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Trace operator and a nonlinear boundary value problem in a new space

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Abstract
We develop a new function space and discuss trace operator on the same genealogical spaces. We also prove that the nonlinear boundary value problem with Dirichlet condition: $-\Delta u = f(|u|)\operatorname{sgn} u$ in the given domain, $u = 0$ on the boundary, possesses only a trivial solution if f obeys the slope condition: $\alpha'(x) > \frac{2n}{n-2}\frac{\alpha(x)}{x}$, where α is the anti-derivative of f with $\alpha(0) = 0$.

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1 Introduction

We are interested in the nonlinear boundary value problem with Dirichlet condition:

$$\begin{cases} -\Delta u = f(|u|)\operatorname{sgn} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open subset of the n-dimensional Euclidean space $\mathbb{R}^n$. Questions of this kind occur in many problems of mathematical physics, in the theory of traveling waves, homogenization, stationary states, boundary layer theory, biology, flame propagation, probability theory, and so on. This problem has been one of the most important and most discussed topics in the theory of partial differential equations during the past several decades.

Many physicists and mathematicians have studied the simplest form

$$-\Delta u = |u|^p \operatorname{sgn} u = |u|^{p-1}u, \quad p > 1,$$

which is a special case of (1) with $f(|x|) = |x|^p$. A critical reason for studying this special case is that the function spaces that have been used to deal with these problems are just the Lebesgue spaces $L^p(\Omega)$ (especially for the existence theory). However, it is too good to be true in reality!

In this paper we build up a new function space which has been designed to handle solutions of the general nonlinear boundary value problem (1) without imposing too much assumption on the function f. As a matter of fact, in [1], one can find series of attempts to construct new function spaces which generalize classical Lebesgue spaces. We extend those ideas to obtain a better space. The motivation of these research comes from taking a close look at the L^p-norm: $\|f\|_{L^p} = (\int_X |f(x)|^p\,d\mu)^{1/p}$ of the classical Lebesgue spaces $L^p(X)$,

$1 \le p < \infty$. It can be rewritten as

$$\|f\|_{L^p} := \alpha^{-1}\left(\int_X \alpha\left(\left|f(x)\right|\right) d\mu\right), \qquad \alpha(x) := x^p.$$

Even though the positive real-variable function $\alpha(x) := x^p$ has very beautiful and convenient algebraic and geometric properties, it also has some practical limitations for the theory of differential equations, as pointed out above. The new space is devised to overcome these limitations without hurting the beauty of L^p-norm too much.

There are two different attempts to generalize the classical Lebesgue spaces - Orlicz spaces $L_A(X)$ and Lorentz spaces $L^{p,r}(X,\mu)$. The theory of Orlicz spaces has been well developed which is similar to our new spaces. The Orlicz space $L_A(X)$ requires the convexity of the N-function A for the triangle inequality of the norm, whereas the norm for the space $L_\alpha(X)$ does not require the convexity of the Hölder function and it has indeed inherited the beautiful and convenient properties from the classical Lebesgue norm.

Based on the new function spaces $L_\alpha(X)$, we present two main results. One of them is the trace theorem in the space $L_\alpha(X)$. The trace theorem is one of the basic requirements to deal with the boundary value problems. We state and prove it in Section 3. We also discuss the non-existence of non-trivial solutions for the problem (1) if the given function f is of fast growth. To be more precise, we prove that the boundary value problem (1) possesses only a trivial solution if f obeys the following slope condition:

$$\alpha'(x) > \frac{2n}{n-2}\frac{\alpha(x)}{x},$$

where α is the anti-derivative of f with $\alpha(0) = 0$ and $n > 2$ (see Section 4 for details).

Throughout this paper, Ω represents an open subset of $\mathbb{R}^n$ and $(X, \mathfrak{M}, \mu)$ is an abstract measure space (Section 2). Also, C denotes various real positive constants.

2 The space $L_\alpha(X)$

We introduce some terminologies to define the Lebesgue type function spaces $L_\alpha(X)$ which improve the original version introduced in [1]. In this section, $\bar{\mathbb{R}}_+ = \{x \in \mathbb{R} : x \ge 0\}$.

A *pre-Hölder function* $\alpha : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ is an *absolutely continuous bijective* function satisfying $\alpha(0) = 0$. If there exists a pre-Hölder function β satisfying

$$\alpha^{-1}(x)\beta^{-1}(x) = x \tag{2}$$

for all $x \in \bar{\mathbb{R}}_+$, then β is called the *conjugate* (*pre-Hölder*) *function* of α. In the relation (2), the notations α^{-1}, β^{-1} are meant to be the inverse functions of α, β, respectively. Examples of pre-Hölder pairs are $(\alpha(x), \beta(x)) = (x^p, x^q)$ for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $(\alpha(x), \beta(x)) = (x\bar{\beta}^{-1} \circ \bar{\alpha}(x), x\bar{\alpha}^{-1} \circ \bar{\beta}(x))$ for $\bar{\alpha}(x) = 2e^x - 2x - 2$, $\bar{\beta}(x) = 2(1+x)\log(1+x) - 2x$. In fact, for any Orlicz N-function A together with complementary N-function $\tilde{A}$, $(\lambda \circ A, \lambda \circ \tilde{A})$ is a pre-Hölder pair with $\lambda(x) = A^{-1}(x)\tilde{A}^{-1}(x)$.

Some basic identities for a pre-Hölder pair (α, β) are listed:

$$x = \beta\left(\frac{x}{\alpha^{-1}(x)}\right) \quad \text{or} \quad \alpha(x) = \beta\left(\frac{\alpha(x)}{x}\right), \tag{3}$$

$$x = \frac{\alpha(x)}{\beta^{-1}(\alpha(x))} \quad \text{or} \quad \frac{\alpha(x)}{x} = \beta^{-1} \circ \alpha(x), \tag{4}$$

$$\frac{\beta^{-1}(x)}{\alpha'(\alpha^{-1}(x))} + \frac{\alpha^{-1}(x)}{\beta'(\beta^{-1}(x))} = 1, \tag{5}$$

$$\frac{y}{\alpha'(x)} + \frac{x}{\beta'(y)} = 1 \quad \text{for } y := \frac{\alpha(x)}{x}, \tag{6}$$

$$\alpha'(x) = \frac{\alpha(x)}{x} + \frac{\alpha(x)}{\beta'(\frac{\alpha(x)}{x}) - x}. \tag{7}$$

In the following discussion, a function Φ represents the two-variable function on $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$ defined by

$$\Phi(x,y) := \alpha^{-1}(x)\beta^{-1}(y),$$

provided that a pre-Hölder pair (α, β) exists.

Definition 2.1 Let $\hbar > 0$. A pre-Hölder function α with the conjugate function β is said to be a Hölder function if for any positive constants $a, b > 0$, there exist constants θ_1, θ_2 (depending on a, b) such that

$$\theta_1 + \theta_2 \le \hbar$$

and that a *comparable condition*

$$\Phi(x,y) \le \theta_1 \frac{ab}{\alpha(a)} x + \theta_2 \frac{ab}{\beta(b)} y \tag{8}$$

holds for all $(x,y) \in \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$.

The following proposition and the proof may illustrate that the comparable condition (8) is not far-fetched.

Proposition 2.2 *Let α be a convex pre-Hölder function together with the convex conjugate function β. Suppose that for any $a, b \ge 0$, there are constants p_1, p_2, q_1, q_2 (depending on a, b) with $\frac{1}{p_1} + \frac{1}{p_2} \le \hbar$ and $\frac{1}{q_1} + \frac{1}{q_2} \ge 1$ satisfying the slope conditions*;

$$p_1 \frac{\alpha(a)}{a} \le \alpha'(a) \le q_1 \frac{\alpha(a)}{a}, \tag{9}$$

$$p_2 \frac{\beta(b)}{b} \le \beta'(b) \le q_2 \frac{\beta(b)}{b}. \tag{10}$$

Then α is a Hölder function. (*So is β.*)

Proof The equation of the tangent plane of the graph of Φ at $(\bar{a}, \bar{b})$ reads

$$\begin{aligned} z &= \Phi_x(\bar{a},\bar{b})(x - \bar{a}) + \Phi_y(\bar{a},\bar{b})(y - \bar{b}) + \Phi(\bar{a},\bar{b}) \\ &= \frac{\beta^{-1}(\bar{b})}{\alpha'(\alpha^{-1}(\bar{a}))}(x - \bar{a}) + \frac{\alpha^{-1}(\bar{a})}{\beta'(\beta^{-1}(\bar{b}))}(y - \bar{b}) + \alpha^{-1}(\bar{a})\beta^{-1}(\bar{b}) \\ &\equiv T_{\bar{a},\bar{b}}(x,y). \end{aligned}$$

Then for $\alpha^{-1}(\bar{a}) := a$ and $\beta^{-1}(\bar{b}) := b$, $T_{\bar{a},\bar{b}}(x,y)$ can be rewritten as

$$T_{\bar{a},\bar{b}}(x,y) = \frac{b}{\alpha'(a)}x + \frac{a}{\beta'(b)}y + ab - \frac{b\alpha(a)}{\alpha'(a)} - \frac{a\beta(b)}{\beta'(b)}. \tag{11}$$

From the slope conditions (9), (10) together with the observation that

$$\frac{b\alpha(a)}{\alpha'(a)} + \frac{a\beta(b)}{\beta'(b)} \geq \frac{1}{q_1}ab + \frac{1}{q_2}ab \geq ab,$$

we have

$$T_{\bar{a},\bar{b}}(x,y) \leq \frac{1}{p_1}\frac{ab}{\alpha(a)}x + \frac{1}{p_2}\frac{ab}{\beta(b)}y. \tag{12}$$

On the other hand, we observe that every point on the surface $z = \Phi(x,y)$ is an elliptic point since the Gaussian curvature of a point on the surface $z = \Phi(x,y)$ is positive from the convexity hypotheses on α and β (we refer to p.162 in [2]). Hence the tangent planes $z = T_{\bar{a},\bar{b}}(x,y)$ touch the graph at $(\bar{a},\bar{b})$ and nowhere lie below the graph $z = \Phi(x,y)$, that is, for any $\bar{a}$, $\bar{b}$,

$$\Phi(x,y) \leq T_{\bar{a},\bar{b}}(x,y). \tag{13}$$

In fact, since the restriction $z = T_{\bar{a},\bar{b}}(x,\bar{b})$ of the tangent plane $z = T_{\bar{a},\bar{b}}(x,y)$ is a tangent line to the graph $\Phi(x,\bar{b}) = b\alpha^{-1}(x)$ $(\beta^{-1}(\bar{b}) = b)$ in the x-z plane and α^{-1} is concave up on $\bar{\mathbb{R}}_+$, we get $\Phi(x,\bar{b}) \leq T_{\bar{a},\bar{b}}(x,\bar{b})$. Furthermore, since $(\bar{a},\bar{b})$ is an elliptic point, a local neighborhood of $(\bar{a},\bar{b})$ in the surface $z = \Phi(x,y)$ belongs to the same side of $z = T_{\bar{a},\bar{b}}(x,y)$ (p.158 in [2]). So on a local neighborhood of $(\bar{a},\bar{b})$, the graph $z = \Phi(x,y)$ lies below the tangent plane $z = T_{\bar{a},\bar{b}}(x,y)$. This holds for all $(\bar{a},\bar{b})$. Hence $\Phi(x,y)$ is concave up on $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$, which, in turn, illustrates (13). Combining (12) and (13), we conclude that

$$\Phi(x,y) \leq \theta_1\frac{ab}{\alpha(a)}x + \theta_2\frac{ab}{\beta(b)}y,$$

where we set $\theta_1 := \frac{1}{p_1}$ and $\theta_2 := \frac{1}{p_2}$. □

We now define the *Lebesgue-Orlicz type function spaces* $L_\alpha(X)$: for a Hölder function α,

$$L_\alpha(X) := \{f \mid f \text{ is a measurable function on } X \text{ satisfying } \|f\|_{L_\alpha} < \infty\},$$

where we set

$$\|f\|_{L_\alpha} := \alpha^{-1}\left(\int_X \alpha\left(\left|f(x)\right|\right) d\mu\right). \tag{14}$$

A Hölder type inequality and a Minkowski inequality on the new space $L_\alpha(X)$ are presented as follows:

Let α be a Hölder function and β be the corresponding Hölder conjugate function. Then for any $f \in L_\alpha(X)$ and $g \in L_\beta(X)$, we have

$$\left|\int_X f(x)g(x)\, d\mu\right| \leq \hbar\|f\|_{L_\alpha}\|g\|_{L_\beta}. \tag{15}$$

The name of Hölder functions originates from the Hölder inequality (15). So we briefly sketch the idea. Let $a := \|g\|_{L_\beta}$ $(\neq 0)$, $b := \|f\|_{L_\alpha}$ $(\neq 0)$, and then there exist θ_1, θ_2 such that $\theta_1 + \theta_2 \leq \hbar$ and

$$\begin{aligned}\left|f(x)g(x)\right| &= \alpha^{-1}\bigl(\alpha\bigl(\left|f(x)\right|\bigr)\bigr)\beta^{-1}\bigl(\beta\bigl(\left|g(x)\right|\bigr)\bigr)\\ &\leq \theta_1\frac{ab}{\alpha(b)}\alpha\bigl(\left|f(x)\right|\bigr) + \theta_2\frac{ab}{\beta(a)}\beta\bigl(\left|g(x)\right|\bigr).\end{aligned}$$

Integration of both sides yields

$$\begin{aligned}\int_X \left|f(x)g(x)\right| d\mu &\leq \theta_1\frac{ab}{\alpha(b)}\int_X \alpha\bigl(\left|f(x)\right|\bigr)\, d\mu + \theta_2\frac{ab}{\beta(a)}\int_X \beta\bigl(\left|g(x)\right|\bigr)\, d\mu\\ &\leq \hbar\|f\|_{L_\alpha}\|g\|_{L_\beta}.\end{aligned}$$

As an important application of a Hölder inequality, we have the Minkowski inequality on $L_\alpha(X)$. We omit the proof.

Remark 2.3 (Generalized Minkowski inequality) Let $(\Gamma, \overline{\mathfrak{M}}, \nu)$ be a σ-finite measure space. Suppose that α is a Hölder function and f is a nonnegative measurable function on $X \times \Gamma$ satisfying $f(\cdot, y) \in L_\alpha(X)$ for almost every $y \in \Gamma$. Then

$$\left\| \int_\Gamma f(\cdot, y)\, d\nu(y) \right\|_{L_\alpha(X)} \leq \hbar \int_\Gamma \left\|f(\cdot, y)\right\|_{L_\alpha(X)} d\nu(y).$$

In particular, for any $f_1, f_2 \in L_\alpha(X)$, we have

$$\|f_1 + f_2\|_{L_\alpha} \leq \hbar\bigl\{\|f_1\|_{L_\alpha} + \|f_2\|_{L_\alpha}\bigr\}.$$

We can also show that the metric space $L_\alpha(X)$ is complete with respect to the metric:

$$d(f,g) := \|f - g\|_{L_\alpha} \quad \text{for } f, g \in L_\alpha(X).$$

We now present some remarks on the dual space of $L_\alpha(X)$. To each $g \in L_\beta(X)$ is associated a bounded linear functional $\mathcal{F}_g$ on $L_\alpha(X)$ by

$$\mathcal{F}_g(f) := \int_X f(x)g(x)\, d\mu,$$

and the operator (inhomogeneous) norm of $\mathcal{F}_g$ is at most $\|g\|_{L_\beta}$:

$$\|\mathcal{F}_g\|_{L'_\alpha} := \sup\left\{ \frac{|\int_X fg\, d\mu|}{\|f\|_{L_\alpha}} : f \in L_\alpha(X), f \neq 0 \right\} \leq \hbar\|g\|_{L_\beta}. \tag{16}$$

For $0 \neq g \in L_\beta(X)$, if we put $f(x) := \frac{\beta(|g(x)|)\operatorname{sgn}(g(x))}{|g(x)|}$, we have $f \in L_\alpha(X)$ and

$$\|\mathcal{F}_g\|_{L'_\alpha} = \sup\left\{ \frac{|\int_X fg\, d\mu|}{\|f\|_{L_\alpha}} : f \in L_\alpha(X), f \neq 0 \right\} \geq \frac{|\int_X fg\, d\mu|}{\|f\|_{L_\alpha}} = \|g\|_{L_\beta}. \tag{17}$$

This implies that the mapping $g \mapsto \mathcal{F}_g$ is isomorphic from $L_\beta(X)$ into the space of continuous linear functionals $L_\alpha(X)'$. Furthermore, it can be shown that the linear transformation $\mathcal{F} : L_\beta(X) \to L_\alpha(X)'$ is *onto*.

Remark 2.4 (Dual space of $L_\alpha(X)$) Let β be the conjugate Hölder function of a Hölder function α. Then the dual space $L_\alpha(X)'$ is isomorphic to $L_\beta(X)$.

The proof is quite parallel to the classical Riesz representation theorem, so we omit the proof.

The two inequalities (16) and (17) explain the quasi-homogeneity of $\|\cdot\|_{L_\alpha}$. That is, we have the following.

Proposition 2.5 *For all* $k \geq 0$ *and* $f \in L_\alpha(X)$,

$$\frac{k}{\hbar}\|f\|_{L_\alpha} \leq \|kf\|_{L_\alpha} \leq k\hbar\|f\|_{L_\alpha}.$$

In particular, when $\hbar = 1$, *we have homogeneity*:

$$\|kf\|_{L_\alpha} = k\|f\|_{L_\alpha}.$$

The metric space $L_\alpha(X)$ and the classical Orlicz space $L_A(X)$ differ by the choice of the conjugate function. In fact, for the Orlicz space $L_A(X)$, the complementary N-function $\tilde{A}$ of A is designed to satisfy the relation

$$\tilde{A}' = (A')^{-1},$$

which implies, in turn,

$$c_1 x \leq A^{-1}(x)\tilde{A}^{-1}(x) \leq c_2 x$$

for some constants $c_1, c_2 > 0$. Also, the *Luxemburg norm*

$$\|u\|_A = \inf\left\{k > 0 : \int_\Omega A\left(\frac{|u(x)|}{k}\right) dx \leq 1\right\}$$

for the Orlicz space $L_A(X)$ requires the convexity of the N-function A for the triangle inequality of the norm. On the other hand, the (inhomogeneous) norm for the space $L_\alpha(X)$ does not require the convexity of the Hölder function and it has indeed inherited the beautiful and convenient properties from the classical Lebesgue norm.

3 Trace operator on Sobolev type space W^1_α

Let Ω be an open subset of $\mathbb{R}^n$. The Sobolev type space $W^1_\alpha(\Omega)$ is employed in

$$W^1_\alpha(\Omega) := \{u \in L_\alpha(\Omega) \mid \partial_{x_j} u \in L_\alpha(\Omega), j = 1, 2, \ldots, n\}$$

together with the norm

$$\|u\|_{W^1_\alpha} := \|u\|_{L_\alpha} + \sum_{j=1}^{n} \|\partial_{x_j} u\|_{L_\alpha},$$

where $\partial_{x_j} := \frac{\partial}{\partial x_j}$. Then it can be shown that the function space $W^1_\alpha(\Omega)$ is a separable complete metric space, and that $C^\infty(\Omega) \cap W^1_\alpha(\Omega)$ is dense in $W^1_\alpha(\Omega)$.

The completion of the space $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{W^1_\alpha}$ is denoted by $W^1_{\alpha,0}(\Omega)$, where $C^\infty_c(\Omega)$ is the space of smooth functions on Ω with compact support.

We introduce the trace operator on $W^1_\alpha(\Omega)$, which is important by itself and also useful in Section 4. We want to point out that the trace operator we present here is an improvement and a completion of the one briefly introduced in [1].

We say that a pre-Hölder function β satisfies a *slope condition* if there exists a positive constant $c > 1$ for which

$$\beta'(x) \geq c\frac{\beta(x)}{x} \tag{18}$$

holds for almost every $x > 0$. The slope condition (18), in fact, corresponds to the Δ_2-condition for Orlicz spaces.

We prove that the boundary trace on $C^\infty(\bar{\Omega}) \cap W^1_\alpha(\Omega)$ can be extended to the space $W^1_\alpha(\Omega)$ as follows.

Theorem 3.1 (Trace map on W^1_α) *Let* (α, β) *be a Hölder pair obeying the slope condition* (18) *and* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with smooth boundary. Then the trace operator* $\Im : W^1_\alpha(\Omega) \to L_\alpha(\partial\Omega)$ *is continuous and uniquely determined by* $\Im(u) = u|_{\partial\Omega}$ *for* $u \in C^\infty(\bar{\Omega}) \cap W^1_\alpha(\Omega)$.

Proof We first prove the theorem for the special case of flat boundary, and by using it, we take care of the general cases.

A special case - $\Omega = \mathbb{R}^n_+$. For the case $\Omega = \mathbb{R}^n_+ := \{(x', x_n) \mid x' \in \mathbb{R}^{n-1}, x_n > 0\}$ and for a smooth function $f \in C^\infty(\overline{\mathbb{R}^n_+}) \cap W^1_\alpha(\mathbb{R}^n_+)$, we observe that

$$\begin{aligned} \alpha\left(\left|f\left(x', 0\right)\right|\right) &= -\int_0^\infty \partial_{x_n}\alpha\left(\left|f\left(x', x_n\right)\right|\right) dx_n \\ &\leq \int_0^\infty \alpha'\left(\left|f\left(x', x_n\right)\right|\right)\left|\partial_{x_n} f\left(x', x_n\right)\right| dx_n \\ &\leq \hbar \left\|\partial_{x_n} f\left(x', \cdot\right)\right\|_{L_\alpha(0,\infty)} \left\|\alpha'\left(\left|f\left(x', \cdot\right)\right|\right)\right\|_{L_\beta(0,\infty)}. \end{aligned} \tag{19}$$

Owing to the identity (6), we have

$$\alpha'(t) = s + t\frac{\alpha'(t)}{\beta'(s)}, \quad s = \frac{\alpha(t)}{t}. \tag{20}$$

On the other hand, using the identity (3), we can notice that the slope condition (18) is equivalent to

$$\beta'\left(\frac{\alpha(t)}{t}\right) \geq ct. \tag{21}$$

Reflecting (21) to the identity (20), we have

$$\alpha'\left(\left|f(x)\right|\right) \leq \frac{c}{c-1}\frac{\alpha(|f(x)|)}{|f(x)|}, \quad x = \left(x', x_n\right).$$

Therefore we have

$$\begin{aligned}\beta^{-1}\left(\int_0^\infty \beta\left(\alpha'\left(\left|f\left(x',x_n\right)\right|\right)\right)dx_n\right) &\le \beta^{-1}\left(\int_0^\infty \beta\left(\frac{c}{c-1}\frac{\alpha(|f(x',x_n)|)}{|f(x',x_n)|}\right)dx_n\right)\\ &\le \hbar\left(\frac{c}{c-1}\right)\beta^{-1}\left(\int_0^\infty \beta\left(\frac{\alpha(|f(x',x_n)|)}{|f(x',x_n)|}\right)dx_n\right)\\ &= \hbar\left(\frac{c}{c-1}\right)\beta^{-1}\left(\int_0^\infty \alpha\left(\left|f\left(x',x_n\right)\right|\right)dx_n\right).\end{aligned}$$

Inserting this into the right side of (19), we obtain

$$\begin{aligned}\alpha\left(\left|f\left(x',0\right)\right|\right) &\le C\left\|\partial_{x_n}f\left(x',\cdot\right)\right\|_{L_\alpha(0,\infty)}\beta^{-1}\circ\alpha\left(\left\|f\left(x',\cdot\right)\right\|_{L_\alpha(0,\infty)}\right)\\ &\le C\left[\alpha\left(\left\|\partial_{x_n}f\left(x',\cdot\right)\right\|_{L_\alpha(0,\infty)}\right)+\alpha\left(\left\|f\left(x',\cdot\right)\right\|_{L_\alpha(0,\infty)}\right)\right]\end{aligned}$$

for some positive constants C (two C's may be different). The comparable condition (8) has been used in the second inequality. Taking integrations on both sides over $\mathbb{R}^{n-1}$, we obtain

$$\alpha\left(\|f\|_{L_\alpha(\mathbb{R}^{n-1})}\right) \le C\left[\alpha\left(\|\partial_{x_n}f\|_{L_\alpha(\mathbb{R}^n_+)}\right)+\alpha\left(\|f\|_{L_\alpha(\mathbb{R}^n_+)}\right)\right]. \tag{22}$$

This inequality says that the trace on $C^\infty(\overline{\mathbb{R}^n_+}) \cap W^1_\alpha(\mathbb{R}^n_+)$ can be uniquely extended to the space $W^1_\alpha(\mathbb{R}^n_+)$.

The general case - Ω being bounded open in $\mathbb{R}^n$. In this section we restrict our attention to the case of Ω being a bounded open subset. However, Ω can be more general, such as unbounded domains satisfying the uniform C^m-regularity condition (p.84 in [3]).

Assume that $\partial\Omega$ is an $n-1$ dimensional C^m-manifold. Letting

$$\begin{aligned}Q &:= \left\{y \in \mathbb{R}^n \mid |y_i| \le 1\right\},\\ Q_0 &:= \{y \in Q \mid y_n = 0\},\\ Q_+ &:= \{y \in Q \mid y_n > 0\},\end{aligned}$$

the last condition can be stated as follows. There is a finite collection of open bounded sets in $\mathbb{R}^n$; $\Omega_1, \Omega_2, \ldots, \Omega_N$ with $\bigcup\{\Omega_j \mid 1 \le i \le N\} \supset \partial\Omega$ and corresponding $\iota_j \in C^m(Q;\Omega_j)$ which are bijections satisfying Q, Q_+, and Q_0 mapping onto Ω_j, $\Omega_j \cap \Omega$, and $\Omega_j \cap \partial\Omega$, respectively, and each Jacobian $J(\iota_j)$ is positive. Each pair (ι_j, Ω_i) is a *coordinate patch*.

Let $\Omega_0 := \Omega$. We can construct $\wp_j \in C_0^\infty(\Omega_j)$, $0 \le j \le N$, with $\wp_j(x) \ge 0$, and

$$\sum_{j=0}^N \wp_j(x) = 1 \quad \text{for } x \in \bar{\Omega}.$$

Thus, $\{\wp_j \mid 1 \le j \le N\}$ is a *partition-of-unity* subordinate to the open cover $\{\Omega_j \mid 1 \le j \le N\}$ of $\partial\Omega$, and $\{\wp_j \mid 0 \le j \le N\}$ is a partition-of-unity subordinate to the open cover $\{\Omega_j \mid 0 \le j \le N\}$ of $\bar{\Omega}$.

If f is a function defined on $\partial\Omega$, then we have

$$\int_{\partial\Omega} f\,dS \equiv \sum_{j=1}^N \int_{\partial\Omega\cap\Omega_j} \wp_j f\,dS = \sum_{j=1}^n \int_{Q_0} (\wp_j f)\circ\iota_j\left(y',0\right)J_j\left(y'\right)dy', \tag{23}$$

where $s = (s_1, s_2, \dots, s_n) = \iota_j(y', 0)$ and $J_j(y')$ is the magnitude of the vector

$$\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ (s_1)_{y_1} & (s_2)_{y_1} & \cdots & (s_n)_{y_1} \\ (s_1)_{y_2} & (s_2)_{y_2} & \cdots & (s_n)_{y_2} \\ & & \ddots & \\ (s_1)_{y_{n-1}} & (s_2)_{y_{n-1}} & \cdots & (s_n)_{y_{n-1}} \end{pmatrix} \tag{24}$$

at $y_n = 0$. Here $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ represents the standard basis for $\mathbb{R}^n$. From the fact that

$$\begin{aligned}(24) &= \frac{\partial(s_2, \dots, s_n)}{\partial(y_1, \dots, y_{n-1})}\mathbf{e}_1 + \frac{\partial(s_1, s_3, \dots, s_n)}{\partial(y_1, \dots, y_{n-1})}\mathbf{e}_2 + \cdots + \frac{\partial(s_1, \dots, s_{n-1})}{\partial(y_1, \dots, y_{n-1})}\mathbf{e}_n \\ &= \left(\frac{\partial(s_2, \dots, s_n)}{\partial(y_1, \dots, y_{n-1})}, \dots, \frac{\partial(s_1, \dots, s_{n-1})}{\partial(y_1, \dots, y_{n-1})}\right),\end{aligned}$$

we notice that

$$J_j(y') = \left\{\sum_{k=1}^{N}\left(\frac{\partial(s_1, \dots, \hat{s}_k, \dots, s_n)}{\partial(y_1, \dots, y_{n-1})}\right)^2\bigg|_{y_n=0}\right\}^{1/2}.$$

Then by the smoothness property,

$$\left|J_j(y')\right| \le K, \quad 1 \le j \le N, y' \in Q_0,$$

since $m \ge 1$. Finally, we construct the trace on $\partial\Omega$ as indicated. First, we represent

$$\begin{aligned}&W^1_\alpha(\Omega) \to W^1_{\alpha,0}(\Omega) \times W^1_\alpha(Q_+)^N, \\ &u = \sum_{j=0}^{N} \wp_j u \mapsto \left(\wp_0 u, (\wp_1 u) \circ \iota_1, \dots, (\wp_N u) \circ \iota_N\right).\end{aligned}$$

We take the trace operator on each component except for the first component $\wp_0 u$, to say $\Im_j$, $1 \le j \le N$:

$$\begin{aligned}&W^1_{\alpha,0}(\Omega) \times W^1_\alpha(Q_+)^N \to L_\alpha(Q_0)^N, \\ &\left(\wp_0 u, (\wp_1 u) \circ \iota_1, \dots, (\wp_N u) \circ \iota_n\right) \mapsto \left(\Im_1(\wp_1 u \circ \iota_1), \dots, \Im_N(\wp_N u \circ \iota_N)\right).\end{aligned}$$

We note that $\Im_j(\wp_j u \circ \iota_j) = (\wp_j \Im_j(u)) \circ \iota_j$, $1 \le j \le N$. Finally, summing up all components, we obtain

$$\begin{aligned}&L_\alpha(Q_0)^N \to L_\alpha(\partial\Omega), \\ &\left(\left(\wp_1 \Im_1(u)\right) \circ \iota_1, \dots, \left(\wp_N \Im_N(u)\right) \circ \iota_N\right) \mapsto \Im(u) \equiv \sum_{j=1}^{N} \wp_j \Im_j u.\end{aligned}$$

The fact $\Im(u) \in L_\alpha(\partial\Omega)$ follows from the estimates

$$\begin{aligned}\int_{\partial\Omega} \alpha\big(\big|\Im(u)\big|\big)\,dS &\le \sum_{j=1}^{N} \int_{\partial\Omega\cap\Omega_j} \alpha\big(|\Im_j u|\big)\,dS \le K \sum_{j=1}^{N} \int_{Q_0} \alpha\big(\big|\Im_j(u)\circ\iota_j\big|\big)\,dy' \\ &\le KC_\alpha \sum_{j=1}^{N} \alpha\big(\|u\circ\iota_j\|_{W_\alpha^1(Q_+)}\big) \\ &\le KC_\alpha \sum_{j=1}^{N} \alpha\big(k_\iota \|u\|_{W_\alpha^1(\Omega\cap\Omega_j)}\big) \\ &\le KC_\alpha N\alpha\big(k_\iota \|u\|_{W_\alpha^1(\Omega)}\big),\end{aligned}$$

where K is the maximum of all Jacobians, C_α is the norm of the trace from half-space as in (22), and k_ι is the largest norm in W_α^1 under a change of variables $\iota_j : W_\alpha^1(Q_+) \to W_\alpha^1(\Omega\cap\Omega_j)$. Clearly, if $u \in C^\infty(\bar{\Omega}) \cap W_\alpha^1(\Omega)$ then $\Im(u) = u|_{\partial\Omega}$. The proof is now completed. □

4 Ill-posedness of boundary value problems

In this section we investigate a nonlinear elliptic partial differential equation, namely the nonlinear boundary value problem:

$$\begin{cases} -\Delta u = f(|u|)\operatorname{sgn} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

Here we assume that f is the derivative of a Hölder function α satisfying a slope condition; $f := \alpha'$. Hence there exists a positive constant $c > 1$ for which

$$\alpha'(x) \ge c\frac{\alpha(x)}{x} \tag{26}$$

holds for almost every $x > 0$. Our goal is to demonstrate that the slope condition (26) with large constant c implies that $u \equiv 0$ is the only strong solution of (25) under a certain geometric condition on Ω. We are going to find such a constant c explicitly. $\frac{2n}{n-2}$ turns out to be exact for $n > 2$.

In the following discussion, Ω is assumed to be a bounded open set with smooth boundary. We multiply the PDE (25) by u and integrate over Ω to find

$$\int_\Omega -\Delta u u\,dx = \int_\Omega f\big(|u|\big)u\,dx = \int_\Omega \alpha'\big(|u|\big)|u|\,dx.$$

The left side can be rewritten as

$$\begin{aligned}\int_\Omega -\Delta u u\,dx &= -\int_\Omega \operatorname{div}(\nabla u)u\,dx \\ &= -\int_{\partial\Omega} (\nabla u)u\cdot\vec{n}\,dS + \int_\Omega \nabla u\cdot\nabla u\,dx \\ &= \int_\Omega \nabla u\cdot\nabla u\,dx,\end{aligned}$$

since $u = 0$ on $\partial\Omega$. In the above, $\vec{n}$ represents the unit outward normal vector. Therefore we get

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega \alpha'\big(|u|\big)|u|\,dx \geq c\int_\Omega \alpha\big(|u|\big)\,dx. \tag{27}$$

On the other hand, multiplying the PDE (25) by $x \cdot \nabla u$ and integrating over Ω, we get

$$\int_\Omega (-\Delta u)(x\cdot\nabla u) = \int_\Omega \alpha'\big(|u|\big)\big(x\cdot\nabla|u|\big)\,dx. \tag{28}$$

We take a close look at the left side:

$$-\int_\Omega \operatorname{div}(\nabla u)(x\cdot\nabla u)\,dx = -\int_{\partial\Omega}(\nabla u)(x\cdot\nabla u)\cdot d\vec{S} + \int_\Omega (\nabla u)\cdot\nabla(x\cdot\nabla u)\,dx,$$

and we consider I and II separately which are defined as

$$I = -\int_{\partial\Omega}(\nabla u)(x\cdot\nabla u)\cdot d\vec{S} \quad \text{and} \quad II = \int_\Omega (\nabla u)\cdot\nabla(x\cdot\nabla u)\,dx.$$

We first take care of the second term II. For it, we observe that

$$(\nabla u)\cdot\nabla(x\cdot\nabla u) = (\nabla u)\cdot(\nabla u) + (\nabla u)\cdot\big(x\big(\nabla^2 u\big)\big), \tag{29}$$

and the second term on the right side of (29) becomes

$$(\nabla u)\cdot\big(x\big(\nabla^2 u\big)\big) = x\cdot\big(\nabla^2 u\big)\cdot(\nabla u) = x\cdot\frac{1}{2}\nabla\big((\nabla u)\cdot(\nabla u)\big) = x\cdot\frac{1}{2}\nabla\big(|\nabla u|^2\big).$$

This says that II can be rewritten as

$$II = \int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega x\cdot\nabla\big(|\nabla u|^2\big)\,dx. \tag{30}$$

The second term of (30) is, in turn,

$$\begin{aligned}\frac{1}{2}\int_\Omega x\cdot\nabla\big(|\nabla u|^2\big)\,dx &= \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\vec{n})\,dS - \frac{1}{2}\int_\Omega(\operatorname{div}\vec{x})|\nabla u|^2\,dx \\ &= \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\vec{n})\,dS - \frac{n}{2}\int_\Omega |\nabla u|^2\,dx.\end{aligned}$$

Therefore we obtain

$$II = \left(1-\frac{n}{2}\right)\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\vec{n})\,dS.$$

We now take care of I. We rewrite it as

$$I = -\int_{\partial\Omega}[\nabla u\cdot\vec{n}](x\cdot\nabla u)\,dS.$$

Since $u = 0$ on $\partial\Omega$, $\nabla u(x)$ is parallel to the normal vector $\vec{n}(x)$ at each point $x \in \partial\Omega$. Thus we get $\nabla u(x) = \pm|\nabla u|\vec{n}$. This identity makes it possible to rewrite I as

$$\begin{aligned} I &= -\int_{\partial\Omega} \bigl[\{\pm|\nabla u|\vec{n}\} \cdot \vec{n}\bigr]\bigl[x \cdot \{\pm|\nabla u|\vec{n}\}\bigr]\,dS \\ &= -\int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{n})\,dS, \end{aligned}$$

where we count on the fact that $|\vec{n}|^2 = 1$. In all, the left side of (28) becomes

$$\begin{aligned} &-\int_\Omega \operatorname{div}(\nabla u)(x \cdot \nabla u)\,dx \\ &\quad= -\int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{n})\,dS + \left(1 - \frac{n}{2}\right)\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{n})\,dS \\ &\quad= \left(1 - \frac{n}{2}\right)\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{n})\,dS. \end{aligned} \tag{31}$$

Now we consider the right side of (28):

$$\begin{aligned} \int_\Omega \alpha'\bigl(|u(x)|\bigr)\bigl(x \cdot \nabla|u(x)|\bigr)\,dx &= \int_\Omega \nabla\bigl(\alpha\bigl(|u(x)|\bigr)\bigr) \cdot x\,dx \\ &= \int_{\partial\Omega} \alpha\bigl(|u(x)|\bigr)(x \cdot \vec{n})\,dS - n\int_\Omega \alpha\bigl(|u(x)|\bigr)\,dx \\ &= -n\int_\Omega \alpha\bigl(|u(x)|\bigr)\,dx. \end{aligned} \tag{32}$$

The last equality follows from the fact that $\alpha(|u|) = 0$ on $\partial\Omega$. In view of (28) together with (31) and (32), we get

$$\begin{aligned} \left(\frac{n-2}{2}\right)\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{n})\,dS &= n\int_\Omega \alpha\bigl(|u(x)|\bigr)\,dx \\ &\le \frac{n}{c}\int_\Omega |\nabla u|^2\,dx, \end{aligned}$$

which can be written as

$$\left(\frac{n-2}{2} - \frac{n}{c}\right)\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{n})\,dS \le 0. \tag{33}$$

Hence if we suppose that Ω is a connected convex domain containing the origin, for example, an open ball $\Omega = \{x : |x| < r\}$, then $x \cdot \vec{n}(x) \ge 0$ for all $x \in \partial\Omega$. From this, we see that (33) implies

$$\left(\frac{n-2}{2} - \frac{n}{c}\right)\int_\Omega |\nabla u|^2\,dx \le 0,$$

which says, in turn, that the constant c should be less than or equal to $\frac{2n}{n-2}$. We summarize.

Theorem 4.1 *Let α be a Hölder function with the slope condition* (26) *and $f := \alpha'$ and let Ω be a connected and bounded open convex subset of $\mathbb{R}^n$ $(n > 2)$ containing the origin with* *smooth boundary. Then the nonlinear boundary value problem*:

$$\begin{cases} -\Delta u = f(|u|)\operatorname{sgn} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ (in the sense of a trace map)}, \end{cases}$$

has only a trivial solution in $W^1_\alpha(\Omega)$ *if* $c > \frac{2n}{n-2}$, *where* c *is the constant appearing in* (26).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HCP organized and wrote this paper. YJP contributed to all the steps of the proofs in this research. All authors read and approved the final manuscript.

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