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New hybrid shrinking projection algorithm for common fixed points of a family of countable quasi-Bregman strictly pseudocontractive mappings with equilibrium and variational inequality and optimization problems

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Abstract

The purpose of this paper is to introduce and consider a new hybrid shrinking projection algorithm for finding a common element of the set of solutions of a system of equilibrium problems, the set of solutions of a system of variational inequality problems, the set of solutions of a system of optimization problems, the common fixed point set of a uniformly closed family of countable quasi-Bregman strictly pseudocontractive mappings in reflexive Banach spaces. Strong convergence theorems have been proved under the appropriate conditions. The main innovative points in this paper are as follows: (1) the notion of the uniformly closed family of countable quasi-Bregman strictly pseudocontractive mappings is presented and the useful conclusions are given; (2) the relative examples of the uniformly closed family of countable quasi-Bregman strictly pseudocontractive mappings are given in classical Banach spaces l^2 and L^2; (3) the hybrid shrinking projection method presented in this paper modified some mistakes in the recent result of Ugwunnadi *et al.* (Fixed Point Theory Appl. 2014:231, 2014). These new results improve and extend the previously known ones in the literature.

MSC: 47H05; 47H09; 47H10

Keywords: Bregman distance; quasi-Bregman strictly pseudocontractive mapping; generalized projection; hybrid algorithm; equilibrium problem; variational inequality problem; optimization problem; fixed point

1 Introduction

Let C be a nonempty subset of a real Banach space and T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Recall that T is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, n \geq 1.$$

It is well known that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In the framework of Hilbert spaces, Takahashi *et al.* [2] have introduced a new hybrid iterative scheme called a shrinking projection method for nonexpansive mappings. It is an advantage of projection methods that the strong convergence of iterative sequences is guaranteed without any compact assumption. Moreover, Schu [3] has introduced a modified Mann iteration to approximate fixed points of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Motivated by [2, 3], Inchan [4] has introduced a new hybrid iterative scheme by using the shrinking projection method with the modified Mann iteration for asymptotically nonexpansive mappings. The mapping T is said to be asymptotically nonexpansive in the intermediate sense (*cf.* [5]) if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| T^n x - T^n y \right\| - \|x - y\| \right) \leq 0. \tag{1.1}$$

If $F(T)$ is nonempty and (1.1) holds for all $x \in C$ and $y \in F(T)$, then T is said to be asymptotically quasi-nonexpansive in the intermediate sense. It is worth mentioning that the class of asymptotically nonexpansive mappings in the intermediate sense contains properly the class of asymptotically nonexpansive mappings since the mappings in the intermediate sense are not Lipschitz continuous in general.

Recently, many authors have studied further new hybrid iterative schemes in the framework of real Banach spaces; for instance, see [6–8]. Qin and Wang [9] have introduced a new class of mappings which are asymptotically quasi-nonexpansive with respect to the Lyapunov functional (*cf.* [10]) in the intermediate sense. By using the shrinking projection method, Hao [11] has proved a strong convergence theorem for an asymptotically quasi-nonexpansive mapping with respect to the Lyapunov functional in the intermediate sense.

In 1967, Bregman [12] discovered an elegant and effective technique for using of the so-called Bregman distance function (see Section 2) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, and for computing fixed points of nonlinear mappings.

Many authors have studied iterative methods for approximating fixed points of mappings of nonexpansive type with respect to the Bregman distance; see [13–17]. In [18], the author introduced a new class of nonlinear mappings which is an extension of asymptotically quasi-nonexpansive mappings with respect to the Bregman distance in the intermediate sense and proved the strong convergence theorems for asymptotically quasi-nonexpansive mappings with respect to Bregman distances in the intermediate sense by using the shrinking projection method.

Recently, Zegeye and Shahzad [19] have proved a strong convergence theorem for the common fixed point of a finite family of right Bregman strongly nonexpansive mappings in a reflexive Banach space. Alghamdi *et al.* [20] proved a strong convergence theorem for the common fixed point of a finite family of quasi-Bregman nonexpansive mappings. Pang *et al.* [21] proved weak convergence theorems for Bregman relatively nonexpansive mappings. Shahzad and Zegeye [22] proved a strong convergence theorem for multivalued Bregman relatively nonexpansive mappings, while Zegeye and Shahzad [23] proved a strong convergence theorem for a finite family of Bregman weak relatively nonexpansive mappings.

Motivated and inspired by the above works, in 2015 Ugwunnadi *et al.* [24] proved a new strong convergence theorem for a finite family of closed quasi-Bregman strictly pseudocontractive mappings and a system of equilibrium problems in a real reflexive Banach space.

The purpose of this paper is to introduce and consider a new hybrid shrinking projection algorithm for finding a common element of the set of solutions of a system of equilibrium problems, the set of solutions of a system of variational inequality problems, the set of solutions of a system of optimization problems, the common fixed point set of a uniformly closed family of countable quasi-Bregman strictly pseudocontractive mappings in reflexive Banach spaces. Strong convergence theorems have been proved under the appropriate conditions. The main innovative points in this paper are as follows: (1) the notion of uniformly closed family of countable quasi-Bregman strictly pseudocontractive mappings is presented and the useful conclusions are given; (2) the relative examples of the uniformly closed family of countable quasi-Bregman strictly pseudocontractive mappings are given in classical Banach spaces l^2 and L^2; (3) the hybrid shrinking projection method presented in this paper modified some mistakes in the recent result of Ugwunnadi *et al.* [24]. These new results improve and extend the previously known ones in the literature.

2 Preliminaries

Throughout this paper, we assume that E is a real reflexive Banach space with the dual space of E^* and $\langle\cdot,\cdot\rangle$ is the pairing between E and E^*.

Let $f: E \to (-\infty, +\infty]$ be a function. The effective domain of f is defined by

$$\operatorname{dom} f := \{x \in E : f(x) < +\infty\}.$$

When $\operatorname{dom} f \neq \emptyset$, we say that f is proper. We denote by $\operatorname{int}\operatorname{dom} f$ the interior of the effective domain of f. We denote by $\operatorname{ran} f$ the range of f.

The function f is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Given a proper and convex function $f: E \to (-\infty, +\infty]$, the subdifferential of f is a mapping $\partial f: E \to E^*$ defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle x^*, y - x\rangle, \forall y \in E\}$$

for all $x \in E$.

The Fenchel conjugate function of f is the convex function $f^*: E \to (-\infty, +\infty)$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

We know that $x^* \in \partial f(x)$ if and only if

$$f(x) + f^*(x^*) = \langle x^*, x\rangle$$

for all $x \in E$ (see [18]).

Proposition 2.1 ([25]) *Let $f : E \to (-\infty, +\infty]$ be a proper, convex, and lower semi-continuous function. Then the following conditions are equivalent*:

(i) $\operatorname{ran} \partial f = E^*$ *and* $\partial f^* = (\partial f)^{-1}$ *is bounded on bounded subsets of* E^*;

(ii) f *is strongly coercive*.

Let $f : E \to (-\infty, +\infty]$ be a convex function and $x \in \operatorname{int}\operatorname{dom} f$. For any $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^\circ(x, y) = \lim_{t\downarrow 0} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be Gâteaux differentiable at x if the limit (2.1) exists for any y. In this case, the gradient of f at x is the function $\nabla f(x) : E \to E^*$ defined by $\langle \nabla f(x), y\rangle = f^\circ(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \operatorname{int}\operatorname{dom} f$. If the limit (2.1) is attained uniformly in $\|y\| = 1$, then the function f is said to be Fréchet differentiable at x. The function f is said to be uniformly Fréchet differentiable on a subset C of E if the limit (2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. We know that if f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (*cf.* [25, 26]). We will need the following results.

Proposition 2.2 ([27]) *If a function $f : E \to R$ is convex, uniformly Fréchet differentiable, and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Proposition 2.3 ([27]) *Let $f : E \to R$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent*:

(i) f *is strongly coercive and uniformly convex on bounded subsets of* E;

(ii) f^* *is Fréchet differentiable and* ∇f^* *is uniformly norm-to-norm continuous on bounded subsets of* $\operatorname{dom} f^* = E^*$.

A function $f : E \to (-\infty, +\infty]$ is said to be admissible if it is proper, convex, and lower semi-continuous on E and Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. Under these conditions we know that f is continuous in $\operatorname{int}\operatorname{dom} f$, ∂f is single-valued and $\partial f = \nabla f$; see [17, 22]. An admissible function $f : E \to (-\infty, +\infty]$ is called Legendre (*cf.* [17]) if it satisfies the following two conditions:

(L1) the interior of the domain of f, $\operatorname{int}\operatorname{dom} f$, is nonempty, f is Gâteaux differentiable, and $\operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f$;

(L2) the interior of the domain of f^*, $\operatorname{int}\operatorname{dom} f^*$, is nonempty, f^* is Gâteaux differentiable, and $\operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*$.

Let f be a Legendre function on E. Since E is reflexive, we always have $\nabla f = (\nabla f^*)^{-1}$. This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\operatorname{ran}\nabla f = \operatorname{dom} f^* = \operatorname{int}\operatorname{dom} f^* \quad \text{and} \quad \operatorname{ran}\nabla f^* = \operatorname{dom} f = \operatorname{int}\operatorname{dom} f.$$

Conditions (L1) and (L2) imply that the functions f and f^* are strictly convex on the interior of their respective domains. In [23], authors gave an example of the Legendre function.

Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The bifunction $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ given by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$$

is called the Bregman distance with respect to f (*cf.* [28]). In general, the Bregman distance is not a metric since it is not symmetric and does not satisfy the triangle inequality. However, it has the following important property, which is called the three point identity (*cf.* [29]): for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle x - y, \nabla f(z) - \nabla f(y)\rangle. \tag{2.2}$$

With a Legendre function $f : E \to (-\infty, +\infty]$, we associate the bifunction $W_f : \operatorname{dom} f^* \times \operatorname{dom} f \to [0, +\infty)$ defined by

$$W^f(w, x) = f(x) - \langle w, x\rangle + f^*(w).$$

Proposition 2.4 ([14]) *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of* $\operatorname{int}\operatorname{dom} f^*$. *Let $x \in \operatorname{int}\operatorname{dom} f$. If the sequence $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Proposition 2.5 ([14]) *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Then the following statements hold*:
- (i) *the function $W^f(\cdot, x)$ is convex for all $x \in \operatorname{dom} f$*;
- (ii) *$W^f(\nabla f(x), y) = D_f(y, x)$ for all $x \in \operatorname{int}\operatorname{dom} f$ and $y \in \operatorname{dom} f$.*

Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The function f is said to be totally convex at a point $x \in \operatorname{int}\operatorname{dom} f$ if its modulus of total convexity at x, $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$, defined by

$$\nu_f(x, t) = \inf\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\},$$

is positive whenever $t > 0$. The function f is said to be totally convex when it is totally convex at every point of $\operatorname{int}\operatorname{dom} f$. The function f is said to be totally convex on bounded sets if, for any nonempty bounded set $B \subset E$, the modulus of total convexity of f on B, $\nu_f(B, t)$ is positive for any $t > 0$, where $\nu_f(B, \cdot) : [0, +\infty) \to [0, +\infty]$ is defined by

$$\nu_f(B, t) = \inf\{\nu_f(x, t) : x \in B \cap \operatorname{int}\operatorname{dom} f\}.$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets; see [26, 27].

Proposition 2.6 ([30]) *Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points. If f is lower semi-continuous, then f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets.*

Proposition 2.7 ([32]) *Let $f : E \to R$ be a totally convex function. If $x \in E$ and the sequence $\{D_f(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Let $f : E \to [0, +\infty)$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The function f is said to be sequentially consistent (*cf.* [31]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Proposition 2.8 ([24]) *A function $f : E \to [0, +\infty)$ is totally convex on bounded subsets of E if and only if it is sequentially consistent.*

Let C be a nonempty, closed, and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a convex function on E which is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$. The Bregman projection $\operatorname{proj}_C^f(x)$ with respect to f (*cf.* [23]) of $x \in \operatorname{int}\operatorname{dom} f$ onto C is the minimizer over C of the functional $D_f(\cdot, x) :\to [0, +\infty]$, that is,

$$\operatorname{proj}_C^f(x) = \operatorname{argmin}\{D_f(y, x) : y \in C\}.$$

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E is smooth, then J is single-valued.

Proposition 2.9 ([33]) *Let $f : E \to R$ be an admissible, strongly coercive, and strictly convex function. Let C be a nonempty, closed, and convex subset of* $\operatorname{dom} f$. *Then* $\operatorname{proj}_C^f(x)$ *exists uniquely for all $x \in \operatorname{int}\operatorname{dom} f$.*

Let $f(x) = \frac{1}{2}\|x\|^2$.

(i) If E is a Hilbert space, then the Bregman projection is reduced to the metric projection onto C.

(ii) If E is a smooth Banach space, then the Bregman projection is reduced to the generalized projection $\Pi_C(x)$ which is defined by

$$\Pi_C(x) = \operatorname{argmin}\{\phi(y, x) : y \in C\},$$

where ϕ is the Lyapunov functional (*cf.* [10]) defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $y, x \in E$.

Proposition 2.10 ([31]) *Let $f : E \to (-\infty, +\infty]$ be a totally convex function. Let C be a nonempty, closed, and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and $x \in \operatorname{int}\operatorname{dom} f$. If $x^* \in C$, then the following statements are equivalent*:

(i) *The vector x^* is the Bregman projection of x onto C.*

(ii) *The vector x^* is the unique solution z of the variational inequality*

$$\langle z - y, \nabla f(x) - \nabla f(z) \rangle \geq 0, \quad \forall y \in C.$$

(iii) *The vector x^* is the unique solution z of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \quad \forall y \in C.$$

In recent years, the following notions have been presented by some authors.

A point $p \in C$ is said to be asymptotic fixed point of a map T if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T. A point $p \in C$ is said to be strong asymptotic fixed point [34] of a mapping T if there exists a sequence $\{x_n\}$ in C which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widetilde{F}(T)$ the set of strong asymptotic fixed points of T. Let $f : E \to R$, a mapping $T : C \to C$ is said to be Bregman relatively nonexpansive [17] if $F(T) = \widehat{F}(T)$ and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$. The mapping $T : C \to C$ is said to be Bregman weak relatively nonexpansive if $F(T) = \widetilde{F}(T)$ and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$. The mapping $T : C \to C$ is said to be quasi-Bregman relatively nonexpansive [24] if $F(T) \neq \emptyset$ and $D_f(p, T(x)) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$. In [24] quasi-Bregman relatively nonexpansive is called left quasi-Bregman relatively nonexpansive. A mapping $T : C \to C$ is said to be right quasi-Bregman relatively nonexpansive [24] if $F(T) \neq \emptyset$ and $D_f(T(x), p) \leq D_f(x, p)$ for all $x \in C$ and $p \in F(T)$.

In [24], authors presented the definition of quasi-Bregman strictly pseudocontractive mapping. In this paper, we extend this definition to the quasi-Bregman pseudocontractive mapping as follows.

Definition 2.11 Let C be a nonempty, closed, and convex subset of E and $f : E \to (-\infty, +\infty]$ be an admissible function. Let T be a mapping from C into itself with a nonempty fixed point set $F(T)$. The mapping T is said to be quasi-Bregman k-pseudocontractive if there exists a constant $k \in [0, +\infty)$ such that

$$D_f(p, Tx) \leq D_f(p, x) + kD_f(x, Tx), \quad \forall p \in F(T), \forall x \in C.$$

If $k \in [0, 1)$, the mapping T is said to be quasi-Bregman strictly pseudocontractive. If $k = 1$, the mapping T is said to be quasi-Bregman pseudocontractive. The mapping T is said to be Bregman quasi-nonexpansive if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall p \in F(T), \forall x \in C.$$

In this paper, we will use the following definition.

Definition 2.12 ([34]) Let C be a nonempty, closed, and convex subset of E. Let $\{T_n\}$ be a sequence of mappings from C into itself with a nonempty common fixed point set $F = \bigcap_{n=1}^{\infty} F(T_n)$. $\{T_n\}$ is said to be uniformly closed if for any convergent sequence $\{z_n\} \subset C$ such that $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

The next lemmas have been proved in [24], which is useful for the results of [24], but in this paper we do not use Lemma 2.13 and Lemma 2.15.

Lemma 2.13 ([24]) *Let $f : E \to R$ be a Legendre function which is uniformly Fréchet differentiable and bounded on subsets of E, let C be a nonempty, closed, and convex subset of E, and let $T : C \to C$ be a quasi-Bregman strictly pseudocontractive mapping with respect to f. Then, for any $x \in C$, $p \in F(T)$ and $k \in [0,1)$, the following holds*:

$$D_f(x, Tx) \leq \frac{1}{1-k}\langle \nabla f(x) - \nabla f(Tx), x - p\rangle.$$

Lemma 2.14 ([24]) *Let $f : E \to R$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E, let C be a nonempty, closed, and convex subset of E, and let $T : C \to C$ be a quasi-Bregman strictly pseudocontractive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 2.15 ([24]) *Let E be a real reflexive Banach space, $f : E \to (-\infty, +\infty]$ be a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla^* f\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i).$$

Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. Then consider the following generalized equilibrium problem of finding $u \in C$ such that

$$\varphi(y) - \varphi(u) + F(u,y) + \langle Au, y - u\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

The set of solutions of (2.3) is denoted by EP, *i.e.*,

$$EP = \left\{u \in C : \varphi(y) - \varphi(u) + F(u,y) + \langle Au, y - u\rangle \geq 0, \forall y \in C\right\}.$$

Whenever $A \equiv 0$, $\varphi(x) \equiv 0$, problem (2.3) is equivalent to finding $u \in C$ such that

$$F(u,y) \geq 0, \quad \forall y \in C, \tag{2.4}$$

which is called the equilibrium problem. The set of its solutions is denoted by $EP(F)$.

Whenever $F \equiv 0$, $\varphi(x) \equiv 0$, problem (2.3) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u\rangle \geq 0, \quad \forall y \in C,$$

which is called the variational inequality of Browder type. The set of its solutions is denoted by $\mathrm{VI}(C, A)$.

Whenever $F \equiv 0$, $A \equiv 0$, problem (2.3) is equivalent to finding $u \in C$ such that

$$\varphi(y) \geq \varphi(u), \quad \forall y \in C,$$

which is called the convex optimization problem. The set of its solutions is denoted by $\mathrm{MIN}(\varphi)$.

Problem (2.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, *e.g.*, [31, 32].

In order to solve the equilibrium problem for finding an element $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C,$$

let us assume that $F : C \times C \to (-\infty, +\infty)$ satisfies the following conditions [33]:

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$,

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$,

(A4) for all $x \in C$, $F(x, \cdot)$ is convex and lower semi-continuous.

For $r > 0$, we define a mapping $K_r : E \to C$ as follows:

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C \right\} \tag{2.5}$$

for all $x \in E$. The following two lemmas were proved in [14].

Lemma 2.16 *Let E be a reflexive Banach space and let $f : E \to R$ be a Legendre function. Let C be a nonempty, closed, and convex subset of E and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$, let $T_r : E \to C$ be the mapping defined by* (2.5). *Then* $\operatorname{dom} T_r = E$.

Lemma 2.17 *Let E be a reflexive Banach space and let $f : E \to R$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$, let $T_r : E \to C$ be the mapping defined by* (2.5). *Then the following statements hold*:

(i) *T_r is single-valued.*

(ii) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, \nabla f(T_r x) - \nabla f(T_r y)\rangle \leq \langle T_r x - T_r y, \nabla f(x) - \nabla f(y)\rangle.$$

(iii) $F(T_r) = \widehat{F}(T_r) = EP(F)$.

(iv) *$EP(F)$ is closed and convex.*

(v) $D_f(p, T_r x) + D_f(T_r x, x) \leq D_f(p, x)$, $\forall p \in EP(F)$, $\forall x \in E$.

Lemma 2.18 *Let E be a reflexive Banach space and let $f : E \to R$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *Let $A : C \to E^*$ be a monotone mapping, i.e.,*

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

Let $\varphi(x) : C \to R$ be a convex lower semi-continuous functional. For $r > 0$, let $K_r : E \to C$ be the mapping defined by

$$K_r(x) = \left\{z \in C : G(z, y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C\right\},$$

where

$$G(x, y) = \varphi(y) - \varphi(x) + F(x, y) + \langle Ax, y - x\rangle.$$

Then the following statements hold:

(i) *K_r is single-valued.*
(ii) *K_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle K_r x - K_r y, \nabla f(K_r x) - \nabla f(K_r y)\rangle \leq \langle K_r x - K_r y, \nabla f(x) - \nabla f(y)\rangle.$$

(iii) $F(K_r) = \widehat{F}(K_r) = EP$.
(iv) *EP is closed and convex.*
(v) $D_f(p, K_r x) + D_f(K_r x, x) \leq D_f(p, x)$, $\forall p \in EP(F)$, $\forall x \in E$.

Proof Let

$$G(x, y) = \varphi(y) - \varphi(x) + F(x, y) + \langle Ax, y - x\rangle, \quad \forall x, y \in C.$$

It is easy to show that $G(x, y)$ satisfies conditions (A1)-(A4). Replacing $F(x, y)$ by $G(x, y)$ in Lemma 2.17, we can get the conclusions. □

From [36] we have the following conclusion.

Theorem 2.19 *Let E be a p-uniformly convex Banach space with $p \geq 2$. Then for all $x, y \in E$, $j(x) \in J_p(x)$, $j(y) \in J_p(y)$,*

$$\langle j(x) - j(y), x - y\rangle \geq \frac{c^p}{c^{p-2}p}\|x - y\|^p,$$

where J_p is the generalized duality mapping from E into E^* and $1/c$ is the p-uniformly convexity constant of E.

From Theorem 2.19, we know that the generalized duality mapping $J_p : E \to E^*$ is a monotone operator. It is well known that if E is also smooth and 2-uniformly convex, the normalized duality mapping $J = J_2 : E \to E^*$ is a single-valued monotone operator.

3 Main results

We now prove the following theorem.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to R$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on a bounded subset of E. Let $\{F_j\}_{j=1}^m$ be finite bifunctions from $C \times C$ to R satisfying* (A1)-(A4) *and let $\{A_j\}_{j=1}^m : C \to E^*$ be finite monotone mappings, i.e.,*

$$\langle A_j x - A_j y, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

Let $\{\varphi_j(x)\}_{j=1}^m : C \to R$ be finite convex lower semi-continuous functionals. Let $\{T_n\}_{n=1}^\infty$ be a uniformly closed family of countable quasi-Bregman strictly pseudocontractive mappings from C into itself with uniformly $k \in [0,1)$ such that $F = \bigcap_{j=1}^m EP_j \cap (\bigcap_{n=1}^\infty F(T_n))$ is nonempty. For given $x_0 \in C$, let $\{T_n\}_{n=1}^\infty$ be a sequence generated by

$$\begin{cases} x_1 = x_0 \in C_1 = C, \\ y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_n x_n)), \\ G_j(u_{j,n}, y) + \frac{1}{r_n}\langle \nabla f(u_{j,n}) - \nabla f(y_n), y - u_{n,j},\rangle \geq 0, \quad \forall y \in C, j = 1,2,3,\dots,m, \\ C_{n+1} = \{z \in C_n : D_f(z, u_{j,n}) \leq D_f(z, y_n) \leq D_f(z, x_n) \\ \qquad + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle, j = 1,2,3,\dots,m\}, \\ x_{n+1} = P^f_{C_{n+1}} x_0, \end{cases}$$

where

$$G_j(x,y) = \varphi_j(y) - \varphi_j(x) + F_j(x,y) + \langle A_j x, y - x\rangle,$$
$$EP_j = \{u \in C : G_j(x,y) \geq 0, \forall y \in C\}$$

for $j = 1,2,3,\dots,m$, and $\{\alpha_n\}$, $\{\beta_{j,n}\}$ are sequences satisfying $\limsup_{n\to\infty} \alpha_n < 1$, $\{r_n\}$ is a sequence satisfying $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges to $q = P^f_F x_0$.

Proof We divide the proof into six steps.

Step 1. We show that C_n is closed and convex for all $n \geq 1$. Let

$$D_n = \left\{z \in E : D_f(z, y_n) \leq D_f(z, x_n) + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle\right\},$$
$$E_{j,n} = \{z \in E : D_f(z, u_{j,n}) \leq D_f(z, y_n)\}, \quad j = 1,2,3,\dots,m,$$

then

$$C_{n+1} = C \cap C_n \cap D_n \cap \left(\bigcap_{j=1}^m E_{j,n}\right).$$

Since $C_1 = C$ is closed and convex, it is sufficient to prove that the sets D_n, $E_{j,n}$ are closed and convex for all $n \geq 1$. We show that D_n is closed and convex for all $n \geq 1$. We rewrite

D_n as follows:

$$\begin{aligned} D_n &= \left\{z \in E : D_f(z, y_n) \le D_f(z, x_n) + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle\right\} \\ &= \left\{z \in E : D_f(z, y_n) - D_f(z, x_n) \le \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle\right\} \\ &= \left\{z \in E : f(x_n) - f(y_n) + \langle z - x_n, \nabla f(x_n)\rangle - \langle z - y_n, \nabla f(y_n)\rangle \right. \\ &\quad \left. \le \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle\right\} \\ &= \left\{z \in E : \langle z - x_n, \nabla f(x_n)\rangle - \langle z - y_n, \nabla f(y_n) \le f(y_n) - f(x_n)\rangle \right. \\ &\quad \left. + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle\right\} \\ &= \left\{z \in E : \left\langle z, \frac{1}{1-k}\nabla f(x_n) - \nabla f(y_n) - \frac{k}{1-k}\nabla f(T_n x_n)\right\rangle \le f(y_n) - f(x_n) \right. \\ &\quad \left. + \left\langle x_n, \frac{1}{1-k}\nabla f(x_n)\right\rangle - \langle x_n, \nabla f(y_n)\rangle - \left\langle x_n, \frac{k}{1-k}\nabla f(T_n x_n)\right\rangle\right\}. \end{aligned}$$

From the above expression, we know that D_n is closed and convex for all $n \ge 1$.

Next we show that $E_{j,n}$ is closed and convex for all $n \ge 1$, $j = 1, 2, 3, \dots, m$. We rewrite $E_{j,n}$ as follows:

$$\begin{aligned} E_{j,n} &= \{z \in E : D_f(z, u_{j,n}) \le D_f(z, y_n)\} \\ &= \{z \in E : f(y_n) - f(u_{j,n}) \le \langle \nabla f(u_{j,n}, z - u_{j,n})\rangle - \langle \nabla f(y_n), z - y_n\rangle\} \\ &= \{z \in E : f(y_n) - f(u_{j,n}) + \langle \nabla f(u_{j,n}), u_{j,n}\rangle - \langle \nabla f(y_n), y_n\rangle \le \langle \nabla f(u_{j,n}) - \nabla f(y_n), z\rangle\}. \end{aligned}$$

From the above expression, we know that $E_{j,n}$ is closed and convex for all $n \ge 1$, $j = 1, 2, 3, \dots, m$. Therefore C_n is closed and convex for all $n \ge 1$.

Step 2. We show that $F \subset C_n$ for all $n \ge 1$. Note that $F \subset C_1 = C$. Suppose $F \subset C_n$ for $n \ge 1$, then for all $p \in F \subset C_n$, since $u_{j,n} = K_r^{(j)}(y_n)$ for all $n \ge 1$, $j = 1, 2, 3, \dots, m$, from Lemma 2.18, we have

$$D_f(p, u_{j,n}) = D_f\big(p, K_r^{(j)}(y_n)\big) \le D_f(p, y_n), \quad j = 1, 2, 3, \dots, m, \tag{3.1}$$

where

$$K_r^{(j)}(x) = \left\{z \in C : G_j(z, y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \ge 0, \forall y \in C\right\}.$$

Since

$$\begin{aligned} D_f(p, y_n) &= D_f\big(p, \nabla f^*\big(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(T_n x_n)\big)\big) \\ &= \alpha_n D_f(p, x_n) + (1 - \alpha_n) D_f(p, T_n x_n) \\ &\le \alpha_n D_f(p, x_n) + (1 - \alpha_n)\big(D_f(p, x_n) + \lambda D_f(x_n, T_n x_n)\big) \end{aligned}$$

$$\begin{aligned} &\le D_f(p,x_n) + \lambda D_f(x_n, T_n x_n) \\ &\le D_f(p,x_n) + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - p\rangle, \end{aligned} \tag{3.2}$$

from (3.1) and (3.2) we know that $p \in C_{n+1}$, which implies $F \subset C_n$ for all $n \ge 1$.

Step 3. We show that $\{x_n\}$ converges to a point $p \in C$.

Since $x_n = P^f_{C_n} x_0$ and $C_{n+1} \subset C_n$, then we get

$$D_f(x_n,x_0) \le D_f(x_{n+1},x_0) \quad \text{for all } n \ge 1. \tag{3.3}$$

Therefore $\{D_f(x_n,x_0)\}$ is nondecreasing. On the other hand, by Proposition 2.10, we have

$$D_f(x_n,x_0) = D_f\bigl(P^f_{C_n} x_0, x_0\bigr) \le D_f(p,x_0) - D_f(p,x_n) \le D_f(p,x_0)$$

for all $p \in F \subset C_n$ and for all $n \ge 1$. Therefore, $D_f(x_n,x_0)$ is also bounded. This together with (3.3) implies that the limit of $\{D_f(x_n,x_0)\}$ exists. Put

$$\lim_{n\to\infty} D_f(x_n,x_0) = d. \tag{3.4}$$

From Proposition 2.10, we have, for any positive integer m, that

$$\begin{aligned} D_f(x_{n+m},x_n) &= D_f\bigl(x_{n+m}, P^f_{C_n} x_0\bigr) \\ &\le D_f(x_{n+m},x_0) - D_f\bigl(P^f_{C_n} x_0, x_0\bigr) \\ &= D_f(x_{n+m},x_0) - D_f(x_n,x_0) \end{aligned}$$

for all $n \ge 1$. This together with (3.4) implies that

$$\lim_{n\to\infty} D_f(x_{n+m},x_n) = 0$$

holds uniformly for all m. Therefore, we get that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0$$

holds uniformly for all m. Then $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $p \in C$ such that $x_n \to p$.

Step 4. We show that the limit of $\{x_n\}$ belongs to $\bigcap_{n=1}^{\infty} F(T_n)$.

Since $x_{n+1} \in C_{n+1}$, we have from the definition of C_{n+1} that

$$D_f(x_{n+1},y_n) \le D_f(x_{n+1},x_n) + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - x_{n+1}\rangle,$$

which implies that $\lim_{n\to\infty} D_f(x_{n+1},y_n) = 0$. Since f is totally convex on bounded subsets of E, f is sequentially consistent (see [35]). It follows that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.5}$$

From the uniform continuity of ∇f, we have

$$\lim_{n\to\infty}\left\|\nabla f(x_n)-\nabla f(y_n)\right\|=0.$$

Since

$$y_n=\nabla f^*\bigl(\alpha_n\nabla f(x_n)+(1-\alpha_n)\nabla f(T_nx_n)\bigr),$$

we obtain that

$$\lim_{n\to\infty}\left\|\nabla f(T_nx_n)-\nabla f(x_n)\right\|=\lim_{n\to\infty}\frac{1}{1-\alpha_n}\left\|\nabla f(x_n)-\nabla f(y_n)\right\|=0. \tag{3.6}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Fréchet differentiable on bounded sets. Moreover, f^* is bounded on bounded sets, and from (3.6) we obtain

$$\lim_{n\to\infty}\|T_nx_n-x_n\|=0.$$

Since $\{T_n\}$ is uniformly closed and $x_n\to p$, we have $p\in\bigcap_{n=1}^{\infty}F(T_n)$.

Step 5. We show that the limit of $\{x_n\}$ belongs to EP_j for all $j=1,2,3,\dots,m$.

We have proved that $x_n\to p$ as $n\to\infty$. Now let us show that $p\in EP_j$ for any $j=1,2,3,\dots,m$. Since $x_{n+1}\in C_{n+1}$, we have from the definition of C_{n+1} that

$$D_f(x_{n+1},u_{j,n})\le D_f(x_{n+1},y_n),\quad j=1,2,3,\dots,m.$$

Since $\lim_{n\to\infty}D_f(x_{n+1},y_n)=0$, we have

$$\lim_{n\to\infty}D_f(x_{n+1},u_{j,n})=0,\quad j=1,2,3,\dots,m.$$

Since f is totally convex on bounded subsets of E, f is sequentially consistent (see [35]). It follows that

$$\lim_{n\to\infty}\|x_n-u_{j,n}\|=0,\quad j=1,2,3,\dots,m.$$

This together with (3.5) implies that

$$\lim_{n\to\infty}\|y_n-u_{j,n}\|=0,\quad j=1,2,3,\dots,m.$$

Since ∇f is uniformly norm-to-norm continuous on bounded subsets of E, from (3.5) we have $\lim_{n\to\infty}\|\nabla f(u_{j,n})-\nabla f(y_n)\|=0$. From $\liminf_{n\to\infty}r_n>0$ it follows that

$$\lim_{n\to\infty}\frac{\|\nabla f(u_{j,n})-\nabla f(y_n)\|}{r_n}=0.$$

By the definition of $u_{j,n}:=K_{r_n}^{(j)}y_n$, we have

$$G(u_{j,n},y)+\frac{1}{r_n}\bigl\langle y-u_{j,n},\nabla f(u_{j,n})-\nabla f(y_n)\bigr\rangle\ge 0,\quad\forall y\in C,$$

where

$$G(u_{j,n},y) = \varphi(y) - \varphi(u_{j,n}) + F(u_{j,n},y) + \langle Au_{j,n}, y - u_{j,n}\rangle.$$

We have from (A2) that

$$\frac{1}{r_n}\bigl\langle y - u_{j,n}, \nabla f(u_{j,n}) - \nabla f(y_n)\bigr\rangle \geq -G(u_{j,n},y) \geq G(y,u_{j,n}), \quad \forall y \in C.$$

Since $y \mapsto f(x,y) + \langle Ax, y - x\rangle$ is convex and lower semi-continuous, letting $n \to \infty$ in the last inequality, from (A4) we have

$$G_j(y,p) \leq 0, \quad \forall y \in C.$$

For t, with $0 < t < 1$, and $y \in C$, let $y_t = ty + (1-t)p$. Since $y \in C$ and $p \in C$, then $y_t \in C$ and hence $G_j(y_t,p) \leq 0$. So, from (A1) we have

$$0 = G_j(y_t,y_t) \leq tG_j(y_t,y) + (1-t)G_j(y_t,p) \leq tG_j(y_t,y).$$

Dividing by t, we have

$$G_j(y_t,y) \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$, from (A3) we can get

$$G_j(p,y) \geq 0, \quad \forall y \in C, j = 1,2,3,\ldots,m.$$

So, $p \in EP_j$ for all $j = 1,2,3,\ldots,m$.

Step 6. Finally, we prove that $p = P^f_F x_0$, from Proposition 2.10 we have

$$D_f\bigl(p, P^f_F x_0\bigr) + D_f\bigl(P^f_F x_0, x_0\bigr) \leq D_f(p,x_0). \tag{3.7}$$

On the other hand, since $x_n = P^f_{C_n} x_0$ and $F \subset C_n$ for all n, also from Proposition 2.10, we have

$$D_f\bigl(P^f_F x_0, x_{n+1}\bigr) + D_f(x_{n+1},x_0) \leq D_f\bigl(P^f_F x_0, x_0\bigr). \tag{3.8}$$

By the definition of $D_f(x,y)$, we know that

$$\lim_{n\to\infty} D_f(x_{n+1},x_0) = D_f(p,x_0). \tag{3.9}$$

Combining (3.7), (3.8), and (3.9), we know that $D_f(p,x_0) = D_f(P^f_F x_0, x_0)$. Therefore, it follows from the uniqueness of $P^f_F x_0$ that $p = P^f_F x_0$. This completes the proof. □

Remark 3.2 Theorem 3.1 includes the following three special cases.

(1) Take $T_n \equiv I$, $\varphi(x) \equiv 0$, $F(x,y) \equiv 0$, where I denotes the identity operator, then the iterative sequence $\{x_n\}$ converges strongly to a solution of the system of variational in-

equalities

$$\begin{cases} \langle A_1 u, y - u\rangle \geq 0, \\ \langle A_2 u, y - u\rangle \geq 0, \\ \langle A_3 u, y - u\rangle \geq 0, \qquad \forall y \in C. \\ \dots, \\ \langle A_m u, y - u\rangle \geq 0, \end{cases}$$

In this case, the iterative sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 = x_0 \in C_1 = C, \\ \langle A_j u_{j,n}, y - u_{j,n}\rangle + \frac{1}{r_n}\langle \nabla f(u_{j,n}) - \nabla f(x_n), y - u_{n,j},\rangle \geq 0, \quad \forall y \in C, j = 1,2,3,\dots,m, \\ C_{n+1} = \{z \in C_n : D_f(z, u_{j,n}) \leq D_f(z, y_n) \leq D_f(z, x_n) \\ \qquad + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle, j = 1,2,3,\dots,m\}, \\ x_{n+1} = P^f_{C_{n+1}} x_0. \end{cases}$$

(2) Take $T_n \equiv I$, $\varphi(x) \equiv 0$, $A \equiv 0$, where I denotes the identity operator, then the iterative sequence $\{x_n\}$ converges strongly to a solution of the system of equilibrium problems

$$\begin{cases} F_1(u, y) \geq 0, \\ F_2(u, y) \geq 0, \\ F_3(u, y) \geq 0, \qquad \forall y \in C. \\ \dots, \\ F_m(u, y) \geq 0, \end{cases}$$

In this case, the iterative sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 = x_0 \in C_1 = C, \\ F(u_{j,n}, y) + \frac{1}{r_n}\langle \nabla f(u_{j,n}) - \nabla f(x_n), y - u_{n,j},\rangle \geq 0, \quad \forall y \in C, j = 1,2,3,\dots,m, \\ C_{n+1} = \{z \in C_n : D_f(z, u_{j,n}) \leq D_f(z, y_n) \leq D_f(z, x_n) \\ \qquad + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle, j = 1,2,3,\dots,m\}, \\ x_{n+1} = P^f_{C_{n+1}} x_0. \end{cases}$$

(3) Take $T_n \equiv I$, $F(x, y) \equiv 0$, $A \equiv 0$, where I denotes the identity operator, then the iterative sequence $\{x_n\}$ converges strongly to a solution of the system of convex optimization problems

$$\begin{cases} \varphi_1(u) = \min_{y \in C} \varphi_1(y), \\ \varphi_2(u) = \min_{y \in C} \varphi_2(y), \\ \varphi_3(u) = \min_{y \in C} \varphi_3(y), \\ \dots, \\ \varphi_m(u) = \min_{y \in C} \varphi_m(y). \end{cases}$$

In this case, the iterative sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 = x_0 \in C_1 = C, \\ \varphi(y) - \varphi(u_{j,n}) + \frac{1}{r_n}\langle \nabla f(u_{j,n}) - \nabla f(x_n), y - u_{n,j}, \rangle \geq 0, \quad \forall y \in C, j = 1,2,3,\ldots,m, \\ C_{n+1} = \{z \in C_n : D_f(z, u_{j,n}) \leq D_f(z, y_n) \leq D_f(z, x_n) \\ \qquad + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - z\rangle, j = 1,2,3,\ldots,m\}, \\ x_{n+1} = P^f_{C_{n+1}} x_0. \end{cases}$$

4 Examples

Let E be a Hilbert space and C be a nonempty closed convex and balanced subset of E. Let $\{x_n\}$ be a sequence in C such that $\|x_n\| = r > 0$, $\{x_n\}$ converges weakly to $x_0 \neq 0$, and $\|x_n - x_m\| \geq r > 0$ for all $n \neq m$. Define a countable family of mappings $\{T_n\} : C \to C$ as follows:

$$T_n(x) = \begin{cases} \frac{n+1}{n} x_n & \text{if } x = x_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n \ (\forall n \geq 1). \end{cases}$$

Conclusion 4.1 *$\{T_n\}$ has a unique common fixed point* 0*, that is,* $F = \bigcap_{n=1}^{\infty} F(T_n) = \{0\}$ *for all* $n \geq 0$.

Proof The conclusion is obvious. □

Conclusion 4.2 *$\{T_n\}$ is a uniformly closed family of countable quasi-Bregman* $(2n+1)$*-pseudocontractive mappings.*

Proof Take $f(x) = \frac{\|x\|^2}{2}$, then

$$D_f(x,y) = \phi(x,y) = \|x-y\|^2$$

for all $x, y \in C$ and

$$D_f(0, T_n x) = \|T_n x\|^2 = \begin{cases} (\frac{n+1}{n})^2 \|x_n\|^2 & \text{if } x = x_n, \\ \|x\|^2 & \text{if } x \neq x_n. \end{cases}$$

Therefore,

$$\begin{aligned} D_f(0, T_n x_n) &\leq \left(\frac{n+1}{n}\right)^2 \|x_n\|^2 \\ &= \frac{n^2 + 2n + 1}{n^2}\|x_n\|^2 \\ &= \|x_n\|^2 + \frac{2n+1}{n^2}\|x_n\|^2 \\ &= \|x_n\|^2 + (2n+1)\frac{\|x_n\|^2}{n^2} \\ &= \|x_n\|^2 + (2n+1)\left\|\frac{x_n}{n}\right\|^2 \end{aligned}$$

$$= \|x_n\|^2 + (2n+1)\|x_n - T_n x_n\|^2$$
$$= D_f(0, x_n) + (2n+1)D_f(x_n, T_n x_n)$$

for all $x \in C$. On the other hand, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$, it is easy to see that there exists a sufficiently large nature number N such that $z_n \neq x_m$ for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, it follows from $\|z_n - T_n z_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. That is, $z_0 \in F$. □

Example 4.3 Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$
$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{\frac{1}{2}}, \quad \forall \xi \in l^2,$$
$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots), \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n, \ldots) \in l^2.$$

Let $\{x_n\} \subset E$ be a sequence defined by

$$x_0 = (1, 0, 0, 0, \ldots),$$
$$x_1 = (1, 1, 0, 0, \ldots),$$
$$x_2 = (1, 0, 1, 0, 0, \ldots),$$
$$x_3 = (1, 0, 0, 1, 0, 0, \ldots),$$
$$\ldots,$$
$$x_n = (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \ldots),$$
$$\ldots,$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1 \end{cases}$$

for all $n \geq 1$. It is well known that $\|x_n\| = \sqrt{2}$, $\forall n \geq 1$ and $\{x_n\}$ converges weakly to x_0. Define a countable family of mappings $T_n : E \to E$ as follows:

$$T_n(x) = \begin{cases} \frac{n+1}{n} x_n & \text{if } x = x_n, \\ -x & \text{if } x \neq x_n \end{cases}$$

for all $n \geq 0$. By using Conclusions 4.1 and 4.2, $\{T_n\}$ is a uniformly closed family of countable quasi-Bregman $(2n+1)$-pseudocontractive mappings.

Example 4.4 Let $E = L^p[0,1]$ $(1 < p < +\infty)$ and

$$x_n = 1 - \frac{1}{2^n}, \quad n = 1,2,3,\ldots.$$

Define a sequence of functions in $L^p[0,1]$ by the following expression:

$$f_n(x) = \begin{cases} \frac{2}{x_{n+1}-x_n} & \text{if } x_n \le x < \frac{x_{n+1}+x_n}{2}, \\ \frac{-2}{x_{n+1}-x_n} & \text{if } \frac{x_{n+1}+x_n}{2} \le x < x_{n+1}, \\ 0 & \text{otherwise} \end{cases}$$

for all $n \ge 1$. Firstly, we can see, for any $x \in [0,1]$, that

$$\int_0^x f_n(t)\,dt \to 0 = \int_0^x f_0(t)\,dt, \tag{4.1}$$

where $f_0(x) \equiv 0$. It is well known that the above relation (4.1) is equivalent to $\{f_n(x)\}$ converges weakly to $f_0(x)$ in a uniformly smooth Banach space $L^p[0,1]$ $(1 < p < +\infty)$. On the other hand, for any $n \ne m$, we have

$$\begin{aligned} \|f_n - f_m\| &= \left(\int_0^1 \left|f_n(x) - f_m(x)\right|^p dx\right)^{\frac{1}{p}} \\ &= \left(\int_{x_n}^{x_{n+1}} \left|f_n(x) - f_m(x)\right|^p dx + \int_{x_m}^{x_{m+1}} \left|f_n(x) - f_m(x)\right|^p dx\right)^{\frac{1}{p}} \\ &= \left(\int_{x_n}^{x_{n+1}} \left|f_n(x)\right|^p dx + \int_{x_m}^{x_{m+1}} \left|f_m(x)\right|^p dx\right)^{\frac{1}{p}} \\ &= \left(\left(\frac{2}{x_{n+1}-x_n}\right)^p (x_{n+1}-x_n) + \left(\frac{2}{x_{m+1}-x_m}\right)^p (x_{m+1}-x_m)\right)^{\frac{1}{p}} \\ &= \left(\frac{2^p}{(x_{n+1}-x_n)^{p-1}} + \frac{2^p}{(x_{m+1}-x_m)^{p-1}}\right)^{\frac{1}{p}} \\ &\ge \left(2^p + 2^p\right)^{\frac{1}{p}} > 0. \end{aligned}$$

Let

$$u_n(x) = f_n(x) + 1, \quad \forall n \ge 1.$$

It is obvious that u_n converges weakly to $u_0(x) \equiv 1$ and

$$\|u_n - u_m\| = \|f_n - f_m\| \ge \left(2^p + 2^p\right)^{\frac{1}{p}} > 0, \quad \forall n \ge 1. \tag{4.2}$$

Define a mapping $T : E \to E$ as follows:

$$T_n(x) = \begin{cases} \frac{n+1}{n} u_n & \text{if } x = u_n \ (\exists n \ge 1), \\ -x & \text{if } x \ne u_n \ (\forall n \ge 1). \end{cases}$$

Since (4.2) holds, by using Conclusions 4.1 and 4.2, we know that $\{T_n\}$ is a uniformly closed family of countable quasi-Bregman $(2n+1)$-pseudocontractive mappings.

5 The mistakes in the result of Ugwunnadi *et al.* [24]

In [24], from page 10, line –3 to page 11, line 2, there exists a mistake ratiocination as follows.

Mistake ratiocination 1 *Since* $x_{n+1} \in C_{n+1}$, *it follows from* (3.6), (3.7) *that*

$$\begin{aligned}(*)\quad f(x_{n+1}) - f(w_n) - \langle \nabla f(w_n), x_{n+1} - w_n \rangle &= D_f(x_{n+1}, w_n) \\ &\le D_f(x_{n+1}, y_n) \\ &\le D_f(x_{n+1}, x_n) + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - x_{n+1}\rangle,\end{aligned}$$

which implies from (3.20), (3.18), (3.13), *and* (3.14) *that*

$$\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0.$$

However, (3.6), (3.7) are the following:

$$\begin{aligned}(3.6)\quad D_f(w, w_n) &= D_f\left(w, \nabla f^*\left(\sum_{j=1}^m \beta_{j,n} \nabla f(u_{j,n})\right)\right) \\ &\le \sum_{j=1}^m \beta_{j,n} D_f(w, u_{j,n}) \\ &\le \sum_{j=1}^m \beta_{j,n} D_f(w, y_n) \\ &= D_f(w, y_n)\end{aligned}$$

for any $w \in F$,

$$\begin{aligned}(3.7)\quad D_f(w, y_n) &= D_f\big(w, \nabla f^*\big(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_n x_n)\big)\big) \\ &\le \alpha_n D_f(w, x_n) + (1-\alpha_n) D_f(w, T_n x_n) \\ &\le \alpha_n D_f(w, x_n) + (1-\alpha_n)\big(D_f(w, x_n) + k D_f(x_n, T_n x_n)\big) \\ &\le D_f(w, x_n) + k D_f(x_n, T_n x_n) \\ &\le D_f(w, x_n) + \frac{k}{1-k}\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - w\rangle\end{aligned}$$

for any $w \in F$.

In fact, the authors attempt taking $w = x_{n+1}$ in (3.6) and (3.7) to get the $(*)$. This is an obvious mistake since (3.6) and (3.7) are right for only $w \in F$, but x_{n+1} does not belong to F. Therefore, the definition of an iterative sequence $\{x_n\}$ must be modified so that $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$ implies $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$.

In [24], page 12, line 3, there exists another mistake ratiocination as follows.

Mistake ratiocination 2 *Also, since $y_n \to p$ as $n \to \infty$, we have from Lemma* 2.3, *for each* $j = 1, 2, 3, \ldots, m$,

$$0 \le D_f(p, u_{j,n}) = D_f\bigl(p, \operatorname{Res}^f_{g_j} y_n\bigr) \le D_f(p, y_n) \to 0 \quad \text{as } n \to \infty.$$

In fact, we are proving that $p \in EP_j$ for any $j = 1, 2, 3, \ldots, m$, therefore, if we do not know whether $p \in EP_j$, then the above inequalities are not right since if $p \in EP_j$, the above inequalities are right. In this paper, we have overcome these shortcomings by modifying the iterative scheme.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by the corresponding author YS, and YS prepared the manuscript initially for basic structure. YX performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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