta} I_z^{\varepsilon}(1)(\theta)\big) \\
&\quad + \frac{\Omega_1}{|\Omega|} T^{\alpha-1}\big(P_0 I_k^{\nu} I_q^{\alpha}(1)(\tau) + P_1\|x\| I_k^{\nu} I_q^{\alpha}(1)(\tau) + P_2\|y\| I_k^{\nu} I_q^{\alpha} I_r^{\delta}(1)(\tau)\big) \\
&\quad + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1}\big(P_0 I_m^{\gamma} I_q^{\alpha}(1)(\eta) + P_1\|x\| I_m^{\gamma} I_q^{\alpha}(1)(\eta) + P_2\|y\| I_m^{\gamma} I_q^{\alpha} I_r^{\delta}(1)(\eta)\big) \\
&\quad + \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1}\big(Q_0 I_n^{\kappa} I_p^{\beta}(1)(\xi) + Q_1\|y\| I_n^{\kappa} I_p^{\beta}(1)(\xi) + Q_2\|x\| I_n^{\kappa} I_p^{\beta} I_z^{\varepsilon}(1)(\xi)\big) \\
&\le E_0 + E_1\|x\| + E_2\|y\|
\end{aligned}
$$

and

$$
\begin{aligned}
|y(t)| &\le Q_0 I_p^{\beta}(1)(t) + Q_1\|y\| I_p^{\beta}(1)(t) + Q_1\|x\| I_p^{\beta} I_z^{\varepsilon}(1)(t) \\
&\quad + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1}\big(P_0 I_m^{\gamma} I_q^{\alpha}(1)(\eta) + P_1\|x\| I_m^{\gamma} I_q^{\alpha}(1)(\eta) + P_2\|y\| I_m^{\gamma} I_q^{\alpha} I_r^{\delta}(1)(\eta)\big) \\
&\quad + \frac{\Omega_2}{|\Omega|} T^{\beta-1}\big(Q_0 I_n^{\kappa} I_p^{\beta}(1)(\xi) + Q_1\|y\| I_n^{\kappa} I_p^{\beta}(1)(\xi) + Q_2\|x\| I_n^{\kappa} I_p^{\beta} I_z^{\varepsilon}(1)(\xi)\big) \\
&\quad + \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1}\big(Q_0 I_h^{\mu} I_p^{\beta}(1)(\theta) + Q_1\|y\| I_h^{\mu} I_p^{\beta}(1)(\theta) + Q_2\|x\| I_h^{\mu} I_p^{\beta} I_z^{\varepsilon}(1)(\theta)\big) \\
&\quad + \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1}\big(P_0 I_k^{\nu} I_q^{\alpha}(1)(\tau) + P_1\|x\| I_k^{\nu} I_q^{\alpha}(1)(\tau) + P_2\|y\| I_k^{\nu} I_q^{\alpha} I_r^{\delta}(1)(\tau)\big) \\
&\le F_0 + F_1\|x\| + F_2\|y\|,
\end{aligned}
$$

which yields

$$\|x\| \le E_0 + E_1\|x\| + E_2\|y\|$$

and

$$\|y\| \le F_0 + F_1\|x\| + F_2\|y\|.$$

Therefore, we have

$$\|x\| + \|y\| \le (E_0 + F_0) + (E_1 + F_1)\|x\| + (E_2 + F_2)\|y\|,$$

and, consequently,

$$\|(x,y)\| \le \frac{E_0 + F_0}{G^*}$$

for any $t \in [0,T]$, which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 4.2, the operator $\mathcal{K}$ has at least one fixed point. Hence the system (1.1) has at least one solution. The proof is complete. □

4.1 Examples

In this subsection, we present some examples to illustrate our results.

Example 4.4 Consider the following coupled system of fractional q-integro-difference equations:

$$\begin{cases} D_{1/2}^{3/2}x(t) = \frac{\cos^2 \pi t}{(e^t+4)^2} \cdot \frac{|x(t)|}{4+|x(t)|} + \frac{e^{-t^2}}{(t+8)^2} \cdot I_{1/4}^{\pi} y(t) + \frac{\sqrt{2}}{2}, \quad 0 < t < 2, \\ D_{1/3}^{4/3}y(t) = \frac{\sin^2 \pi t}{(11+t)^2} \cdot \frac{|y(t)|}{1+|y(t)|} + \frac{1}{(e^t+8)^2} \cdot I_{1/5}^{\pi/2} x(t) + \sqrt{3}, \\ x(0) = 0, \qquad \sqrt{2} I_{1/8}^{7/6} x(\frac{3}{2}) = I_{1/6}^{\sqrt{2}} y(\frac{1}{2}), \\ y(0) = 0, \qquad \frac{\sqrt{3}}{2} I_{1/9}^{e} y(\frac{1}{3}) = I_{1/7}^{\sqrt{3}} x(\frac{5}{3}). \end{cases} \tag{4.5}$$

Here $\alpha = 3/2$, $\delta = \pi$, $\beta = 4/3$, $\varepsilon = \pi/2$, $\gamma = 7/6$, $\kappa = \sqrt{2}$, $\mu = e$, $\nu = \sqrt{3}$, $q = 1/2$, $r = 1/4$, $p = 1/3$, $z = 1/5$, $m = 1/8$, $n = 1/6$, $h = 1/9$, $k = 1/7$, $\eta = 3/2$, $\xi = 1/2$, $\theta = 1/3$, $\tau = 5/3$, $\lambda_1 = \sqrt{2}$, $\lambda_2 = \sqrt{3}/2$, $T = 2$, $f(t,x,I_r^{\delta}y) = (|x|\cos^2 \pi t)/((e^t+4)^2(4+|x|)) + (e^{-t^2}/((t+8)^2))I_{1/4}^{\pi}y + \sqrt{2}/2$, and $g(t,y,I_r^{\delta}x) = (|y|\sin^2 \pi t)/((11+t)^2(1+|y|)) + (1/(e^t+8)^2)I_{1/5}^{\pi/2}x + \sqrt{3}$. Since

$$\left|f(t,u_1,u_2) - f(t,v_1,v_2)\right| \leq \frac{1}{100}|u_1 - v_1| + \frac{1}{64}|u_2 - v_2|$$

and

$$\left|g(t,u_1,u_2) - g(t,v_1,v_2)\right| \leq \frac{1}{121}|u_1 - v_1| + \frac{1}{81}|u_2 - v_2|,$$

then the assumptions of Theorem 4.1 are satisfied with $M_1 = 1/100$, $M_2 = 1/64$, $N_1 = 1/121$, and $N_2 = 1/81$. By using the Maple program, we can find that

$$\Omega = \Omega_1\Omega_2 - \lambda_1\lambda_2\Omega_3\Omega_4 \approx 0.61154471 \neq 0$$

and

$$\begin{aligned} B_1 &= M_1A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1N_2A_{11} + |\lambda_1|\Omega_4M_1A_5 + \Omega_4N_2A_{10}\bigr) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}M_1A_8 \\ &\approx 0.0577709, \\ B_2 &= M_2A_3 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1N_1A_7 + |\lambda_1|\Omega_4M_2A_9 + \Omega_4N_1A_6\bigr) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}M_2A_{12} \\ &\approx 0.1489994, \\ C_1 &= N_2A_4 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\bigl(\Omega_2M_1A_5 + |\lambda_2|\Omega_3N_2A_{11} + \Omega_3M_1A_8\bigr) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}N_2A_{10} \\ &\approx 0.22629179, \\ C_2 &= N_1A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\bigl(\Omega_2M_2A_9 + |\lambda_2|\Omega_3N_1A_7 + \Omega_3M_2A_{12}\bigr) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}N_1A_6 \\ &\approx 0.28656994. \end{aligned}$$

Therefore, we get

$$B_1 + B_2 + C_1 + C_2 \approx 0.71963204 < 1.$$

Hence, by Theorem 4.1, the problem (4.5) has a unique solution on $[0,2]$.

Example 4.5 Consider the following coupled system of fractional q-integro-difference equations with fractional q-integral conditions:

$$\begin{cases} D_{1/3}^{4/3}x(t) = \frac{4e^{-t}}{(t+11)^2}\cdot\frac{|x(t)|}{2+|x(t)|} + \frac{1}{(e^{-t^2}+8)^2}\cdot I_{1/5}^{\sqrt{3}}y(t) + \sqrt{2}, \quad 0 < t < \pi, \\ D_{1/4}^{7/5}y(t) = \frac{\cos^2 2\pi t}{(10+t)^2}\cdot\frac{|y(t)|}{1+|y(t)|} + \frac{1}{(e^t+7)^2}\cdot I_{1/6}^{\sqrt{2}}x(t) + \frac{1}{2}, \\ x(0) = 0, \qquad I_{1/3}^{1/3}y(\frac{2\pi}{5}) + \sqrt{5}I_{1/2}^{\sqrt{5}}x(\frac{\pi}{5}) = 0, \\ y(0) = 0, \qquad I_{1/8}^{\sqrt{2}/2}x(\frac{4\pi}{5}) + \sqrt{2}I_{1/7}^{\sqrt{\pi}}y(\frac{3\pi}{5}) = 0. \end{cases} \tag{4.6}$$

Here $\alpha = 4/3$, $\delta = \sqrt{3}$, $\beta = 7/5$, $\varepsilon = \sqrt{2}$, $\gamma = \sqrt{5}$, $\kappa = 1/3$, $\mu = \sqrt{\pi}$, $\nu = \sqrt{2}/2$, $q = 1/3$, $r = 1/5$, $p = 1/4$, $z = 1/6$, $m = 1/2$, $n = 1/3$, $h = 1/7$, $k = 1/8$, $\eta = \pi/5$, $\xi = 2\pi/5$, $\theta = 3\pi/5$, $\tau = 4\pi/5$, $\lambda_1 = -\sqrt{5}$, $\lambda_2 = -\sqrt{2}$, $T = \pi$, $f(t,x,I_r^\delta y) = (4e^{-t}/(t+11)^2)(|x|/(2+|x|)) + (1/(e^{-t^2} + 8)^2)I_{1/5}^{\sqrt{3}}y + \sqrt{2}$, and $g(t,y,I_r^\delta x) = (\cos^2 2\pi t/(10+t)^2)(|y|/(1+|y|)) + (1/(e^t+7)^2)I_{1/6}^{\sqrt{2}}x + (1/2)$. Since

$$\bigl|f(t,u_1,u_2)\bigr| \le \sqrt{2} + \frac{1}{72}|u_1| + \frac{1}{81}|u_2|$$

and

$$\bigl|g(t,v_1,v_2)\bigr| \le \frac{1}{2} + \frac{1}{100}|v_1| + \frac{1}{81}|v_2|,$$

then the assumptions of Theorem 4.3 are satisfied with $P_0 = \sqrt{2}$, $P_1 = 1/72$, $P_2 = 1/81$, $Q_0 = 1/2$, $Q_1 = 1/100$, and $Q_2 = 1/81$. By using the Maple program, we can find that

$$\Omega = \Omega_1\Omega_2 - \lambda_1\lambda_2\Omega_3\Omega_4 \approx -4.18385985 \neq 0$$

and

$$\begin{aligned} E_0 &= P_0A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1Q_0A_7 + |\lambda_1|\Omega_4P_0A_5 + \Omega_4Q_0A_6\bigr) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_0A_8 \\ &\approx 11.08984581, \\ E_1 &= P_1A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1Q_2A_{11} + |\lambda_1|\Omega_4P_1A_5 + \Omega_4Q_2A_{10}\bigr) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_1A_8 \\ &\approx 0.1629615871, \\ E_2 &= P_2A_3 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}\bigl(\Omega_1Q_1A_7 + |\lambda_1|\Omega_4P_2A_9 + \Omega_4Q_1A_6\bigr) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_2A_{12} \\ &\approx 0.34091157, \\ F_0 &= Q_0A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\bigl(\Omega_2P_0A_5 + |\lambda_2|\Omega_3Q_0A_7 + \Omega_3P_0A_8\bigr) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_0A_6 \\ &\approx 25.68580671, \\ F_1 &= Q_2A_4 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\bigl(\Omega_2P_1A_5 + |\lambda_2|\Omega_3Q_2A_{11} + \Omega_3P_1A_8\bigr) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_2A_{10} \\ &\approx 0.68153261, \end{aligned}$$

$$F_2 = Q_1A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}\big(\Omega_2P_2A_9 + |\lambda_2|\Omega_3Q_1A_7 + \Omega_3P_2A_{12}\big) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_1A_6$$
$$\approx 0.5902944$$

and

$$G^* = \max\big\{1-(E_1+F_1), 1-(E_2+F_2)\big\} = \max\{0.1555058, 0.06879403\} = 0.1555058.$$

Therefore, we get

$$E_1 + F_1 \approx 0.8444942 < 1 \quad \text{and} \quad E_2 + F_2 \approx 0.93120597 < 1.$$

Hence, by Theorem 4.3, the problem (4.6) has at least one solution on $[0,\pi]$.

5 Uncoupled integral boundary conditions case

In this section we consider the following system:

$$\begin{cases} D_q^\alpha x(t) = f(t, x(t), I_r^\delta y(t)), & t \in [0,T], 1 < \alpha \le 2, \\ D_p^\beta y(t) = g(t, y(t), I_z^\varepsilon x(t)), & t \in [0,T], 1 < \beta \le 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa x(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\nu y(\tau). \end{cases} \tag{5.1}$$

Lemma 5.1 (Auxiliary lemma, see [20]) *For $h \in C([0,T],\mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_q^\alpha x(t) = h(t), & t \in [0,T], 1 < \alpha \le 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa x(\xi), \end{cases} \tag{5.2}$$

is given by

$$x(t) = I_q^\alpha h(t) + \frac{\lambda_1 t^{\alpha-1}}{\Lambda} I_m^\gamma I_q^\alpha h(\eta) - \frac{t^{\alpha-1}}{\Lambda} I_n^\kappa I_q^\alpha h(\xi), \tag{5.3}$$

where

$$\Lambda = \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa+\alpha)}\xi^{\kappa+\alpha-1} - \lambda_1\frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma+\alpha)}\eta^{\gamma+\alpha-1} \neq 0.$$

5.1 Existence results for uncoupled case

In view of Lemma 5.1, we define an operator $\mathcal{T} : X \times Y \to X \times Y$ by

$$\mathcal{T}(x,y)(t) = \begin{pmatrix} \mathcal{T}_1(x,y)(t) \\ \mathcal{T}_2(x,y)(t) \end{pmatrix},$$

where

$$\mathcal{T}_1(x,y)(t) = I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(t) + \frac{\lambda_1 t^{\alpha-1}}{\Lambda} I_m^\gamma I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\eta)$$
$$- \frac{t^{\alpha-1}}{\Lambda} I_n^\kappa I_q^\alpha f\big(s, x(s), I_r^\delta y(s)\big)(\xi)$$

and

$$\begin{aligned}\mathcal{T}_2(x,y)(t) &= I_p^{\beta} g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(t) + \frac{\lambda_2 t^{\beta-1}}{\Psi} I_h^{\mu} I_p^{\beta} g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(\theta)\\ &\quad - \frac{t^{\beta-1}}{\Psi} I_k^{\nu} I_p^{\beta} g\big(s,y(s),I_z^{\varepsilon}x(s)\big)(\tau),\end{aligned}$$

where

$$\Psi = \frac{\Gamma_k(\beta)}{\Gamma_k(\nu+\beta)}\tau^{\nu+\beta-1} - \lambda_2 \frac{\Gamma_h(\beta)}{\Gamma_h(\mu+\beta)}\theta^{\mu+\beta-1} \neq 0.$$

In the sequel, we set constants

$$\begin{aligned}
A_{13} &= \frac{\Gamma_n(\alpha+1)\xi^{\kappa+\alpha}}{\Gamma_n(\kappa+\alpha+1)\Gamma_q(\alpha+1)},\\
A_{14} &= \frac{\Gamma_n(\alpha+\delta+1)\Gamma_q(\delta+1)\xi^{\kappa+\alpha+\delta}}{\Gamma_n(\kappa+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)},\\
A_{15} &= \frac{\Gamma_k(\beta+1)\tau^{\nu+\beta}}{\Gamma_k(\nu+\beta+1)\Gamma_p(\beta+1)},\\
A_{16} &= \frac{\Gamma_k(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\tau^{\nu+\beta+\varepsilon}}{\Gamma_k(\nu+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)},\\
H_1 &= \overline{M}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|}\overline{M}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|}\overline{M}_1 A_{13},\\
H_2 &= \overline{M}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|}\overline{M}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|}\overline{M}_2 A_{14},\\
L_1 &= \overline{N}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|}\overline{N}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|}\overline{N}_2 A_{16},\\
L_2 &= \overline{N}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|}\overline{N}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|}\overline{N}_1 A_{15}.
\end{aligned}$$

Now we present the existence and the uniqueness result for the problem (5.1). We do not provide the proof of this result as it is similar to the one for Theorem 4.1.

Theorem 5.2 *Assume that $f,g:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions and there exist constants $\overline{K}_i$, $\overline{L}_i$, $i=1,2$ such that for all $t\in[0,T]$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$,*

$$\big|f(t,u_1,u_2)-f(t,v_1,v_2)\big| \le \overline{M}_1|u_1-v_1| + \overline{M}_2|u_2-v_2|$$

and

$$\big|g(t,u_1,u_2)-g(t,v_1,v_2)\big| \le \overline{N}_1|u_1-v_1| + \overline{N}_2|u_2-v_2|.$$

In addition, assume that

$$H_1+H_2+L_1+L_2<1.$$

Then the boundary value problem (5.1) *has a unique solution on* $[0,T]$.

The second result deals with the existence of solutions for the problem (5.1), is analogous to Theorem 4.3 and is given below.

Theorem 5.3 *Assume that there exist real constants* $\bar{m}_i, \bar{n}_i \geq 0$ $(i = 1, 2)$, *and* $\bar{m}_0 > 0$, $\bar{n}_0 > 0$ *such that* $\forall x_i \in \mathbb{R}$ $(i = 1, 2)$ *we have*

$$\begin{aligned} \left|f(t, x_1, x_2)\right| &\leq \bar{m}_0 + \bar{m}_1|x_1| + \bar{m}_2|x_2|, \\ \left|g(t, x_1, x_2)\right| &\leq \bar{n}_0 + \bar{n}_1|x_1| + \bar{n}_2|x_2|. \end{aligned}$$

In addition it is assumed that

$$U_1 + V_1 < 1 \quad \textit{and} \quad U_2 + V_2 < 1,$$

where U_i, V_i, $i = 1, 2$, *are given by*

$$\begin{aligned} U_1 &= \bar{m}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_{13}, \\ U_2 &= \bar{m}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_{14}, \\ V_1 &= \bar{n}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{16}, \\ V_2 &= \bar{n}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_1 A_{15}. \end{aligned}$$

Then the boundary value problem (5.1) *has at least one solution on* $[0, T]$.

Proof By setting

$$G^* = \min\left\{1 - (U_1 + V_1), 1 - (U_2 + V_2)\right\},$$

the proof is similar to that of Theorem 4.3. So we omit it. □

5.2 Examples

In this subsection, we present two examples of uncoupled case of nonlocal conditions.

Example 5.4 Consider the following system of fractional q-integro-difference equations with q-integral conditions:

$$\begin{cases} D_{1/9}^{3/2} x(t) = \frac{e^{-t\sin\pi t}}{(t+3)^2} \cdot \frac{|x(t)|}{9+|x(t)|} + \frac{\cos^2 \pi t}{\pi(t+7)^2} \cdot I_{1/8}^{\sqrt{2}} y(t) - \frac{1}{3}, \quad 0 < t < 3, \\ D_{1/7}^{5/4} y(t) = \frac{2\pi e^{-2t}}{(7\pi+t)^2} \cdot \frac{|y(t)|}{2+|y(t)|} + \frac{\sin 2\pi t}{(3e^t+5)^2} \cdot I_{1/6}^{\sqrt{3}} x(t) + \frac{1}{3}, \\ x(0) = 0, \qquad \frac{1}{2} I_{1/3}^{\frac{\sqrt{2}}{2}} x(\frac{3}{4}) = I_{1/4}^{\sqrt{3}} x(\frac{9}{4}), \\ y(0) = 0, \qquad I_{1/6}^{\sqrt{\pi}} y(3) + \frac{1}{3} I_{1/5}^{\pi} y(\frac{3}{2}) = 0. \end{cases} \tag{5.4}$$

Here $\alpha = 3/2$, $\delta = \sqrt{2}$, $\beta = 5/4$, $\varepsilon = \sqrt{3}$, $\gamma = \sqrt{2}/2$, $\kappa = \sqrt{3}/2$, $\mu = \pi$, $\nu = \sqrt{\pi}$, $q = 1/9$, $r = 1/8$, $p = 1/7$, $z = 1/6$, $m = 1/3$, $n = 1/4$, $h = 1/5$, $k = 1/6$, $\eta = 3/4$, $\xi = 9/4$, $\theta = 3/2$, $\tau = 3$, $\lambda_1 =$

$1/2$, $\lambda_2 = -1/3$, $T = 3$, $f(t,x,I_r^\delta y) = (e^{-t\sin\pi t}/(t+3)^2)(|x|/(9+|x|)) + (\cos^2\pi t/\pi(t+7)^2)I_{1/8}^{\sqrt{2}}y - (1/3)$, and $g(t,y,I_r^\delta x) = (2\pi e^{-2t}/(7\pi+t)^2)(|y|/(2+|y|)) + (\sin 2\pi t/(3e^t+5)^2)I_{1/6}^{\sqrt{3}}x + (1/3)$. Since

$$\bigl|f(t,u_1,u_2) - f(t,v_1,v_2)\bigr| \le \frac{1}{81}|u_1 - v_1| + \frac{1}{49\pi}|u_2 - v_2|$$

and

$$\bigl|g(t,u_1,u_2) - g(t,v_1,v_2)\bigr| \le \frac{1}{36\pi}|u_1 - v_1| + \frac{1}{64}|u_2 - v_2|,$$

then the assumptions of Theorem 5.2 are satisfied with $\overline{M}_1 = 1/81$, $\overline{M}_2 = 1/49\pi$, $\overline{N}_1 = 1/36\pi$, and $\overline{N}_2 = 1/64$. By using the Maple program, we can find that

$$\Lambda = \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa+\alpha)}\xi^{\kappa+\alpha-1} - \lambda_1\frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma+\alpha)}\eta^{\gamma+\alpha-1} \approx 1.9245172 \neq 0,$$
$$\Psi = \frac{\Gamma_k(\beta)}{\Gamma_k(\nu+\beta)}\tau^{\nu+\beta-1} - \lambda_2\frac{\Gamma_h(\beta)}{\Gamma_h(\mu+\beta)}\theta^{\mu+\beta-1} \approx 8.37494759 \neq 0$$

and

$$H_1 = \overline{M}_1 A_1 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\overline{M}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|}\overline{M}_1 A_{13} \approx 0.11268247,$$
$$H_2 = \overline{M}_2 A_3 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\overline{M}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|}\overline{M}_2 A_{14} \approx 0.19713212,$$
$$L_1 = \overline{N}_2 A_4 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\overline{N}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|}\overline{N}_2 A_{16} \approx 0.58490031,$$
$$L_2 = \overline{N}_1 A_2 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\overline{N}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|}\overline{N}_1 A_{15} \approx 0.06286768.$$

Therefore, we get

$$H_1 + H_2 + L_1 + L_2 \approx 0.95758259 < 1.$$

Hence, by Theorem 5.2, the problem (5.4) has a unique solution on $[0,3]$.

Example 5.5 Consider the following system of fractional q-integro-difference equations:

$$\begin{cases} D_{\sqrt{2}/2}^{\sqrt{\pi}}x(t) = \frac{25e^{-t}}{(e^{-t}+4)^2}\cdot\frac{|x(t)|}{5+|x(t)|} + \frac{3\pi^2}{(t+3\pi)^2}\cdot I_{\sqrt{3}/2}^{1/2}y(t) + \frac{1}{\sqrt{5}}, \quad 0 < t < 1, \\ D_{\pi/4}^{\pi/2}y(t) = \frac{9e^{-t\cos^2\pi t}}{(t+6)^2}\cdot\frac{|y(t)|}{1+|y(t)|} + \frac{6}{(t+6)^2}\cdot I_{\pi/5}^{3/2}x(t) + \frac{\sqrt{2}}{3}, \\ x(0) = 0, \qquad \frac{\sqrt{3}}{2}I_{\pi/6}^{4/5}x(1) + I_{\pi/7}^{2/3}x(\frac{3}{4}) = 0, \\ y(0) = 0, \qquad 5I_{\pi/8}^{\sqrt{3}}y(\frac{1}{2}) = I_{\pi/9}^{1/3}y(\frac{1}{4}). \end{cases} \tag{5.5}$$

Here $\alpha = \sqrt{\pi}$, $\delta = 1/2$, $\beta = \pi/2$, $\varepsilon = 3/2$, $\gamma = 4/5$, $\kappa = 2/3$, $\mu = \sqrt{3}$, $\nu = 1/3$, $q = \sqrt{2}/2$, $r = \sqrt{3}/2$, $p = \pi/4$, $z = \pi/5$, $m = \pi/6$, $n = \pi/7$, $h = \pi/8$, $k = \pi/9$, $\eta = 1$, $\xi = 3/4$, $\theta = 1/2$, $\tau = 1/4$, $\lambda_1 = -\sqrt{3}/2$, $\lambda_2 = 5$, $T = 1$, $f(t,x,I_r^\delta y) = (25e^{-t}/(e^{-t}+4)^2)(|x|/(5+|x|)) + (3\pi^2/(t+3\pi)^2)I_{\sqrt{3}/2}^{1/2}y + (1/\sqrt{5})$, and $g(t,y,I_r^\delta x) = (9e^{-t\cos^2\pi t}/(t+6)^2)(|y|/(1+|y|)) + (6/(t+6)^2)I_{\pi/5}^{3/2}x +$

$(\sqrt{2}/3)$. Since

$$\left|f(t,x_1,x_2)\right| \leq \frac{1}{\sqrt{5}} + \frac{1}{5}|x_1| + \frac{1}{3}|x_2|$$

and

$$\left|g(t,x_1,x_2)\right| \leq \frac{\sqrt{2}}{3} + \frac{1}{4}|x_1| + \frac{1}{6}|x_2|,$$

then the assumptions of Theorem 5.3 are satisfied with $\bar{m}_0 = 1/\sqrt{5}$, $\bar{m}_1 = 1/5$, $\bar{m}_2 = 1/3$, $\bar{n}_0 = \sqrt{2}/3$, $\bar{n}_1 = 1/4$, and $\bar{n}_2 = 1/6$. By using the Maple program, we can find that

$$\Lambda = \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa+\alpha)}\xi^{\kappa+\alpha-1} - \lambda_1\frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma+\alpha)}\eta^{\gamma+\alpha-1} \approx 1.21235918 \neq 0,$$

$$\Psi = \frac{\Gamma_k(\beta)}{\Gamma_k(\nu+\beta)}\tau^{\nu+\beta-1} - \lambda_2\frac{\Gamma_h(\beta)}{\Gamma_h(\mu+\beta)}\theta^{\mu+\beta-1} \approx -0.32647283 \neq 0$$

and

$$U_1 = \bar{m}_1A_1 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\bar{m}_1A_5 + \frac{T^{\alpha-1}}{|\Lambda|}\bar{m}_1A_{13} \approx 0.23965603,$$

$$U_2 = \bar{m}_2A_3 + \frac{|\lambda_1|T^{\alpha-1}}{|\Lambda|}\bar{m}_2A_9 + \frac{T^{\alpha-1}}{|\Lambda|}\bar{m}_2A_{14} \approx 0.27471434,$$

$$V_1 = \bar{n}_2A_4 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\bar{n}_2A_{11} + \frac{T^{\beta-1}}{|\Psi|}\bar{n}_2A_{16} \approx 0.04758258,$$

$$V_2 = \bar{n}_1A_2 + \frac{|\lambda_2|T^{\beta-1}}{|\Psi|}\bar{n}_1A_7 + \frac{T^{\beta-1}}{|\Psi|}\bar{n}_1A_{15} \approx 0.36424461.$$

Therefore, we get

$$U_1 + V_1 \approx 0.28723861 < 1 \quad \text{and} \quad U_2 + V_2 \approx 0.63895895 < 1.$$

Hence, by Theorem 5.3, the problem (5.5) has at least one solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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