Chaoui et al. *Boundary Value Problems* (2015) 2015:133
DOI 10.1186/s13661-015-0396-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Full discretization of wave equation

Abderrazak Chaoui[1*], Fateh Ellaggoune[2] and Assia Guezane-Lakoud[3]

*Correspondence:
rchaoui@outlook.fr
[1]Department of Mathematics, University 8 Mai 1945 Guelma, P.O. Box 401, Guelma, 24000, Algeria
Full list of author information is available at the end of the article

Abstract

Rothe's method for time discretization and Crouseix-Raviart nonconforming finite element method to the spatial variable. After introducing error estimators, we prove the equivalence between the error and its indicators.

MSC: Primary 35K55; secondary 35A35

Keywords: Rothe's method; Crouseix-Raviart nonconforming finite element method; wave equation; *a posteriori* estimate

1 Introduction

Among commonly used methods for the numerical approach of problems which arise in engineering, for example, Laplace equation and Maxwell system, the finite element method is one of the most relied on methods because it is much more interested in the analysis of the error committed between the exact solution and the approximate solution. In many of these applications, adaptive techniques using *a posteriori* error estimators have become an indispensable tool. These estimators allow to measure the quality of the computed solution and provide information to control the mesh adaptation algorithm. There are a lot of works on the *a posteriori* estimators for the elliptic partial differential equations and dynamic partial differential equations. Of these works, it is possible to refer to [1] where the authors considered an elliptic second order boundary value problem approximated by a discontinuous Galerkin method. Time dependent Stokes equations in [2] and second order wave equations in [3] are discretized by Euler's implicit scheme in time and standard finite elements in space. Using Rothe's method in [4] and [5], the authors studied the equation of telegraph and integrodifferential equation with integral conditions (resp.).

The purpose of this work is to combine Rothe's method with nonconforming finite element method of Crouseix-Raviart and to introduce *a posteriori* error estimators suitable for the wave equation assumptions on the mesh. These indicators can give a good overview of the local distribution of the error and a useful tool for mesh adaptation.

Let Ω be a bounded open domain of $\mathbb{R}^d$, $d = 2$ or 3 with Lipschitz boundary Γ that we suppose to be polygonal ($d = 2$) or polyhedral ($d = 3$). We further assume that Ω is simply connected and that Γ is connected. Let T be a fixed positive number,

$$\begin{aligned}
&\partial_{tt}u - \triangle u = f \quad \text{in } \Omega\times\,]0,T[,\\
&u(x,0) = u_0(x) \quad \text{in } \Omega \text{ at } t = 0,\\
&u = 0 \quad \text{at } \Gamma \times I, I = (0,T),\\
&\partial_t u(x,0) = u_1(x) \quad \text{in } \Omega \text{ at } t = 0,
\end{aligned} \tag{1}$$

Springer

where $f \in L^2((0,T),L^2(\Omega))$, $U_0 \in H_0^1(\Omega)$ and $U_1 \in L^2(\Omega)$. Under these conditions, problem (1) is equivalent to

$$(\partial_{tt}u, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega) \text{ a.e. } t \in (0,T) \tag{2}$$

has a unique weak solution $C((0,T),H_0^1(\Omega)) \cap C^1((0,T),L^2(\Omega))$. If we put $U = \binom{u}{\partial_t u}$ and $F = \binom{0}{f}$, then problem (1) can be rewritten as follows:

$$\begin{cases} \partial_t U - \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix} U = F & \text{in } \Omega \times]0,T[, \\ U = 0 & \text{on } \Gamma \times I, \\ U(\cdot,0) = U_0 \end{cases} \tag{3}$$

with $U_0 = \binom{u_0}{u_1}$.

2 Time discretization using Rothe's method

We divide the interval $(0,T)$ into subintervals of length $\tau = \frac{T}{n}$ and denote $u^j = u(j\tau, x)$, $x \in \Omega$, $j = 1,\dots,n$. Successively, for $j = 1,\dots,n$, we solve the linear stationary boundary value problem

$$\begin{cases} \frac{u^j - 2u^{j-1} + u^{j-2}}{\tau^2} - \triangle u^j = f^j & \text{in } \Omega, \\ u^j = 0 & \text{on } \Gamma, \\ u^0(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{4}$$

where $f^j = f(x,t_j) = f(x,j\tau)$. Setting $u^{-1}(x) = u_0(x) - \tau u_1(x)$, define $\delta u^j = \frac{u^j - u^{j-1}}{\tau}$, $\delta^2 u^j = \frac{\delta u^j - \delta u^{j-1}}{\tau}$, $j = 1,\dots,n$.

This problem has a unique weak solution $u^j \in H_0^1(\Omega)$ by the Lax-Milgram lemma whose variational formulation is

$$\left(\delta^2 u^j, v\right) + \left(\nabla u^j, \nabla v\right) = \left(f^j, v\right), \quad \forall v \in H_0^1(\Omega). \tag{5}$$

We define Rothe's functions by a piecewise linear interpolation with respect to the time t,

$$u^n(t,x) = u^{j-1} + (t - t_{j-1})\delta u^j \quad \text{for } t_{j-1} \le t \prec t_j, j = 1,\dots,n, \tag{6}$$

$$\delta u^n(t,x) = \delta u^{j-1} + (t - t_{j-1})\delta^2 u^j \quad \text{for } t_{j-1} \le t \prec t_j, j = 1,\dots,n, \tag{7}$$

the auxiliary functions are

$$\overline{u(t,x)^n} = \begin{cases} u^j & \text{if } t \in (t_{j-1}, t_j), j = 1,\dots,n, \\ u_0 & \text{if } t \in [-h, 0], \end{cases} \tag{8}$$

$$\overline{\delta u(t,x)^n} = \begin{cases} \delta u^j & \text{if } t \in (t_{j-1}, t_j), j = 1,\dots,n, \\ u_1 & \text{if } t \in [-h, 0]. \end{cases} \tag{9}$$

3 Full discretization

We consider the following nonconforming finite element method to approximate our problem. For all $j = 1,\dots,n$, we consider a triangulation Υ_{jh} made of triangles T if $d = 2$

and of tetrahedra if $d = 3$ whose edges/faces are denoted by e. We assume that this triangulation is regular in Ciarlet's sense ([6], p.124), *i.e.* , $\exists\sigma \succ 0$ such that $\frac{h_T}{\rho_T} \leq \sigma$, $\forall T \in \Upsilon_{jh}$, where h_T is the diameter of T and ρ_T is the diameter of its largest inscribed bull. We define $h_j = \max_{T\in\Upsilon_{jh}} h_T$. Let $\zeta_{jh}^{\mathrm{int}}$ be the set of interior edges/faces of Υ_{jh} and ζ_T be the set of edges/faces of the element T. For an edge/face $e \in \zeta_T \cap \zeta_K$, we denote by $h_e = \frac{1}{2}(\frac{d(T)}{|e|} + \frac{d(K)}{|e|})$ its mean height. Problem (5) is approximated by the Crouseix-Raviart nonconforming finite element space

$$X_{jh}^0 = \left\{ v \in L^2(\Omega); v\backslash T \in P_1, \forall T \in \Upsilon_{jh}, \int_e v\backslash T = \int_e v\backslash K, \forall e \in \zeta_T \cap \zeta_K \cap \zeta_{jh}^{\mathrm{int}}, \right.$$
$$\left. T, K \in \Upsilon_{jh}, \int_e v\backslash T = 0, \forall e \in \zeta_T \cap \Gamma, T \in \Upsilon_{jh} \right\}.$$

We consider the fully discrete scheme for problem (1): for each $j = 1, \ldots, n$, find $u_h^j \in X_{jh}^0$ such that

$$\tau^2 \sum_{T\in\Upsilon_{jh}} \int_T \nabla u_h^j \nabla v_h = \tau^2 \int_\Omega f^j v_h - \int_\Omega \left(u_h^j - 2u_h^{j-1} + u_h^{j-2}\right) v_h. \tag{10}$$

We will use the following Crouseix-Raviart property:

$$\int_e [u_h]_e = 0, \quad \forall e \in \zeta_{jh}, \forall u_h \in X_{jh}^0, \tag{11}$$

where the jump of some function v across an edge/face at point x is defined by

$$\left[v(x)\right]_e = \begin{cases} \lim_{\alpha\to 0^+} v(x + \alpha\eta_e) - v(x - \alpha\eta_e) & \text{if } e \in \zeta_{jh}^{\mathrm{int}}, \\ v(x) & \text{if } e \in \zeta_{jh}\backslash\zeta_{jh}^{\mathrm{int}}, \end{cases}$$

η_e denotes the outward normal vector for a boundary edge/face e and $t_e = (-\eta_{e_2}, \eta_{e_1})$ is the tangent vector if $\eta_e = (\eta_{e_1}, \eta_{e_2})$.

Since $[v_h]$ is linear on e, the condition $\int_e [v_h]_e = 0$ is equivalent to the continuity of v_h at e barycenter.

For $v_h \in X_{jh}^0$, we define its broken gradient $\nabla_h v$ in Ω by

$$(\nabla_h v)\backslash T = \nabla(v\backslash T), \quad \forall T \in \Upsilon_{jh}.$$

We will need local subdomain, also called patches. For any $T \in \Upsilon_{jh}$, let w_T be the union of all elements having a common edge/face with T. Similarly let w_e be the union of all elements having e as an edge/face. Finally, let w_x be the union of elements having x as a node, and $\widetilde{w}_T$ (resp. $\widetilde{w}_e$) be the union of all triangles sharing a node with T (resp. e).

Later on, we also need the standard P_1 conforming finite element spaces:

$$V_{jh} = \left\{ v \in H^1(\Omega); v\backslash T \in P_1, \forall T \in \Upsilon_{jh} \right\},$$
$$V_{jh}^0 = V_{jh} \cap H_0^1(\Omega).$$

We further need

$$Y_{jh} = \left\{ v \in L^2(\Omega); v\backslash T \in H^1(\Omega), \forall T \in \Upsilon_{jh}, \int_e v\backslash T = \int_e v\backslash K, \forall e \in \zeta_T \cap \zeta_K \cap \zeta_{jh}^{\text{int}}, \right.$$
$$\left. T, K \in \Upsilon_{jh} \right\},$$
$$Y_{jh}^0 = \left\{ v \in L^2(\Omega); v\backslash T \in H^1(\Omega), \forall T \in \Upsilon_{jh}, \int_e v\backslash T = \int_e v\backslash K, \forall e \in \zeta_T \cap \zeta_K \cap \zeta_{jh}^{\text{int}}, \right.$$
$$\left. T, K \in \Upsilon_{jh}, \int_e v\backslash T = 0, \forall e \in \zeta_T \cap \Gamma, T \in \Upsilon_{jh} \right\}.$$

We recall that for a node $x \in N_{jh}$, we denote by λ_x the standard hat function such that $\lambda_x(y) = \delta_{xy}, \forall y \in N_{jh}$, where N_{jh} is the set of nodes of Υ_{jh} and N_{jh}^{int} denotes the set of interior nodes of Υ_{jh}.

Definition 3.1 For $v \in Y_{jh}$ and $w \in Y_{jh}^0$, Clément interpolation is defined as follows:

$$I_C v = \sum_{x \in N_{jh}} |w_x|^{-1} \left(\int_{w_x} v \right) \lambda_x, \tag{12}$$

$$I_C^0 w = \sum_{x \in N_{jh}^{\text{int}}} |w_x|^{-1} \left(\int_{w_x} w \right) \lambda_x. \tag{13}$$

We define the gradient jump of u_h^j in normal and tangential direction as follows:

$$J_{e,\eta}^j = \begin{cases} [\nabla u_h^j \cdot \eta_e] & \text{if } e \in \zeta_{jh}^{\text{int}}, \\ 0 & \text{if } e \in \zeta_{jh} \backslash \zeta_{jh}^{\text{int}}. \end{cases} \tag{14}$$

If $d = 2$, then

$$J_{e,t}^j = \begin{cases} [\nabla u_h^j \cdot t_e] & \text{if } e \in \zeta_{jh}^{\text{int}}, \\ -\nabla u_h^j \cdot t_e & \text{if } e \in \zeta_{jh} \backslash \zeta_{jh}^{\text{int}}. \end{cases} \tag{15}$$

If $d = 3$, then

$$J_{e,t}^j = \begin{cases} [\nabla u_h^j \times n_e] & \text{if } e \in \zeta_{jh}^{\text{int}}, \\ -\nabla u_h^j \times n_e & \text{if } e \in \zeta_{jh} \backslash \zeta_{jh}^{\text{int}}. \end{cases}$$

Lemma 3.1 [7] *For all $v \in Y_{jh}$ and $w \in Y_{jh}^0$, we have*

$$\|v - I_C v\|_T \lesssim h_T \|\nabla_h v\|_{\widetilde{w}_T}, \quad \forall T \in \Upsilon_{jh}, \tag{16}$$

$$\|v - I_C v\|_e \lesssim h_e^{\frac{1}{2}} \|\nabla_h v\|_{\widetilde{w}_e}, \quad \forall e \in \zeta_{jh}, \tag{17}$$

$$\left\| w - I_C^0 w \right\|_T \lesssim h_T \|\nabla_h w\|_{\widetilde{w}_T}, \quad \forall T \in \Upsilon_{jh}, \tag{18}$$

$$\left\| w - I_C^0 w \right\|_e \lesssim h_e^{\frac{1}{2}} \|\nabla_h w\|_{\widetilde{w}_e}, \quad \forall e \in \zeta_{jh}^{\text{int}}, \tag{19}$$

$$\left\| \nabla I_C^0 w \right\| \lesssim \|\nabla_h w\|_{\widetilde{w}_T}, \quad \forall T \in \Upsilon_{jh}. \tag{20}$$

Next, we need the following Green's formulas.

If D is open bounded of $\mathbb{R}^2$ and $v, w \in H^1(D)$, then

$$\int_D \nabla v \operatorname{curl} w = \int_{\partial D} v \operatorname{curl} w \cdot \eta = \int_{\partial D} \nabla v \cdot tw, \tag{21}$$

where t is the unit tangent vector along ∂D and $\operatorname{curl} w = \binom{\partial_2 w}{-\partial_1 w}$.

Similarly if D is open bounded of $\mathbb{R}^3$ and $v \in H^1(D)$, $w \in H^1(D)^3$, then we have

$$\int_D \nabla v \operatorname{curl} w = \int_{\partial D} v \operatorname{curl} w \cdot \eta = \int_{\partial D} (\nabla v \times \eta) \cdot w. \tag{22}$$

4 *A posteriori* analysis of time discretization

For each j, $j = 1, \dots, n$, the refinement indicator is defined by

$$\eta_t^j = \tau \left\| \nabla_h \left(u_h^{j+1} - u_h^j \right) \right\| + \tau \left\| \frac{u_h^{j+1} - 2u_h^j + u_h^{j-1}}{\tau} \right\|, \tag{23}$$

$e^\tau = u - u^n$ indicate the error with respect to the discretization time.

Proposition 4.1 (Upper and lower bounds of the error in time) *The following a posteriori error estimate holds for all t_{j+1}, $j = 1, \dots, n-1$:*

$$\begin{aligned} &\left\| (\partial_t u)(t_{j+1}) - \frac{u^{j+1} - u^j}{\tau} \right\|_{H^{-1}(\Omega)} + \left\| u(t_{j+1}) - u^{j+1} \right\| \\ &\quad \lesssim \sum_{m=1}^{j} \eta_t^m + \tau \|\nabla U_0\| + \tau^2 \|\nabla U_1\| + \sum_{m=1}^{j} \tau \sum_{k=0}^{1} \left\| \nabla \left(u^{m+1-k} - u_h^{m+1-k} \right) \right\| \\ &\qquad + \sum_{m=1}^{j} \tau \sum_{k=0}^{1} \left\| \delta u^{m+1-k} - \delta_h^{m+1-k} \right\|, \end{aligned} \tag{24}$$

$$\begin{aligned} \eta_t^j &\leq \sum_{k=0}^{1} \left\| (\partial_t u)(t_{j+1-k}) - \delta u^{j+1-k} \right\|_{H^{-1}(\Omega)} + \left\| u(t_{j+1}) - \overline{u^{j+1-k}} \right\| + \left\| \int_{t_j}^{tj+1} \nabla \left(u - u^n \right)(s)\, ds \right\| \\ &\quad + \left\| \int_{t_j}^{tj+1} \left(\partial_t u - \delta u^n \right)(s)\, ds \right\| + \tau \sum_{k=0}^{1} \left\| \nabla \left(u^{j+1-k} - u_h^{j+1-k} \right) \right\| \\ &\quad + \tau \sum_{k=0}^{1} \left\| \delta u^{j+1-k} - \delta_h u_h^{j+1-k} \right\|. \end{aligned} \tag{25}$$

Proof See [3]. □

5 *A posteriori* analysis of space discretization

The error indicator is defined by

$$\eta_T^j = h_T \left\| f_h^j - \frac{u_h^j - 2u_h^{j-1} + u_h^{j-2}}{\tau^2} \right\| + \sum_{e \in \zeta_{jh}} h_e^{\frac{1}{2}} \left(\left\| J_{e,\eta}^j \right\|_e + \left\| J_{e,t}^j \right\|_e \right),$$

the global error estimator η^j is given by

$$\eta^j = \sqrt{\sum_{T\in\Upsilon_{jh}} \left(\eta_T^j\right)^2},$$

the higher order term depending on the datum f is defined as

$$\operatorname{osc}(f, w_T)^2 = \sum_{T\in\Upsilon_{jh}} h_T^2 \left\| f - f_h^j \right\|_{w_T}^2, \quad \text{where } \left(f_h^j\right)\backslash T := \frac{1}{|T|}\int_T f_h^j, T \in \Upsilon_{jh}.$$

Our main result is the following theorem.

Theorem 5.1 (Upper bound) *The following inequality holds*:

$$\left\|e^n\right\|^2 + \sum_{j=1}^n \tau^2 \left\|\nabla_h e^j\right\|^2 \lesssim \left\|e^0\right\|^2 + \left\|e^1\right\|^2 + \tau^2 \sum_{j=1}^n \operatorname{osc}(f, w_T)^2 + \sum_{j=1}^n \max\left\{h_j^2, \tau^2\right\}\left(\eta^j\right)^2.$$

To prove this theorem we need some lemmas. As our approximated scheme is a non-conforming one, we need to use an appropriate Helmholtz decomposition of the error.

Lemma 5.1 (Helmholtz decomposition of the error) *Let* $e^j = u^j - u_h^j$, *then we have the following decomposition of error* e^j:

$$\nabla_h e^j = \nabla\varphi^j + \operatorname{curl}\chi^j \tag{26}$$

with $\chi^j \in H^1(\Omega)$ *and* $\varphi^j \in H_0^1(\Omega)$; *furthermore, the following inequalities hold*:

$$\left\|\nabla\varphi^j\right\| \le \left\|\nabla_h e^j\right\|, \tag{27}$$

$$\left\|\operatorname{curl}\chi^j\right\| \le \left\|\nabla_h e^j\right\|. \tag{28}$$

Proof We consider the following problem: find $\varphi^j \in H_0^1(\Omega)$, a solution of

$$\begin{cases} \operatorname{div}(\nabla_h e^j - \nabla\varphi^j) = 0 & \text{in } \Omega, \\ \varphi^j = 0 & \text{on } \Gamma. \end{cases} \tag{29}$$

The weak formulation of that problem (29) is

$$\int_\Omega \nabla\varphi^j \nabla v = \int_\Omega \nabla_h e^j \nabla v, \quad \forall v \in H_0^1(\Omega). \tag{30}$$

As the vector field $(\nabla_h e^j - \nabla\varphi^j)$ is divergence-free in Ω, *i.e.*, $\operatorname{div}(\nabla_h e^j - \nabla\varphi^j) = 0$ in Ω, by Theorem 1.3.1 of [8] if $d = 2$ or Theorem 1.3.4 of [8] if $d = 3$, there exists $\chi^j \in H^1(\Omega)$ if $d = 2$ and $\chi^j \in H^1(\Omega)^3$ if $d = 3$ such that

$$\operatorname{curl}\chi^j = \nabla_h e^j - \nabla\varphi^j.$$

Estimate (27) directly follows by taking $v = \varphi^j$ in (30). To prove the inequality, we use identity (26) and we get

$$\int_\Omega |\operatorname{curl} \chi^j|^2 = \int_\Omega \operatorname{curl} \chi^j (\nabla_h e^j - \nabla \varphi^j). \tag{31}$$

Using Green's formula and taking into account that $\varphi^j = 0$ on Γ, we get

$$\int_\Omega |\operatorname{curl} \chi^j|^2 = \int_\Omega \operatorname{curl} \chi^j \nabla_h e^j.$$

The Cauchy-Schwarz inequality implies

$$\|\operatorname{curl} \chi^j\| \le \|\nabla_h e^j\|.$$

□

Lemma 5.2 *The error satisfies the following identity*:

$$\sum_{T\in\Upsilon_{jh}} \int_T \nabla_h e^j \nabla v_h = \int_\Omega \frac{-e^j + 2e^{j-1} - e^{j-2}}{\tau^2} v_h, \quad \forall v_h \in V_{jh}^0(\Omega). \tag{32}$$

Proof We only need to take $v = v_h$ in (5), then taking the difference between (5) and (10) we get the result. □

Lemma 5.3 *Let $\varphi_h \in V_{jh}$ if $d = 2$ and $\varphi_h \in (V_{jh})^3$ if $d = 3$, then the error verifies*

$$\sum_{T\in\Upsilon_{jh}} \int_T \nabla_h e^j \operatorname{curl} \varphi_h = 0. \tag{33}$$

Proof We integrate by parts the expression $\int_\Omega \nabla_h e^j \operatorname{curl} \varphi_h$, using Green's formula and taking into account that $u^j \in H_0^1(\Omega)$, then we use the property of finite elements of Crouseix-Raviart ($\int_e [u_h^j] = 0$) and get (33). □

Lemma 5.4 *Let $\varphi \in H^1(\Omega)$ if $d = 2$ and $\varphi \in (H^1(\Omega))^3$ if $d = 3$, then we have*

$$\int_\Omega \nabla_h e^j \operatorname{curl} \varphi = \sum_{e\in\zeta_{jh}} \int_E J_{E,t}^j \cdot \varphi. \tag{34}$$

Proof The integration by parts and Green's formula give us

$$\int_\Omega \nabla_h e^j \operatorname{curl} \varphi = \int_\Omega \nabla u^j \operatorname{curl} \varphi - \sum_{T\in\Upsilon_{jh}} \int_T \nabla u_h^j \operatorname{curl} \varphi_h = - \sum_{T\in\Upsilon_{jh}} \int_{\partial T} \nabla u_h^j \cdot t_T \varphi,$$

because $u^j \in H_0^1(\Omega)$, and according to the definition of $J_{E,t}^j$ we find (34). □

Lemma 5.5 *Let $\varphi \in H_0^1(\Omega)$, then e^j verifies*

$$\sum_{T\in\Upsilon_{jh}} \int_T \nabla_h e^j \nabla \varphi = \sum_{T\in\Upsilon_{jh}} \int_T \left(f^j - \frac{u^j - 2u^{j-1} + u^{j-2}}{\tau^2} \right) \varphi + \sum_{e\in\zeta_{jh}} \int_e J_{e,\eta}^j \cdot \varphi.$$

Proof We integrate by parts the expression $\sum_{T\in\Upsilon_{jh}}\int_T \nabla_h e^j \nabla\varphi$ with $\triangle u = 0$ on each element $T\in\Upsilon_{jh}$, and from the definition of $J^j_{e,\eta}$ we conclude the proof. □

Remark 5.1 Lemmas 5.4, 5.5 imply that $\forall\varphi\in H^1_0(\Omega)$ and $\chi\in H^1(\Omega)$ if $d=2$ and $\chi\in(H^1(\Omega))^3$ if $d=3$, we have

$$\sum_{T\in\Upsilon_{jh}}\int_T \nabla_h e^j(\nabla\varphi+\operatorname{curl}\chi) = \sum_{T\in\Upsilon_{jh}}\int_T\left(f^j-\frac{u^j-2u^{j-1}+u^{j-2}}{\tau^2}\right)\varphi + \sum_{e\in\zeta_{jh}}\int_e\left(J^j_{e,\eta}\cdot\varphi + J^j_{\eta,t}\cdot\chi\right).$$

Note that the local error estimator η^j_T is inspired by the latter identity.

Proof From what precedes (Lemmas 5.2 and 5.5), we can easily prove that

$$\tau^2\int_\Omega \nabla_h e^j\nabla\varphi^j = \tau^2\int_\Omega\left(f^j-\frac{u^j-2u^{j-1}+u^{j-2}}{\tau^2}\right)\left(\varphi^j-I^0_C\varphi^j\right) + \tau^2\sum_{e\in\zeta_{jh}}\int_E J^j_{e,\eta}\left(\varphi^j-I^0_C\varphi^j\right) - \int_\Omega\left(e^j-2e^{j-1}+e^{j-2}\right)\varphi^j. \tag{35}$$

From Lemmas 5.3 and 5.4 we get

$$\int_\Omega \nabla_h e^j \operatorname{curl}\chi^j = \sum_{E\in\zeta_{jh}}\int_E J^j_{E,t}\left(\chi^j - I_C\chi^j\right). \tag{36}$$

By using the Helmholtz decomposition of the error and identities (35)-(36), we obtain

$$\begin{aligned}\|e^j\| + \tau^2\int_\Omega\left|\nabla_h e^j\right|^2 &= \tau^2\int_\Omega \nabla_h e^j\nabla I^0_C\left(e^j-\varphi^j\right) + \left(-e^j+2e^{j-1}-e^{j-2}, e^j-\varphi^j-I^0_C\left(e^j-\varphi^j\right)\right)\\ &\quad + \tau^2\int_\Omega\left(f^j-\frac{u^j-2u^{j-1}+u^{j-2}}{\tau^2}\right)\left(\varphi^j-I^0_C\varphi^j\right) + \left(2e^{j-1}-e^{j-2}, e^j\right)\\ &\quad + \tau^2\sum_{e\in\zeta_{jh}}\int_e\left(J^j_{e,\eta}\left(\varphi^j-I^0_C\varphi^j\right) + J^j_{e,t}\left(\chi^j-I_C\chi^j\right)\right).\end{aligned} \tag{37}$$

The Cauchy-Schwarz inequality and estimate (19) give

$$\begin{aligned}\sum_{e\in\zeta_{jh}}\int_e J^j_{e,\eta}\left(\varphi^j-I_C\varphi^j\right) &\le \sum_{e\in\zeta_{jh}}\left\|J^j_{e,\eta}\right\|\left\|\varphi^j-I_C\varphi^j\right\|_e \lesssim \sum_{e\in\zeta_{jh}}\left\|J^j_{e,\eta}\right\| h_e^{\frac{1}{2}}\left\|\nabla\varphi^j\right\|_{\widetilde{w}_e}\\ &\lesssim \sum_{e\in\zeta_{jh}}\eta^j_T\left\|\nabla\varphi^j\right\|_{\widetilde{w}_e} \lesssim \eta^j\left\|\nabla\varphi^j\right\|.\end{aligned} \tag{38}$$

Similarly, using the Cauchy-Schwarz inequality and estimate (17), we get

$$\sum_{e\in\zeta_{jh}}\int_E J^j_{E,t}\left(\chi^j-I_C\chi^j\right) \lesssim \eta^j\left\|\nabla\chi^j\right\|. \tag{39}$$

By using the Helmholtz decomposition, Green's formula and identity (36), we find

$$\int_\Omega |\operatorname{curl} \chi^j|^2 = \int_\Omega (\nabla_h e^j - \nabla_h \varphi^j) \operatorname{curl} \chi^j = \int_\Omega \nabla_h e^j \operatorname{curl} \chi^j$$
$$= \sum_{e\in\zeta_{jh}} \int_e J^j_{e,t}(\chi^j - I_C\chi^j) \lesssim \eta^j \|\nabla\chi^j\|. \tag{40}$$

The Cauchy-Schwarz inequality and estimate (17) imply

$$\int_\Omega |\operatorname{curl} \chi^j|^2 \lesssim \eta^j \|\nabla\chi^j\|.$$

Knowing that

$$\|\operatorname{curl} \chi^j\| = \|\nabla\chi^j\|,$$

we get

$$\|\nabla\chi^j\| \lesssim \eta^j,$$

and consequently

$$\sum_{e\in\zeta_{jh}} \int_e J^j_{e,t}(\chi^j - I_C\chi^j) \lesssim (\eta^j)^2. \tag{41}$$

On the other hand, the Cauchy-Schwarz inequality and estimate (18) give

$$\sum_{T\in\Upsilon_{jh}} \int_T (f^j - f^j_h)(\varphi^j - I^0_C\varphi^j) \lesssim \sum_{T\in\Upsilon_{jh}} h_T \|f^j - f^j_h\| \|\nabla\varphi^j\|_{\widetilde{w}_T}$$
$$\lesssim \operatorname{osc}(f,_{\widetilde{w}_T})\|\nabla\varphi\|. \tag{42}$$

Similarly, we have

$$|(-e^j + 2e^{j-1} - e^{j-2}, \varphi^j - e^j - I^0_C(\varphi^j - e^j))| \lesssim h_T \|e^j - 2e^{j-1} + e^{j-2}\| \|\nabla(e^j - \varphi^j)\|.$$

Proceeding as in (40) we can prove that

$$\|\nabla(e^j - \varphi^j)\| \lesssim \eta^j,$$

which implies that

$$|(-e^j + 2e^{j-1} - e^{j-2}, \varphi^j - e^j - I^0_C(\varphi^j - e^j))| \lesssim h_T\eta^j \|e^j - 2e^{j-1} + e^{j-2}\|. \tag{43}$$

To estimate $\|\nabla\varphi^j\|$, we have

$$\|\nabla\varphi^j\| \le \|\nabla_h(\varphi^j - e^j)\| + \|\nabla_h e^j\| \le C(\eta^j)^2 + \|\nabla_h e^j\|^2. \tag{44}$$

For the residual element, using the Cauchy-Schwarz inequality and estimation (18), we get

$$\sum_{T\in\Upsilon_{jh}}\int_T\left(f_h^j-\frac{u^j-2u^{j-1}+u^{j-2}}{\tau^2}\right)\left(\varphi^j-I_C^0\varphi^j\right)\lesssim\sum_{T\in\Upsilon_{jh}}h_T\left\|f_h^j-\frac{u^j-2u^{j-1}+u^{j-2}}{\tau^2}\right\|\left\|\nabla\varphi^j\right\|_{\widetilde{w}_T}$$
$$\lesssim\eta^j\left\|\nabla\varphi^j\right\|. \tag{45}$$

Using the ϵ-inequality and replacing the previous estimates in (37), we find

$$\left\|e^j\right\|^2+\tau^2\int_\Omega\left|\nabla_h e^j\right|^2\leq\frac{1}{2}\left\|e^{j-1}\right\|^2+\frac{1}{2}\left\|e^{j-2}\right\|^2+C\left(\max\left\{h_j^2,\tau^2\right\}\left(\eta^j\right)^2+\tau^2\cdot\operatorname{osc}(f,_{\widetilde{w}_T})^2\right).$$

Summing from $j=2$ until n results in

$$\left\|e^n\right\|^2+\sum_{j=1}^n\tau^2\int_\Omega\left|\nabla_h e^j\right|^2\lesssim\left\|e^0\right\|^2+\left\|e^1\right\|^2+\sum_{j=1}^n\max\left\{h_j^2,\tau^2\right\}\left(\eta^j\right)^2$$
$$+\sum_{j=1}^n\tau^2\cdot\operatorname{osc}(f,_{\widetilde{w}_T})^2.$$

□

Theorem 5.2 (Lower bound of the error) [9, 10] *For each element $T\in\Upsilon_{jh}$, $j=2,\ldots,n$, the following estimate holds*:

$$\eta_T^j\lesssim h_T\left\|\frac{u_h^j-2u_h^{j-1}+u^{j-2}}{\tau^2}\right\|+\left\|\nabla_h e^j\right\|_{w_T}+h_T\left\|f^j-f_h^j\right\|_{w_T}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, University 8 Mai 1945 Guelma, P.O. Box 401, Guelma, 24000, Algeria. [2]Laboratory of Applied Mathematics and Modeling, University 8 Mai 1945 Guelma, P.O. Box 401, Guelma, 24000, Algeria. [3]Advanced Materials Laboratory, University of Badji Mokhtar Annaba, P.O. Box 12, Annaba, 23000, Algeria.

Acknowledgements
The authors are grateful for the referees' careful reading and comments on this paper that led to the improvement of the original manuscript. This work is supported by the MESRS of Algeria (CNEPRU Project B01520130004).

Received: 14 April 2015 Accepted: 19 July 2015 Published online: 07 August 2015

References
1. Cochez-Dhondt, S, Nicaise, S: Equilibrated error estimators for discontinuous Galerkin methods. Numer. Methods Partial Differ. Equ. **24**, 1236-1252 (2008)
2. Bernardi, C, Verfürth, R: *A posteriori* error analysis of the fully discretized time-dependent Stokes equations. M2AN Math. Model. Numer. Anal. **38**, 437-455 (2004)
3. Bernardi, C, Süli, E: Time and space adaptivity for the second-order wave equation. Math. Models Methods Appl. Sci. **15**(2), 199-225 (2005)
4. Guezane-Lakoud, A, Belakroum, D: Rothe's method for a telegraph equation with integral conditions. Nonlinear Anal. **70**, 3842-3853 (2009)
5. Guezane-Lakoud, A, Jasmati, MS, Chaoui, A: Rothe's method for an integrodifferential equation with integral conditions. Nonlinear Anal. **72**, 1522-1530 (2010)
6. Ciarlet, PG: The Finite Element Method for Elliptic Problems. Studies in Mathematics and Its Applications, vol. 4. North-Holland, Amsterdam (1978)
7. Clément, P: Approximation by finite element functions using local regularization. Rev. Fr. Autom. Inform. Rech. Opér., Anal. Numér. **9**(R-2), 77-84 (1975)

8. Girault, V, Raviart, PA: Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms. Springer Series in Computational Mathematics, vol. 5. Springer, Berlin (1986)
9. Dari, E, Duran, R, Padra, C, Vampa, V: *A posteriori* error estimators for nonconforming finite element methods. Modél. Math. Anal. Numér. **30**(4), 385-400 (1996)
10. Nicaise, S, Soualem, N: *A posteriori* error estimates for a nonconforming finite element discretization of the heat equation. M2AN Math. Model. Numer. Anal. **39**(2), 319-348 (2005)