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On (p,q)-analogue of two parametric Stancu-Beta operators

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Abstract

Our purpose is to introduce a two-parametric (p,q)-analogue of the Stancu-Beta operators. We study approximating properties of these operators using the Korovkin approximation theorem and also study a direct theorem. We also obtain the Voronovskaya-type estimate for these operators. Furthermore, we study the weighted approximation results and pointwise estimates for these operators.

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1 Introduction

The q-calculus has attracted attention of many researchers because of its applications in various fields such as numerical analysis, computer-aided geometric design, differential equations, and so on. In the field of approximation theory, the application of q-calculus has been the area of many recent researches.

Lupaş [1] presented the first q-analogue of the classical Bernstein operators in 1987. He studied the approximation and shape-preserving properties of these operators. Another q-companion of the classical Bernstein polynomials is due to Phillips [2]. Inspired by this, several authors produced generalizations of well-known positive linear operators based on q-integers and studied them extensively. For instance, the approximation properties of the Kantorovich-type q-Bernstein operators [3], q-BBH operators [4], q-analogue of generalized Bernstein-Schurer operators [5], weighted statistical approximation by Kantorovich-type q-Szász-Mirakjan operators [6], q-Szász-Durrmeyer operators [7], operators constructed by means of q-Lagrange polynomials and A-statistical approximation [8], statistical approximation properties of modified q-Stancu-Beta operators [9], and q-Bernstein-Schurer-Kantorovich operators [10].

The q-calculus has led to the discovery of the (p,q)-calculus. Recently, Mursaleen et al. have used the (p,q)-calculus in approximation theory. They have applied it to construct a (p,q)-analogue of the classical Bernstein operators [11], a (p,q)-analogue of the Bernstein-Stancu operators [12], and a (p,q)-analogue of the Bernstein-Schurer operators [13] and have studied their approximation properties. Most recently, (p,q)-analogues of some other operators have been studied in [14–18], and [19].

We now give some basic notions of the (p,q)-calculus.

The (p,q)-integer is defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},\quad n=0,1,2,\ldots,0<q<p\leq 1.$$

The (p,q)-companion of the binomial expansion is

$$(ax+by)^n_{p,q}=\sum_{k=0}^{n}\binom{n}{k}_{p,q}q^{\frac{k(k-1)}{2}}p^{\frac{(n-k)(n-k-1)}{2}}a^{n-k}b^kx^{n-k}y^k,$$

$$(x+y)^n_{p,q}=(x+y)(px+qy)\bigl(p^2x+q^2y\bigr)\cdots\bigl(p^{n-1}x+q^{n-1}y\bigr).$$

The (p,q)-analogues of the binomial coefficients are defined by

$$\binom{n}{k}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The (p,q)-analogues of definite integrals of a function f are defined by

$$\int_0^a f(x)\,d_{p,q}x=(q-p)a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right)\quad\text{when }\left|\frac{p}{q}\right|<1$$

and

$$\int_0^a f(x)\,d_{p,q}x=(p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}a\right)\quad\text{when }\left|\frac{q}{p}\right|<1.$$

For $m,n\in N$, the (p,q)-gamma and the (p,q)-beta functions are defined by

$$\Gamma_{p,q}(n)=\int_0^{\infty}p^{\frac{n(n-1)}{2}}E_{p,q}(-qx)\,d_{p,q}x,\qquad \Gamma_{p,q}(n+1)=[n]_{p,q}!$$

and

$$B_{p,q}(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\,d_{p,q}x, \tag{1.1}$$

respectively. These two are related by

$$B_{p,q}(m,n)=q^{\frac{2-m(m-1)}{2}}p^{\frac{-m(m-1)}{2}}\frac{\Gamma_{p,q}(n)\Gamma_{p,q}(m)}{\Gamma_{p,q}(m+n)}. \tag{1.2}$$

For $p=1$, all the concepts of the (p,q)-calculus reduce to those of q-calculus. The details on (p,q)-calculus can be found in [20–22].

Stancu [23] introduced the beta operators to approximate the Lebesgue-integrable functions on $[0,\infty)$ as follows:

$$L_n(f,x)=\frac{1}{B(nx,n+1)}\int_0^{\infty}\frac{t^{nx}}{(1+t)^{nx+n+1}}f(t)\,dt.$$

The q-companion of the Stancu-Beta operators was given by Aral and Gupta [24] as follows:

$$L_n(f,x) = \frac{K(A,[n]_q x)}{B([n]_q x,[n]_q+1)} \int_0^{\infty/A} \frac{u^{[n]_q x-1}}{(1+u)^{[n]_q x+[n]_q+1}} f\bigl(q^{[n]_q x}u\bigr)\, d_q u.$$

Let $0 < q < p < 1$. Mursaleen et al. [25] constructed the (p,q)-Stancu-Beta operators as follows:

$$L_n^{p,q}(f,x) = \frac{1}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \int_0^{\infty} \frac{u^{[n]_{p,q}x-1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}} f\bigl(p^{[n]_{p,q}x} q^{[n]_{p,q}x}u\bigr)\, d_{p,q} u. \tag{1.3}$$

They investigated the approximating properties and estimated the rate of convergence of these operators. Motivated by this work, we introduce the following sequence of operators:

$$\begin{aligned} S_{n,p,q}^{\alpha,\beta}(f;x) &= \frac{1}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \\ &\quad\times \int_0^{\infty} \frac{u^{[n]_{p,q}x-1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}} f\left(\frac{[n]_{p,q} p^{[n]_{p,q}x} q^{[n]_{p,q}x}u+\alpha}{[n]_{p,q}+\beta}\right) d_{p,q} u, \end{aligned} \tag{1.4}$$

where $0 \leq \alpha \leq \beta$. We call them two-parametric (p,q)-Stancu-Beta operators. For $\alpha = 0 = \beta$, the operators (1.4) coincide with the operators (1.3). So the latter is a generalization of the former.

2 Main results

We shall investigate approximation results for the operators (1.4). We calculate the moments of the operators $S_{n,p,q}^{\alpha,\beta}(f;x)$ in the following lemma.

Lemma 2.1 *Let* $S_{n,p,q}^{\alpha,\beta}(f;x)$ *be given by* (1.4). *Then we have the following equalities*:

(i) $S_{n,p,q}^{\alpha,\beta}(1;x) = 1$,

(ii) $S_{n,p,q}^{\alpha,\beta}(t;x) = \frac{[n]_{p,q}}{([n]_{p,q}+\beta)}x + \frac{\alpha}{([n]_{p,q}+\beta)}$,

(iii) $S_{n,p,q}^{\alpha,\beta}(t^2;x) = \frac{[n]_{p,q}^3}{pq([n]_{p,q}-1)([n]_{p,q}+\beta)^2}x^2 + \frac{[n]_{p,q}}{([n]_{p,q}+\beta)^2}(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)} + 2\alpha)x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}$.

Proof Using (1.1), (i) is immediate. Further,

$$\begin{aligned} S_{n,p,q}^{\alpha,\beta}(t;x) &= \frac{1}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \\ &\quad\times \int_0^{\infty} \frac{u^{[n]_{p,q}x-1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}} \left(\frac{[n]_{p,q} p^{[n]_{p,q}x} q^{[n]_{p,q}x}u+\alpha}{([n]_{p,q}+\beta)}\right) d_{p,q} u \\ &= \frac{[n]_{p,q}}{([n]_{p,q}+\beta)} \frac{p^{[n]_{p,q}x} q^{[n]_{p,q}x}}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \int_0^{\infty} \frac{u^{[n]_{p,q}x}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}}\, d_{p,q} u \\ &\quad+ \frac{\alpha}{([n]_{p,q}+\beta)} \frac{1}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \int_0^{\infty} \frac{u^{[n]_{p,q}x-1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}}\, d_{p,q} u \\ &= \frac{[n]_{p,q}}{([n]_{p,q}+\beta)} L_n^{p,q}(t;x) + \frac{\alpha}{([n]_{p,q}+\beta)} L_n^{p,q}(1;x) \\ &= \frac{[n]_{p,q}}{([n]_{p,q}+\beta)}x + \frac{\alpha}{([n]_{p,q}+\beta)}, \end{aligned}$$

and (ii) is proved;

$$\begin{aligned}
S_{n,p,q}^{\alpha,\beta}\bigl(t^2;x\bigr) &= \frac{1}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \\
&\quad\times \int_0^\infty \frac{u^{[n]_{p,q}x-1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}} \left(\frac{[n]_{p,q}p^{[n]_{p,q}x}q^{[n]_{p,q}x}u+\alpha}{([n]_{p,q}+\beta)}\right)^2 d_{p,q}u \\
&= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} \frac{p^{2[n]_{p,q}x}q^{2[n]_{p,q}x}}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \int_0^\infty \frac{u^{[n]_{p,q}x+1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}}\, d_{p,q}u \\
&\quad+ \frac{2\alpha}{[n]_{p,q}}\bigl([n]_{p,q}+\beta\bigr)^2 \frac{q^{[n]_{p,q}x}}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \int_0^\infty \frac{u^{[n]_{p,q}x}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}}\, d_{p,q}u \\
&\quad+ \frac{\alpha^2}{([n]_{p,q}+\beta)^2} \frac{1}{B_{p,q}([n]_{p,q}x,[n]_{p,q}+1)} \int_0^\infty \frac{u^{[n]_{p,q}x-1}}{(1+u)^{[n]_{p,q}x+[n]_{p,q}+1}}\, d_{p,q}u \\
&= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} L_n^{p,q}\bigl(t^2;x\bigr) + \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2} L_n^{p,q}(t;x) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} L_n^{p,q}(1;x) \\
&= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} \left(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)}x^2 + \frac{1}{pq([n]_{p,q}-1)}x\right) \\
&\quad+ \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2} + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} \\
&= \frac{[n]_{p,q}^3}{pq([n]_{p,q}-1)([n]_{p,q}+\beta)^2}x^2 + \frac{n}{([n]_{p,q}+\beta)^2}\left(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)} + 2\alpha\right)x \\
&\quad+ \frac{\alpha^2}{([n]_{p,q}+\beta)^2},
\end{aligned}$$

which proves (iii).

Hence, the lemma is proved. □

We readily obtain the following lemma.

Lemma 2.2 *Let $p,q \in (0,1)$. Then, for $x \in [0,\infty)$, we have:*
(i) $S_{n,p,q}^{\alpha,\beta}((t-x);x) = \frac{\alpha-\beta x}{([n]_{p,q}+\beta)}$,
(ii) $S_{n,p,q}^{\alpha,\beta}((t-x)^2;x) \leq (\frac{[n]_{p,q}}{pq([n]_{p,q}-1)} - \frac{([n]_{p,q}-\beta)}{([n]_{p,q}+\beta)})x^2 + \frac{1}{pq([n]_{p,q}-1)}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} \leq \frac{2(1+\beta)^2x^2+x+\alpha^2}{pq([n]_{p,q}-1)}$.

Proof We have

$$\begin{aligned}
S_{n,p,q}^{\alpha,\beta}\bigl((t-x);x\bigr) &= S_{n,p,q}^{\alpha,\beta}(t;x) - xS_{n,p,q}^{\alpha,\beta}(1;x) \\
&= \frac{[n]_{p,q}}{([n]_{p,q}+\beta)}x + \frac{\alpha}{([n]_{p,q}+\beta)} - x \\
&= \left(\frac{[n]_{p,q}}{([n]_{p,q}+\beta)} - 1\right)x + \frac{\alpha}{([n]_{p,q}+\beta)} \\
&= \frac{-\beta}{([n]_{p,q}+\beta)}x + \frac{\alpha}{([n]_{p,q}+\beta)} \\
&= \frac{\alpha-\beta x}{([n]_{p,q}+\beta)},
\end{aligned}$$

which proves (i). Now

$$\begin{aligned}
&S^{\alpha,\beta}_{n,p,q}\big((t-x)^2;x\big)\\
&\quad= S^{\alpha,\beta}_{n,p,q}\big(t^2;x\big) + x^2 S^{\alpha,\beta}_{n,p,q}(1;x) - 2xS^{\alpha,\beta}_{n,p,q}(t;x)\\
&\quad= \frac{[n]^3_{p,q}}{pq([n]_{p,q}-1)([n]_{p,q}+\beta)^2}x^2 + \frac{[n]_{p,q}}{([n]_{p,q}+\beta)^2}\left(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)} + 2\alpha\right)x\\
&\qquad+ \frac{\alpha^2}{([n]_{p,q}+\beta)^2} - 2x\left(\frac{[n]_{p,q}}{([n]_{p,q}+\beta)}x + \frac{\alpha}{([n]_{p,q}+\beta)}\right) + x^2\\
&\quad= \frac{[n]^3_{p,q}}{pq([n]_{p,q}-1)([n]_{p,q}+\beta)^2} - \frac{2[n]_{p,q}}{([n]_{p,q}+\beta)+1}x^2 + \frac{[n]^2_{p,q}}{pq([n]_{p,q}-1)([n]_{p,q}+\beta)^2}\\
&\qquad+ \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2} - \frac{2\alpha}{([n]_{p,q}+\beta)}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\\
&\quad\le \left(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)} - \frac{([n]_{p,q}-\beta)}{([n]_{p,q}+\beta)}\right)x^2 + \frac{1}{pq([n]_{p,q}-1)}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\\
&\quad= \frac{\{(p-q)[n]^3_{p,q} + ([n]_{p,q}+pq[n]_{p,q}-pq)\beta^2 + (2\beta+pq)[n]^2_{p,q}\}x^2 + ([n]_{p,q}+\beta)^2)x + pq([n]_{p,q}-1)\alpha^2}{pq([n]_{p,q}-1)([n]_{p,q}+\beta)^2}\\
&\quad= \frac{\{(p^n-q^n)[n]^2_{p,q} + ([n]_{p,q}+pq[n]_{p,q}-pq)\beta^2 + (2\beta+pq)[n]^2_{p,q}\}x^2 + ([n]_{p,q}+\beta)^2)x + pq([n]_{p,q}-1)\alpha^2}{pq([n]_{p,q}-1)([n]_{p,q}+\beta)^2}\\
&\quad\le \frac{2(\beta^2+\beta+1)x^2 + x + \alpha^2}{pq([n]_{p,q}-1)}\\
&\quad\le \frac{2(\beta+1)^2x^2 + x + \alpha^2}{pq([n]_{p,q}-1)}
\end{aligned}$$

which gives (ii). Hence, the lemma is proved. □

Next, we present a direct theorem for the operators $S^{\alpha,\beta}_{n,p,q}(f;x)$.

We denote By $C_B[0,\infty)$, the space of all real-valued continuous bounded functions f on the interval $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{0\le x<\infty}\big|f(x)\big|.$$

Let $\delta > 0$ and $W^2 = \{h : h', h'' \in C(I), I = [0,\infty)\}$, then the Peetre K-functional is defined by

$$K_2(f,\delta) = \inf_{h\in W^2}\big\{\|f-h\| + \delta\|h''\|\big\}.$$

The second-order modulus of continuity ω_2 of f is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<p<\delta^{\frac{1}{2}}}\sup_{x\in I}\big|f(x+2p) - 2f(x+p) + f(x)\big|.$$

By DeVore-Lorentz theorem (see [26], p.177, Theorem 2.4) there exists a constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta}). \tag{2.1}$$

Also, by $\omega(f,\delta)$ we denote the first-order modulus of continuity of $f \in C(I)$ defined as

$$\omega(f,\delta) = \sup_{0<p<\delta} \sup_{x\in I} \left| f(x+p) - f(x) \right|.$$

We shall use the notation $\nu^2(x) = x + x^2$.

Theorem 2.3 *Suppose that $f \in C_B[0,\infty)$ and $0 < p,q < 1$. Then for all $x \in [0,\infty)$ and $n \geq 2$, there exists a constant C such that*

$$\left| S_{n,p,q}^{\alpha,\beta}(f;x) - f(x) \right| \leq C\omega_2\left(f, \frac{\delta_n(x)}{\sqrt{pq([n]_{p,q}-1)}} \right) + \omega\left(f, \frac{\gamma_n(x)}{[n]_{p,q}+\beta} \right),$$

where

$$\delta_n^2(x) = \nu^2(x) + \frac{2pq\alpha^2}{([n]_{p,q}+\beta)}$$

and

$$\gamma_{n(x)}^2 = (\alpha - \beta x)^2 + [n]_{p,q}\big([n]_{p,q}+\beta\big)x^2 + \alpha\beta x.$$

Proof Let us define the auxiliary operators

$$S_{n,p,q}^{*\alpha,\beta}(f;x) = S_{n,p,q}^{\alpha,\beta}(f;x) - f\left(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta} \right) + f(x). \tag{2.2}$$

By the Lemma 2.1 it is readily seen that these operators are linear and

$$S_{n,p,q}^{*\alpha,\beta}\big((t-x);x\big) = 0. \tag{2.3}$$

Suppose that $g \in W^2$. By the Taylor expansion we can write

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du, \quad t \in [0,\infty).$$

Operating by $S_{n,p,q}^{*\alpha,\beta}(.;x)$ on both sides of the above and using (2.3), we obtain:

$$S_{n,p,q}^{*\alpha,\beta}(g;x) = g(x) + S_{n,p,q}^{*\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du;x \right),$$

$$S_{n,p,q}^{*\alpha,\beta}(g;x) - g(x) = S_{n,p,q}^{*\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du;x \right),$$

$$\left| S_{n,p,q}^{*\alpha,\beta}(g;x) - g(x) \right| = \left| S_{n,p,q}^{*\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du;x \right) \right|.$$

Using (2.2) in the right-hand side, we get

$$\left| S_{n,p,q}^{*\alpha,\beta}(g;x) - g(x) \right| = \Bigg| S_{n,p,q}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du;x \right) - \int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta} - u \right) g''(u)\,du \Bigg|.$$

So we obtain

$$\begin{aligned}&\left|S^{*\alpha,\beta}_{n,p,q}(g;x)-g(x)\right|\\&\quad\leq\left|S^{\alpha,\beta}_{n,p,q}\left(\int_x^t(t-u)g''(u)\,du;x\right)\right|+\left|\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}}\left(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\right)g''(u)\,du\right|\\&\quad\leq S^{\alpha,\beta}_{n,p,q}\left(\left|\int_x^t(t-u)g''(u)\,du\right|;x\right)+\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}}\left|\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\right|\left|g''(u)\right|du.\end{aligned}$$

Using the linearity of the integral operator and the operator $S^{\alpha,\beta}_{n,p,q}(\cdot;x)$ in the second and first parts of right-hand side, respectively, and using the fact that for all $x\in[0,\infty)$,

$$\left|g(x)\right|\leq\|g\|,$$

we obtain

$$\left|S^{*\alpha,\beta}_{n,p,q}(g;x)-g(x)\right|\leq\left\|g''\right\|S^{\alpha,\beta}_{n,p,q}\left((t-x)^2;x\right)+\left\|g''\right\|\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}}\left|\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\right|du. \tag{2.4}$$

In the first part, solving the integral $\int_x^t|t-u|\,du$ and using the linearity of the operators $S^{\alpha,\beta}_{n,p,q}(\cdot;x)$, we readily see that

$$S^{\alpha,\beta}_{n,p,q}\left(\int_x^t|t-u|\,du\right)\leq S^{\alpha,\beta}_{n,p,q}\left((t-x)^2;x\right),$$

and after some calculations, for the second part of (2.4), we get

$$\begin{aligned}&\int_x^{\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}}\left|\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}-u\right|du\\&\quad\leq\frac{([n]_{p,q}x+\alpha)^2-x([n]_{p,q}x+\alpha)([n]_{p,q}+\beta)+x^2([n]_{p,q}+\beta)^2}{([n]_{p,q}+\beta)^2}\\&\quad=\frac{(\alpha-\beta x)^2+[n]_{p,q}x^2([n]_{p,q}+\beta)+\alpha\beta x}{([n]_{p,q}+\beta)^2}\\&\quad=\left(\frac{\alpha-\beta x}{[n]_{p,q}+\beta}\right)^2+\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x^2+\frac{\alpha\beta}{([n]_{p,q}+\beta)^2}x.\end{aligned}$$

So by (2.4), we obtain

$$\begin{aligned}&\left|S^{*\alpha,\beta}_{n,p,q}(g;x)-g(x)\right|\\&\quad\leq\left\|g''\right\|\left(S^{\alpha,\beta}_{n,p,q}\left((t-x)^2;x\right)+\left(\frac{\alpha-\beta x}{[n]_{p,q}+\beta}\right)^2+\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x^2+\frac{\alpha\beta}{([n]_{p,q}+\beta)^2}x\right).\end{aligned} \tag{2.5}$$

Using Lemma 2.2(ii), we obtain

$$\begin{aligned}&S^{\alpha,\beta}_{n,p,q}\left((t-x)^2;x\right)+\left(\frac{\alpha-\beta x}{[n]_{p,q}+\beta}\right)^2+\frac{[n]_{p,q}}{([n]_{p,q}+\beta)}x^2+\frac{\alpha\beta}{([n]_{p,q}+\beta)^2}x\\&\quad\leq\left(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)}-\frac{([n]_{p,q}-\beta)}{([n]_{p,q}+\beta)}\right)x^2+\frac{1}{pq([n]_{p,q}-1)}x+\frac{\alpha^2}{([n]_{p,q}+\beta)^2}\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\alpha-\beta x}{[n]_{p,q}+\beta}\right)^2+\frac{[n]_{p,q}}{([n]_{p,q}+\beta)}x^2+\frac{\alpha\beta}{([n]_{p,q}+\beta)^2}x\\
&\leq \frac{(p-q)[n]_{p,q}^3}{pq([n]_{p,q}+\beta)^2([n]_{p,q}-1)}x^2+\frac{[n]_{p,q}^2+4pq(1-[n]_{p,q})\alpha\beta}{pq([n]_{p,q}+\beta)^2([n]_{p,q}-1)}x+\frac{2\alpha^2}{([n]_{p,q}+\beta)^2}\\
&\leq \frac{(p-q)[n]_{p,q}^3x^2+[n]_{p,q}^2x+2pq([n]_{p,q}-1)\alpha^2}{pq([n]_{p,q}+\beta)^2([n]_{p,q}-1)}\\
&= \frac{(p^n-q^n)[n]_{p,q}^2x^2+[n]_{p,q}^2x+2pq([n]_{p,q}-1)\alpha^2}{pq([n]_{p,q}+\beta)^2([n]_{p,q}-1)}\\
&\leq \frac{[n]_{p,q}^2x^2+[n]_{p,q}^2x+2pq[n]_{p,q}\alpha^2}{pq([n]_{p,q}+\beta)^2([n]_{p,q}-1)}\\
&\leq \frac{[n]_{p,q}(1+x)x+2pq\alpha^2}{pq([n]_{p,q}+\beta)^2([n]_{p,q}-1)}\\
&\leq \frac{1}{pq([n]_{p,q}-1)}\left(\nu^2(x)+\frac{2pq\alpha^2}{([n]_{p,q}+\beta)}\right)\\
&= \frac{\delta_n^2(x)}{pq([n]_{p,q}-1)},
\end{aligned}
$$

where

$$\delta_n^2(x)=\nu^2(x)+\frac{2pq\alpha^2}{([n]_{p,q}+\beta)}.$$

Therefore, by (2.5) we get

$$\left|S_{n,p,q}^{*\alpha,\beta}(g;x)-g(x)\right|\leq \frac{\delta_n^2(x)}{pq([n]_{p,q}-1)}\left\|g''\right\|. \tag{2.6}$$

On the other hand, by (2.2) we have

$$\left|S_{n,p,q}^{*\alpha,\beta}(f;x)\right|\leq \left|S_{n,p,q}^{\alpha,\beta}(f;x)\right|+2\|f\|\leq 3\|f\|. \tag{2.7}$$

By (2.2), (2.6), and (2.7), we obtain:

$$
\begin{aligned}
\left|S_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\right| &\leq \left|S_{n,p,q}^{*\alpha,\beta}(f-g;x)-(f-g)(x)\right|+\left|S_{n,p,q}^{*\alpha,\beta}(g;x)-g(x)\right|\\
&\quad+\left|f\left(\frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|\\
&\leq 4\|f-g\|+\frac{\delta_n^2(x)}{pq([n]_{p,q}-1)}\left\|g''\right\|\\
&\quad+\omega\left(f,\frac{\sqrt{(\alpha-\beta x)^2+[n]_{p,q}([n]_{p,q}+\beta)x^2+\alpha\beta x}}{[n]_{p,q}+\beta}\right)\\
&= 4\|f-g\|+\frac{\delta_n^2(x)}{pq([n]_{p,q}-1)}\left\|g''\right\|+\omega\left(f,\frac{\gamma_n(x)}{[n]_{p,q}+\beta}\right),
\end{aligned}
\tag{2.8}
$$

where

$$\gamma_{n(x)}^2=(\alpha-\beta x)^2+[n]_{p,q}\big([n]_{p,q}+\beta\big)x^2+\alpha\beta x.$$

Taking the infimum over all $g \in W^2$ on the right-hand side of (2.8), we obtain

$$\left|S_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\right| \le CK_2\left(f,\frac{\delta_n^2(x)}{pq([n]_{p,q}-1)}\right)+\omega\left(f,\frac{(\gamma_n)x}{[n]_{p,q}+\beta}\right).$$

Using relation (2.1), for $p,q \in (0,1)$, we get

$$\left|S_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\right| \le C\omega_2\left(f,\frac{\delta_n(x)}{\sqrt{pq([n]_{p,q}-1)}}\right)+\omega\left(f,\frac{\gamma_n(x)}{[n]_{p,q}+\beta}\right),$$

and this completes the proof. □

3 Rate of approximation

Let $B_{x^2}[0,\infty)$ denote the set of all functions f such that $f(x) \le M_f(1+x^2)$, where M_f is a constant depending on f. By $C_{x^2}[0,\infty)$ we denote the subspace of all continuous functions in the space $B_{x^2}[0,\infty)$. Also, we denote by $C_{x^2}^*[0,\infty)$, the subspace of all functions $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite with

$$\|f\| = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}.$$

For $a > 0$, the modulus of continuity of f over $[0,a]$ is defined by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta}\ \sup_{0\le x,t\le a}\left|f(t)-f(x)\right|.$$

We have the following proposition.

Proposition 3.1

(i) *For $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$, $a > 0$, approaches to zero.*

(ii) *For every $\delta > 0$, we have*

$$\left|f(y)-f(x)\right| \le \left(1+\frac{|y-x|}{\delta}\right)\omega_a(f,\delta)$$

and

$$\left|f(y)-f(x)\right| \le \left(1+\frac{(y-x)^2}{\delta^2}\right)\omega_a(f,\delta).$$

In the following theorem, we estimate the rate of convergence of the operators $S_{n,p,q}^{\alpha,\beta}(f;x)$.

Theorem 3.2 *Let $f \in C_{x^2}[0,\infty)$, $p,q \in (0,1)$, and let $\omega_{a+1}(f,\delta)$ be the modulus of continuity on the interval $[0,1+a] \subseteq [0,\infty)$, $a > 0$. Then, for $n \ge 2$, we have*

$$\left\|S_{n,p,q}^{\alpha,\beta}(f)-f\right\|_{C[0,a]} \le \frac{4M_f(1+a^2)(2(1+\beta)^2a^2+a+\alpha^2)}{pq([n]_{p,q}-1)} + 2\omega_{1+a}\left(f,\left(\frac{2(1+\beta)^2a^2+a+\alpha^2}{pq([n]_{p,q}-1)}\right)^{\frac{1}{2}}\right).$$

Proof Let $x \in [0,a]$ and $t > a+1$. Since $1+x<t$, we have

$$\begin{aligned} |f(t)-f(x)| &\le M_f(x^2+t^2+2) \le M_f(2+3x^2+2(t-x)^2) \\ &\le M_f(4+3x^2)(t-x)^2 \le 4M_f(1+a^2)(t-x)^2. \end{aligned} \tag{3.1}$$

For $\delta > 0$, $x \in [0,a]$, $t-1 \le a$, by Proposition 3.1 we obtain

$$|f(t)-f(x)| \le \omega_{1+a}(f,|t-x|) \le \omega_{1+a}(f,\delta)\left(1+\frac{1}{\delta}|t-x|\right). \tag{3.2}$$

By (3.1) and (3.2), for $x \in [0,a]$ and nonnegative t, we can write

$$|f(t)-f(x)| \le 4M_f(1+a^2)(t-x)^2\omega_{1+a}(f,\delta)\left(1+\frac{1}{\delta}|t-x|\right). \tag{3.3}$$

Therefore,

$$\begin{aligned} &\left|S_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\right| \\ &\quad\le S_{n,p,q}^{\alpha,\beta}(|f(t)-f(x)|;x) \\ &\quad\le 4M_f(1+a^2)S_{n,p,q}^{\alpha,\beta}((t-x)^2;x)+\omega_{1+a}(f,\delta)\left(1+\frac{1}{\delta}\left(S_{n,p,q}^{\alpha,\beta}((t-x)^2;x)\right)^{\frac{1}{2}}\right). \end{aligned}$$

Hence, using the Lemma 2.2(ii) and the Schwarz inequality, for every $p,q \in (0,1)$ and $x \in [0,a]$, we obtain

$$\begin{aligned} \left|S_{n,p,q}^{\alpha,\beta}(f;x)-f(x)\right| &\le 4M_f(1+a^2)\left(\frac{2(1+\beta)^2x^2+x+\alpha^2}{pq([n]_{p,q}-1)}\right) \\ &\quad+\omega_{1+a}(f,\delta)\left(1+\frac{1}{\delta}\left(\frac{2(1+\beta)^2a^2+a+\alpha^2}{pq([n]_{p,q}-1)}\right)^{\frac{1}{2}}\right) \\ &\le \frac{4M_f(1+a^2)(2(1+\beta)^2a^2+a+\alpha^2)}{pq([n]_{p,q}-1)} \\ &\quad+\omega_{1+a}\left(1+\frac{1}{\delta}\left(\frac{2(1+\beta)^2a^2+a+\alpha^2}{pq([n]_{p,q}-1)}\right)^{\frac{1}{2}}\right). \end{aligned}$$

By choosing $\delta^2 = \frac{2(1+\beta)^2a^2+a+\alpha^2}{pq([n]_{p,q}-1)}$ we get the required result. □

4 Weighted approximation

This section is devoted to the study of weighted approximation theorems for the operators (2.2).

Theorem 4.1 *Suppose that $p=p_n$ and $q=q_n$ are two sequences satisfying $0<p_n,q_n<1$ and suppose that $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$. Then, for each $f \in C_{x^2}^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|S_{n,p_n,q_n}^{\alpha,\beta}(f)-f\right\|_{x^2}=0.$$

Proof By the theorem in [27] it suffices to prove that

$$\lim_{n\to\infty}\left\| S^{\alpha,\beta}_{n,p_n,q_n}\left(t^i\right) - x^i \right\|_{x^2} = 0 \quad \text{for } i = 0,1,2. \tag{4.1}$$

By Lemma 2.1(i)-(ii), the conditions of (4.1) are easily verified for $i = 0$ and 1. For $i = 2$, we can write

$$\begin{aligned}
&\left\| S^{\alpha,\beta}_{n,p_n,q_n}\left(t^2\right) - x^2 \right\|_{x^2} \\
&\quad = \sup_{x\in[0,\infty)} \frac{|S^{\alpha,\beta}_{n,p_n,q_n}(t^2) - x^2|}{1+x^2} \\
&\quad \le \left(\frac{[n]^3_{p_n,q_n}}{p_nq_n([n]_{p_n,q_n}-1)([n]_{p_n,q_n}+\beta)^2} - 1 \right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
&\qquad + \frac{[n]^2_{p_n,q_n} + 2p_nq_n[n]_{p_n,q_n}([n]_{p_n,q_n}-1)\alpha}{p_nq_n([n]_{p_n,q_n}-1)([n]_{p_n,q_n}+\beta)^2} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{\alpha^2}{([n]_{p_n,q_n}+\beta)^2} \\
&\quad \le \frac{(p_n^n - q_n^n)[n]^2_{p_n,q_n} - p_nq_n(2\beta-1)[n]^2_{p_n,q_n} - q_n\beta(\beta-1)[n]_{p_n,q_n} + q_n\beta^2}{p_nq_n([n]_{p_n,q_n}-1)([n]_{p_n,q_n}+\beta)^2} \\
&\qquad + \left(\frac{[n]^2_{p_n,q_n} + 2p_nq_n[n]_{p_n,q_n}([n]_{p_n,q_n}-1)\alpha}{p_nq_n([n]_{p_n,q_n}-1)([n]_{p_n,q_n}+\beta)^2} \right) + \frac{\alpha^2}{([n]_{p_n,q_n}+\beta)^2},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\left\| S^{\alpha,\beta}_{n,p_n,q_n}\left(t^2,x\right) - x^2 \right\|_{x^2} = 0.$$

This completes the proof of the theorem. □

Theorem 4.2 *Let $p = (p_n)$ and $q = (q_n)$ be two sequences such that $0 < p_n, q_n < 1$, and let $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$. Then, for each $f \in C_{x^2}[0,\infty)$ and all $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|S^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha^2}} = 0.$$

Proof For $x_0 > 0$ fixed, we have:

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{|S^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha^2}} &= \sup_{x\le x_0} \frac{|S^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha^2}} + \sup_{x\ge x_0} \frac{|S^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha^2}} \\
&\le \left\| S^{\alpha,\beta}_{n,p_n,q_n}(f) - f \right\|_{C[0,a]} + \|f\|_{x^2} \sup_{x\ge x_0} \frac{|S^{\alpha,\beta}_{n,p_n,q_n}(1+t^2;x)|}{(1+x^2)^{1+\alpha^2}} \\
&\quad + \sup_{x\ge x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha^2}}.
\end{aligned}$$

The first term of this inequality goes to zero by Theorem 3.2. Also, for any fixed $x_0 > 0$, it is readily seen from Lemma 2.1 that

$$\sup_{x\ge x_0} \frac{|S^{\alpha,\beta}_{n,p_n,q_n}(1+t^2;x)|}{(1+x^2)^{1+\alpha^2}}$$

approaches zero as $n \to \infty$. If we choose $x_0 > 0$ large enough so that the last part of the last inequality is arbitrarily small, then our theorem is proved. □

5 Voronovskaya-type theorem

This section presents the Voronovskaya-type theorem for the operators $S^{\alpha,\beta}_{n,p,q}(f;x)$. We need the following lemma.

Lemma 5.1 *Suppose that $p_n, q_n \in (0,1)$ are such that $p_n^n \to a, q_n^n \to b$ $(0 \le a, b < 1)$ as $n \to \infty$. Then, for every $x \in [0,\infty)$, simple computations yield*

$$\lim_{n\to\infty} [n]_{p_n,q_n} S^{\alpha,\beta}_{n,p_n,q_n}\bigl((t-x);x\bigr) = \alpha - \beta x,$$
$$\lim_{n\to\infty} [n]_{p_n,q_n} S^{\alpha,\beta}_{n,p_n,q_n}\bigl((t-x)^2;x\bigr) = (1-a)(1-b)x^2 + x.$$

Theorem 5.2 *Assume that $p_n, q_n \in (0,1)$ are such that $p_n^n \to a, q_n^n \to b$ $(0 \le a, b < 1)$ as $n \to \infty$. Then, for $f \in C^*_{x^2}[0,\infty)$ such that $f', f''^{*}_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty} [n]_{p_n,q_n} \bigl(S^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x)\bigr) = (\alpha - \beta x) f'(x) + \frac{(1-a)(1-b)x^2 + x}{2} f''(x)$$

uniformly on $[0,A]$ for any $A > 0$.

Proof Let $f, f', f'' \in C^*_{x^2}[0,\infty)$ and $x \in [0,\infty)$. By the Taylor formula we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + r(t;x)(t-x)^2, \tag{5.1}$$

where $r(t;x)$ is the remainder term, $r(\cdot;x) \in C^*_{x^2}[0,\infty)$, and $\lim_{t\to x} r(t;x) = 0$. Operating by $S^{\alpha,\beta}_{n,p_n,q_n}$ on both sides of (5.1), we get

$$\begin{aligned}
&[n]_{p_n,q_n} \bigl(S^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x)\bigr) \\
&\quad = [n]_{p_n,q_n} S^{\alpha,\beta}_{n,p_n,q_n}\bigl((t-x);x\bigr) f'(x) + \frac{1}{2}[n]_{p_n,q_n} S^{\alpha,\beta}_{n,p_n,q_n}\bigl((t-x)^2;x\bigr) f''(x) \\
&\qquad + [n]_{p_n,q_n} S^{\alpha,\beta}_{n,p_n,q_n}\bigl(r(\cdot;x)(\cdot - x)^2;x\bigr).
\end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$S^{\alpha,\beta}_{n,p_n,q_n}\bigl(r(\cdot;x)(\cdot - x)^2;x\bigr) \le \sqrt{S^{\alpha,\beta}_{n,p_n,q_n}(r^2(\cdot;x);x}\sqrt{S^{\alpha,\beta}_{n,p_n,q_n}(r\bigl((\cdot - x)^4;x\bigr)}. \tag{5.2}$$

Note that $r^2(x;x) = 0$ and $r^2(\cdot;x) \in C^*_{x^2}[0,\infty)$. Therefore, it follows that

$$\lim_{n\to\infty} S^{\alpha,\beta}_{n,p_n,q_n}\bigl(r^2(\cdot;x);x\bigr) = r^2(x;x) = 0 \tag{5.3}$$

uniformly over $[0,A]$.

By Lemma 5.1 and equations (5.2) and (5.3), we obtain

$$\lim_{n\to\infty} [n]_{p_n,q_n} S^{\alpha,\beta}_{n,p_n,q_n}\bigl(r(\cdot;x)(\cdot - x)^2;x\bigr) = 0.$$

Thus, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}\big(S^{\alpha,\beta}_{n,p_n,q_n}(f;x)-f(x)\big)\\
&\quad=\lim_{n\to\infty}\bigg([n]_{p_n,q_n}S^{\alpha,\beta}_{n,p_n,q_n}\big((t-x);x\big)f'(x)+\frac{1}{2}[n]_{p_n,q_n}S^{\alpha,\beta}_{n,p_n,q_n}\big((t-x)^2;x\big)f''(x)\\
&\qquad+[n]_{p_n,q_n}S^{\alpha,\beta}_{n,p_n,q_n}\big(r(\cdot;x)(\cdot-x)^2;x\big)\bigg)\\
&\quad=(\alpha-\beta x)f'(x)+\frac{(1-a)(1-b)x^2+x}{2}f''(x).
\end{aligned}$$

□

6 Pointwise estimates

In this section, we study pointwise estimates of rate of convergence of the operators $S^{\alpha,\beta}_{n,p,q}(f;x)$.

Let $0<\nu\leq$ and $E\subset[0,\infty)$. We say that a function $f\in C[0,\infty)$ belongs to $Lip(\nu)$ if

$$\big|f(t)-f(x)\big|\leq M_f|t-x|^{\nu},\quad t\in[0,\infty),x\in E, \tag{6.1}$$

where M_f is a constant depending on α and f only.

We have the following theorem.

Theorem 6.1 *Let $\nu\in(0,1]$, $f\in Lip(\nu)$, and $E\subset[0,\infty)$. Then, for $x\in[0,\infty)$,*

$$\begin{aligned}
&\big\|S^{\alpha,\beta}_{n,p,q}(f;x)-f(x)\big\|\\
&\quad\leq M_f\Bigg\{\bigg(\bigg(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)}-\frac{([n]_{p,q}-\beta)}{([n]_{p,q}+\beta)}\bigg)x^2+\frac{1}{pq([n]_{p,q}-1)}x+\frac{\alpha^2}{([n]_{p,q}+\beta)^2}\bigg)^{\frac{\nu}{2}}\\
&\qquad+2\big(d(x,E)\big)^{\nu}\Bigg\},
\end{aligned}$$

where $d(x,E)$ denotes the distance of the point x from the set E, defined by

$$d(x,E)=\inf\big\{|x-y|:y\in E\big\}.$$

Proof Taking $y\in\bar{E}$, we can write

$$\big|f(t)-f(x)\big|\leq\big|f(t)-f(y)\big|+\big|f(y)-f(x)\big|,\quad x\in[0,\infty).$$

By (6.1) we have

$$\begin{aligned}
\big|S^{\alpha,\beta}_{n,p,q}(f;x)-f(x)\big|&=\big|S^{\alpha,\beta}_{n,p,q}(f;x)-S^{\alpha,\beta}_{n,p,q}\big(f(x);x\big)\big|\\
&\leq S^{\alpha,\beta}_{n,p,q}\big(\big|f(t)-f(x)\big|;x\big)\\
&\leq S^{\alpha,\beta}_{n,p,q}\big(\big|f(t)-f(y)\big|;x\big)+S^{\alpha,\beta}_{n,p,q}\big(\big|f(y)-f(x)\big|;x\big)\\
&\leq S^{\alpha,\beta}_{n,p,q}\big(\big|f(t)-f(y)\big|;x\big)+\big|f(x)-f(y)\big|\\
&\leq M_fS^{\alpha,\beta}_{n,p,q}\big(|t-y|^{\nu};x\big)+|x-y|^{\nu}\\
&\leq M_fS^{\alpha,\beta}_{n,p,q}\big(|t-x|^{\nu}+|x-y|^{\nu};x\big)+|x-y|^{\nu}\\
&\leq M_fS^{\alpha,\beta}_{n,p,q}\big(|t-x|^{\nu};x\big)+2|x-y|^{\nu}.
\end{aligned}$$

Using the Hölder inequality with $p = \frac{2}{\nu}, q = \frac{2}{2-\nu}$, we obtain

$$\begin{aligned}
&\left|S_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\right| \\
&\quad\leq M_f\left\{\left(S_{n,p,q}^{\alpha,\beta}\left(|t-x|^{p\nu};x\right)\right)^{\frac{1}{p}}\left(S_{n,p,q}^{\alpha,\beta}\left(1^q;x\right)\right)^{\frac{1}{q}} + 2\left(d(x,E)\right)^{\nu}\right)\right\} \\
&\quad= M_f\left\{\left(S_{n,p,q}^{\alpha,\beta}\left(|t-x|^2;x\right)\right)^{\frac{\nu}{2}} + 2\left(d(x,E)\right)^{\nu}\right)\right\} \\
&\quad= \left\{\left(\left(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)} - \frac{([n]_{p,q}-\beta)}{([n]_{p,q}+\beta)}\right)x^2 + \frac{1}{pq([n]_{p,q}-1)}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\right)^{\frac{\nu}{2}}\right. \\
&\qquad\left. + 2\left(d(x,E)\right)^{\nu}\right\},
\end{aligned}$$

and the theorem is proved. □

We now present a theorem regarding a local direct estimate for the operators $S_{n,p,q}^{\alpha,\beta}(f;x)$ in terms of the Lipschitz-type maximal function of order ν as introduced by Lenze [28]. It is defined by

$$\tilde{\omega}_\nu(f;x) = \sup_{y\neq x, y\in[0,\infty)} \frac{|f(y)-f(x)|}{|y-x|^\nu}, \quad x \in [0,\infty), \nu \in (0,1]. \tag{6.2}$$

Theorem 6.2 *Let* $\nu \in (0,1]$ *and* $f \in C[0,\infty)$. *Then, for each* $x \in [0,\infty)$, *we have*

$$\begin{aligned}
&\left|S_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\right| \\
&\quad\leq \tilde{\omega}_\nu(f;x)\left\{\left(\frac{[n]_{p,q}}{pq([n]_{p,q}-1)} - \frac{([n]_{p,q}-\beta)}{([n]_{p,q}+\beta)}\right)x^2 + \frac{1}{pq([n]_{p,q}-1)}x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}\right\}^{\frac{\nu}{2}}.
\end{aligned}$$

Proof By (6.2) we can write

$$\left|f(t) - f(x)\right| \leq \tilde{\omega}_\nu(f;x)|t-x|^\nu$$

and

$$\left|S_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\right| \leq S_{n,p,q}^{\alpha,\beta}\left(\left|f(t)-f(x)\right|;x\right) \leq \tilde{\omega}_\nu(f;x)S_{n,p,q}^{\alpha,\beta}\left(|t-x|^\nu;x\right).$$

Using the Lemma 2.2 and applying the Hölder inequality with $p = \frac{2}{\nu}$, $q = \frac{2}{2-\nu}$, we obtain

$$\left|S_{n,p,q}^{\alpha,\beta}(f;x) - f(x)\right| \leq \tilde{\omega}_\nu(f;x)S_{n,p,q}^{\alpha,\beta}\left(|t-x|^\nu;x\right),$$

which proves the theorem. □

Remark The further properties of the operators such as convergence properties via summability methods (see, e.g., [29–31]) can be studied.

7 Conclusions

In this paper, we have introduced a two-parametric (p,q)-analogue of the Stancu-Beta operators and studied some approximating properties of these operators. We also obtained the Voronovskaya-type estimate and the weighted approximation results for these operators. Furthermore, we obtained a pointwise estimate for these operators.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

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