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# Growth of the solutions of some $q$-difference differential equations

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**Abstract**

In view of Nevanlinna theory, we study the growth and poles of solutions of some complex $q$-difference differential equations. We obtain the estimates on the Nevalinna order, the lower order, and the counting function of poles for meromorphic solutions of such equations.

**MSC:** 39A13; 30D35

**Keywords:** growth; $q$-difference differential equation; pole

## 1 Introduction and main results

In this paper, the fundamental theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used (see Hayman [1], Yang [2] and Yi and Yang [3]). For a meromorphic function $f(z)$, we also use $S(r,f)$ to denote any quantity satisfying $S(r,f) = o(T(r,f))$ for all $r$ outside a possible exceptional set $E$ of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t} < \infty$, and a meromorphic function $a(z)$ is called a small function with respect to $f$, if $T(r,a) = S(r,f) = o(T(r,f))$.

In 1925, Ritt [4] gave the form of solutions of the Schrödinger equation

$$f(cz) = R\bigl(f(z)\bigr),$$

where $c \in \mathbb{C}$, $c \neq 0, 1$, and $R(f)$ is a rational function in $f$. In 1983, Rubel [5] posed the following question:

What can be said about the more general equation

$$f(cz) = R\bigl(z, f(z)\bigr),$$

where $R(z,f)$ is rational in both variables?

Later, Ishizaki [6] and Wittich [7] investigated the existence of meromorphic solutions of the equation of the following form:

$$f(cz) = a(z)f(z) + b(z),$$

where $a(z)$ and $b(z)$ are meromorphic functions.

In 2002, Gundersen *et al.* [8] studied the growth of meromorphic solutions of $q$-difference equations and obtained results as follows.

**Theorem 1.1** ([8], Theorem 3.2) *Suppose that $f$ is a transcendental meromorphic solution of an equation of the form*

$$f(cz) = R\big(z,f(z)\big) = \frac{\sum_{j=0}^{p} a_j(z)f(z)^j}{\sum_{j=0}^{q} b_j(z)f(z)^j}$$

*with meromorphic coefficients $a_j(z)$, $b_j(z)$ are of growth $S(r,f)$, and a constant $c$ ($|c| > 1$), assuming that $d := \max\{p,q\} \geq 1$, $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$, and that $R(z,f(z))$ is irreducible in $f$. Then $\rho(f) = \frac{\log d}{\log |c|}$, where $\rho(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}$.*

**Theorem 1.2** ([8], Theorem 3.4) *Let $c$ be a complex constant satisfying $|c| > 1$, and suppose that $f$ is a nonconstant meromorphic solution of a functional equation of the form*

$$A\big(cz,f(cz)\big) = B\big(z,f(z)\big),$$

*where $A(z,y)$ and $B(z,y)$ are rational functions with meromorphic coefficients of growth $S(r,f)$ such that $A(z,y)$ and $B(z,y)$ are irreducible in $y$. If $0 < a := \deg_f A \leq \deg_f B =: b$, then $\rho(f) = \frac{\log b - \log a}{\log |c|}$.*

In 2012, Beardon [9] studied entire solutions of the generalized function equation

$$f(qz) = qf(z)f'(z), \qquad f(0) = 0, \tag{1}$$

where $q$ is a non-zero complex number. To state the results of Beardon [9], we first introduce some notations as follows.

Let the formal series $\mathcal{O}$ and $\mathcal{I}$ be defined by

$$\mathcal{O} := 0 + 0z + 0z^2 + \cdots, \qquad \mathcal{I} := 0 + 1z + 0z^2 + 0z^3 + \cdots,$$

and the sets $\mathcal{K}_p = \{z : z^p = p + 2\}$ ($p = 1,2,\ldots$), and $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \cdots$. Thus, we see that $\mathcal{K}_p$ contains $p$ elements and $|z| = r_p$, for $z \in \mathcal{K}_p$, where $r_p = (p + 2)^{\frac{1}{p}}$. Since $p \in \mathbb{N}_+$, we have $|z| > 1$. Since $\frac{\log(x+2)}{x}$ is decreasing as $x > 1$, we have $r_1 > r_2 > \cdots > 1$, and $r_p \to 1$ as $p \to \infty$. Based on the above notations, Beardon obtained two main theorems as follows.

**Theorem 1.3** ([9]) *Any transcendental solution $f$ of (1) is of the form*

$$f(z) = z + z\big(bz^p + \cdots\big),$$

*where $p$ is a positive integer, $b \neq 0$ and $q \in \mathcal{K}_p$. In particular, if $q \notin \mathcal{K}$, then the only formal solutions of (1) are $\mathcal{O}$ and $\mathcal{I}$.*

**Theorem 1.4** ([9]) *For each positive integer $p$, there is a unique real entire function*

$$F_p(z) = z\big(1 + z^p + b_2z^{2p} + b_3z^{3p} + \cdots\big),$$

*which is a solution of (1) for each $q \in \mathcal{K}_p$. Further, if $q \in \mathcal{K}_p$, then the only transcendental solutions of (1) are the linear conjugates of $F_p$.*

Recently, Zhang [10] further studied the growth of solutions of (1) and obtained the following theorem.

**Theorem 1.5** ([10], Theorem 1.1) *Suppose that $f$ is a transcendental solution of* (1) *for $q \in \mathcal{K}$, then we have*

$$\rho(f) \le \frac{\log 2}{\log |q|},$$

*where*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Regarding Theorem 1.5, Zhang [10] asked the following question: Is the order of transcendental solutions of (1) exactly $\rho(f) \le \frac{\log 2}{\log |q|}$?

In this paper, we further investigate the growth of solution of some class of $q$-difference differential equation and obtain the following results.

**Theorem 1.6** *Suppose that $f$ is a solution of equation*

$$f(qz)^n = qf(z)\bigl[f^{(j)}(z)\bigr]^s, \tag{2}$$

*where $q \in \mathcal{K}$ and $n, s, j \in N_+$. If $f$ is a transcendental entire function, then $n \le s+1$ and the order of $f$ satisfies*

$$\rho(f) \le \frac{\log(s+1) - \log n}{\log |q|}.$$

The following example shows that (2) has non-transcendental entire function solution.

**Example 1.1** Let $q = 2$, $n = 2$, $j = 1$, and $s = 2$, then $f(z) = 2z^2$ satisfies equation

$$f(2z)^2 = 2f(z)\bigl(f'(z)\bigr)^2.$$

The following example shows that (2) also has a transcendental entire function solution.

**Example 1.2** Let $q = 3$, $n = 2$, $j = 1$, and $s = 5$, then $f(z) = \exp\{3^{-\frac{1}{5}}z\}$ satisfies the equation

$$f(3z)^2 = 3f(z)\bigl(f'(z)\bigr)^5,$$

and

$$\rho(f) = 1 = \frac{\log 6 - \log 2}{\log 3}.$$

**Remark 1.1** Thus, a question arises naturally: Does (2) have a transcendental meromorphic solution?

When the constant $q$ of the right of (2) is replaced by a function, the following example shows that the equation has a transcendental meromorphic solution.

**Example 1.3** Let $f(z) = \frac{e^z}{z^2}$ and $q = 2$, then $f(z)$ satisfies the equation

$$f(2z) = \frac{z^3}{4z-8} f(z)f'(z)$$

and the order is

$$\frac{\log 2 - \log 1}{\log |2|} = \rho(f) = 1 \leq \frac{\log 3 - \log 1}{\log 2}.$$

Thus, we have the following theorems.

**Theorem 1.7** *Let $f$ be a transcendental solution of the equation*

$$f(qz)^n = \varphi_1(z) f(z)\big[f^{(j)}(z)\big]^s, \tag{3}$$

*where $q$ is a non-zero complex number and $|q| > 1$, $n, j, s$ are positive integers and $\varphi_1(z)$ is a rational function. If $f$ is an entire function, then $n \leq s + 1$ and*

$$\rho(f) \leq \frac{\log(s+1) - \log n}{\log |q|}.$$

*Furthermore, if $n = 1$ and $f$ is a meromorphic function with infinitely many poles, then we have*

$$\frac{\log(s+1)}{\log |q|} \leq \rho(f) \leq \frac{\log(sj+s+1)}{\log |q|}.$$

**Theorem 1.8** *Let $f$ be a transcendental solution of the equation*

$$f(qz)^n = \varphi_2(z) f(z)\big[f^{(j)}(z)\big]^s, \tag{4}$$

*where $q$ is a complex number and $|q| > 1$, $n, j, s$ are positive integers and $\varphi_2(z)$ is a small function with respect to $f$. If $f$ is a meromorphic function with $\overline{N}(r,f) = S(r,f)$, then $n < s + 1$ and $f$ satisfies*

$$\rho(f) \leq \frac{\log(s+1) - \log n}{\log |q|}.$$

*Furthermore, if $n = 1$ and $f$ has infinitely many poles with $\overline{N}(r,f) = S(r,f)$, and the number of distinct common poles of $f$ and $\frac{1}{\varphi_2}$ is finite, then we have*

$$\rho(f) = \frac{\log(s+1)}{\log |q|}.$$

The following example shows that (4) has a transcendental meromorphic solution $f$ with the order $\rho(f) = \frac{\log(s+1)}{\log |q|}$.

**Example 1.4** Let $n = j = s = 1$ and $q = 2$, then $f(z) = \frac{(z-1)e^z}{z}$ satisfies the equation

$$f(2z) = \frac{2z^2 - z}{2z-1} f(z)f'(z),$$

where $\varphi_2(z) = \frac{2z^2-z}{2z-1}$ with $T(r,\varphi_2) = S(r,f)$ and the order of $f(z)$ satisfies

$$\rho(f) = 1 = \frac{\log 2 - \log 1}{\log 2}.$$

Let $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_1 z + p_0$, where $p_k (\not\equiv 0), \ldots, p_0$ are complex constants. Now, we investigate the growth of solutions of such equations, where $qz$ is replaced by $p(z)$ in (2)-(4), and we obtain the following result.

**Theorem 1.9** *Let $f$ be a transcendental solution of equation*

$$f\bigl(p(z)\bigr)^n = \varphi_3(z) f(z) \bigl[f^{(j)}(z)\bigr]^s, \tag{5}$$

*where $k \geq 2$, $n$, $j$, $s$ are positive integers and $\varphi_3(z)$ is a small function with respect to $f$. If $f$ is a transcendental meromorphic function and $n < sj + s + 1$, then $f$ satisfies*

$$T(r,f) = O\bigl((\log r)^\alpha\bigr), \qquad \alpha = \frac{\log(sj+s+1) - \log n}{\log k}.$$

Recently, there were many results on meromorphic solutions of complex functional equations (see [11–20]). In 2007, Barnett *et al.* [21] firstly established an analog of the logarithmic derivative lemma on $q$-difference operators. In 2010, by applying their theorems, Zheng and Chen [22] considered the growth of meromorphic solutions of $q$-difference equations and obtained results which extended some theorems given by Heittokangas *et al.* [23].

**Theorem 1.10** ([22], Theorem 2) *Suppose that $f$ is a transcendental meromorphic solution of equation*

$$\sum_{j=1}^{n} a_j(z) f\bigl(q^j z\bigr) = R\bigl(z, f(z)\bigr) = \frac{P(z,f(z))}{Q(z,f(z))},$$

*where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j(z)$ are rational functions and $P$, $Q$ are relatively prime polynomials in $f$ over the field of rational functions satisfying $p = \deg_f P$, $t = \deg_f Q$, $d = p - t \geq 2$. If $f$ has infinitely many poles, then for sufficiently large $r$, $n(r,f) \geq K d^{\frac{\log r}{n \log |q|}}$ holds for some constant $K > 0$. Thus, the lower order of $f$, which has infinitely many poles, satisfies $\mu(f) \geq \frac{\log d}{n \log |q|}$, where $\mu(f) = \liminf_{r \to +\infty} \frac{\log \log T(r,f)}{\log r}$.*

From Theorem 1.10, we further study the growth of the solutions of a class of $q$-difference differential equation and obtain a result as follows.

**Theorem 1.11** *Suppose that $f$ is a transcendental meromorphic solution of the equation*

$$f(qz) f'(z) = R\bigl(z, f(z)\bigr) = \frac{P(z,f(z))}{Q(z,f(z))}, \tag{6}$$

*where $q \in \mathbb{C}$, $|q| > 1$, and $P$, $Q$ are relatively prime polynomials in $f$ over the field of rational functions satisfying $p = \deg_f P$, $t = \deg_f Q$, $d = p - t \geq 4$, where the coefficients of $P$, $Q$ are* *rational functions in z. If $f$ has infinitely many poles, then for sufficiently large r, $n(r,f) \geq K(d-1)^{\frac{\log r}{\log|q|}}$ holds for some constant $K > 0$. Thus, the lower order of $f$, which has infinitely many poles, satisfies $\mu(f) \geq \frac{\log(d-1)}{\log|q|}$.*

**Remark 1.2** Under the conditions of Theorem 1.11, by using the same argument as in Theorem 1.8, we can see that the lower order, the order of $f$, which has infinitely many poles, satisfies

$$\frac{\log(d-1)}{\log|q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(d+2)}{\log|q|}.$$

The following example shows that (6) has a non-transcendental solution.

**Example 1.5** Let $q = 2$ and $d = 3$, then $f(z) = \frac{1}{z^2}$ satisfies the equation

$$f(2z)f'(z) = -\frac{1}{2}zf(z)^3.$$

The following examples show that (6) has transcendental entire and meromorphic solutions.

**Example 1.6** Let $q = 2$ and $d = 3$, then $f(z) = \sin z$ satisfies the equation

$$f(2z)f'(z) = 2f(z) - 2f(z)^3.$$

Then we have $\mu(f) = \rho(f) = 1 = \frac{\log(3-1)}{\log 2}$.

**Example 1.7** Let $q = 2$ and $d = 5$, then $f(z) = \frac{1}{z}e^{z^2}$ satisfies the equation

$$f(2z)f'(z) = z^2\left(z^2 - \frac{1}{2}\right)f(z)^5.$$

Then we see that $f$ has finitely many poles and $\mu(f) = \rho(f) = 2 = \frac{\log(5-1)}{\log 2}$.

**Example 1.8** Let $q = 2$ and $d = 3$, then $f(z) = \frac{1}{\sin z}$ satisfies the equation

$$f(2z)f'(z) = -\frac{1}{2}f(z)^3.$$

So, $f(z)$ has infinitely many poles and $\mu(f) = \rho(f) = 1 = \frac{\log(3-1)}{\log 2}$.

**Remark 1.3** By comparing Example 1.8 and Theorem 1.11, we pose a question as follows: Whether the condition '$d = p - t \geq 4$' may be relaxed to '$d \geq 3$ or $d \geq 2$' in Theorem 1.11?

## 2 Some lemmas

**Lemma 2.1** (Valiron-Mohon'ko [24]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f$,*

$$R(z,f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

*with meromorphic coefficients $a_i(z)$, $b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies*

$$T\bigl(r,R\bigl(z,f(z)\bigr)\bigr)=dT(r,f)+O\bigl(\Psi(r)\bigr),$$

*where $d=\max\{m,n\}$ and $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.*

**Lemma 2.2** ([3], p.37 or [2]) *Let $f(z)$ be a nonconstant meromorphic function in the complex plane and $l$ be a positive integer. Then*

$$N\bigl(r,f^{(l)}\bigr)=N(r,f)+l\overline{N}(r,f),\qquad T\bigl(r,f^{(l)}\bigr)\le T(r,f)+l\overline{N}(r,f)+S(r,f).$$

**Lemma 2.3** ([8]) *Let $\Phi:(1,+\infty)\to(0,+\infty)$ be a monotone increasing function, and let $f$ be a nonconstant meromorphic function. If for some real constant $\alpha\in(0,1)$, there exist real constants $K_1>0$ and $K_2\ge 1$ such that*

$$T(r,f)\le K_1\Phi(\alpha r)+K_2T(\alpha r,f)+S(\alpha r,f),$$

*then the order of growth of $f$ satisfies*

$$\rho(f)\le\frac{\log K_2}{-\log\alpha}+\limsup_{r\to\infty}\frac{\log\Phi(r)}{\log r}.$$

**Lemma 2.4** ([25]) *Let $f(z)$ be a transcendental meromorphic function and $p(z)=p_kz^k+p_{k-1}z^{k-1}+\cdots+p_1z+p_0$ be a complex polynomial of degree $k>0$. For given $0<\delta<|p_k|$, let $\lambda=|p_k|+\delta$, $\mu=|p_k|-\delta$, then for given $\varepsilon>0$ and for $r$ large enough,*

$$(1-\varepsilon)T\bigl(\mu r^k,f\bigr)\le T(r,f\circ p)\le(1+\varepsilon)T\bigl(\lambda r^k,f\bigr).$$

**Lemma 2.5** ([26, 27] or [28]) *Let $g:(0,+\infty)\to R$, $h:(0,+\infty)\to R$ be monotone increasing functions such that $g(r)\le h(r)$ outside of an exceptional set $E$ with finite linear measure, or $g(r)\le h(r)$, $r\notin H\cup(0,1]$, where $H\subset(1,+\infty)$ is a set of finite logarithmic measure. Then, for any $\alpha>1$, there exists $r_0$ such that $g(r)\le h(\alpha r)$ for all $r\ge r_0$.*

**Lemma 2.6** ([29]) *Let $\psi(r)$ be a function of $r$ ($r\ge r_0$), positive and bounded in every finite interval.*

(i) *Suppose that $\psi(\mu r^m)\le A\psi(r)+B$ ($r\ge r_0$), where $\mu$ ($\mu>0$), $m$ ($m>1$), $A$ ($A\ge1$), $B$ are constants. Then $\psi(r)=O((\log r)^\alpha)$ with $\alpha=\frac{\log A}{\log m}$, unless $A=1$ and $B>0$; and if $A=1$ and $B>0$, then for any $\varepsilon>0$, $\psi(r)=O((\log r)^\varepsilon)$.*

(ii) *Suppose that (with the notation of (i)) $\psi(\mu r^m)\ge A\psi(r)$ ($r\ge r_0$). Then for all sufficiently large values of $r$, $\psi(r)\ge K(\log r)^\alpha$ with $\alpha=\frac{\log A}{\log m}$, for some positive constant $K$.*

**Lemma 2.7** (see [12])

$$T\bigl(r,f(qz)\bigr)=T\bigl(|q|r,f\bigr)+O(1)$$

*holds for any meromorphic function $f$ and any non-zero constant $q$.*

## 3 Proofs of Theorems 1.6-1.8

### 3.1 The proof of Theorem 1.6

By Lemma 2.1 and Lemma 2.7, it follows from (2) that

$$T\big(|q|r,f(z)\big) \leq \frac{1}{n}T(r,f) + \frac{s}{n}T\big(r,f^{(j)}(z)\big) + O(1). \tag{7}$$

If $f$ is a transcendental entire function, then we have by Lemma 2.2

$$T\big(|q|r,f(z)\big) \leq \frac{1+s}{n}T(r,f) + S(r,f). \tag{8}$$

Since $|q| > 1$ and $f$ is transcendental, it follows from (8) that $n \leq s+1$. Set $\alpha = \frac{1}{|q|}$, it follows

$$T\big(r,f(z)\big) \leq \frac{1+s}{n}T(\alpha r,f) + S(\alpha r,f).$$

By Lemma 2.3, we have $\rho(f) \leq \frac{\log(s+1)-\log n}{\log|q|}$.

### 3.2 The proof of Theorem 1.7

Since $\varphi_1(z)$ is a rational function, we have $T(r,\varphi_1(z)) = O(\log r)$. If $f$ is a transcendental entire function, similar to the argument as in Theorem 1.6, we easily get $\rho(f) \leq \frac{\log(s+1)-\log n}{\log|q|}$.

If $f$ is a meromorphic function, by Lemma 2.1, Lemma 2.2, and Lemma 2.7, it follows from (3) that

$$T\big(|q|r,f(z)\big) \leq \frac{sj+s+1}{n}T\big(r,f(z)\big) + S(r,f).$$

Since $|q| > 1$, by Lemma 2.3 we have $\rho(f) \leq \frac{\log(sj+s+1)-\log n}{\log|q|}$.

Since $\varphi_1(z)$ is a rational function, we can choose a sufficiently large constant $R$ $(> 0)$ such that $\varphi_1(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$. Since $f$ has infinitely many poles, we can choose a pole $z_0$ of $f$ of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. Then the right side of (3) has a pole of multiplicity $\tau_1 = (s+1)\tau + sj$ at $z_0$. Then $f$ has a pole of multiplicity $\tau_1$ at $qz_0$. Replacing $z$ by $qz_0$ in (3), we see that $f$ has a pole of multiplicity $\tau_2 = (s+1)\tau_1 + sj$ at $q^2z_0$. We proceed to follow the steps above. Since $\varphi_1(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$ and $f$ has infinitely many poles again, we may construct poles $\zeta_k = q^kz_0$, $k \in \mathbb{N}_+$ of $f$ of multiplicity $\tau_k$ satisfying

$$\tau_k = (s+1)\tau_{k-1} + sj = (s+1)^k\tau + sj\big[(s+1)^{k-1} + \cdots + 1\big],$$

as $k \to \infty$, $k \in \mathbb{N}$. Since $|q| > 1$, $|\zeta_k| \to \infty$ as $k \to \infty$. For sufficiently large $k$, we have

$$\begin{aligned} \tau(s+1)^k &\leq (\tau + j)(s+1)^k - j = \tau_k \leq \tau + \tau_1 + \cdots + \tau_k \leq n\big(|\zeta_k|,f\big) \\ &\leq n\big(|q|^k|z_0|,f\big). \end{aligned} \tag{9}$$

Thus, for each sufficiently large $r$, there exists a $k \in \mathbb{N}_+$ such that

$$r \in \big[|q|^k|z_0|, |q|^{(k+1)}|z_0|\big), \quad \textit{i.e. } k > \frac{\log r - \log r_0 - \log|q|}{\log|q|}. \tag{10}$$

Thus, we have

$$n(r,f) \geq \tau(s+1)^k \geq \tau(s+1)^{\frac{\log r-\log r_0-\log|q|}{\log|q|}} \geq K_1(s+1)^{\frac{\log r}{\log|q|}}, \tag{11}$$

where

$$K_1 = \tau(s+1)^{\frac{-\log r_0-\log|q|}{\log|q|}}.$$

Since, for all $r \geq r_0$,

$$K_1(s+1)^{\frac{\log r}{\log|q|}} \leq n(r,f) \leq \frac{1}{\log 2}N(2r,f) \leq \frac{1}{\log 2}T(2r,f),$$

it follows from (11) that

$$\rho(f) \geq \mu(f) \geq \frac{\log(s+1)}{\log|q|}.$$

Thus, this completes the proof of Theorem 1.7.

### 3.3 The proof of Theorem 1.8

Since $\varphi_2(z)$ is a small function, similar to (7), we have

$$T\big(|q|r,f(z)\big) \leq \frac{1}{n}T(r,f) + \frac{s}{n}T\big(r,f^{(j)}(z)\big) + S(r,f). \tag{12}$$

Since $f$ is a transcendental meromorphic function and $\overline{N}(r,f) = S(r,f)$, by Lemma 2.2 we have

$$T\big(r,f^{(j)}(z)\big) \leq T(r,f) + S(r,f). \tag{13}$$

Thus, from (12) and (13), by using the same argument as in Theorem 1.6, we can get

$$\rho(f) \leq \frac{\log(s+1)-\log n}{\log|q|}. \tag{14}$$

If $n = 1$ and $f$ has infinitely many poles, since the number of distinct common poles of $f$ and $\frac{1}{\varphi_2}$ is finite, we can choose a sufficiently large constant $R$ (> 0) such that $f$ and $\frac{1}{\varphi_2(z)}$ have no common poles in $\{z \in \mathbb{C} : |z| > R\}$. Thus, we can take a pole $z_0$ of $f$ of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. By using the same argument as in Theorem 1.7, we can see that

$$\rho(f) \geq \mu(f) \geq \frac{\log(s+1)}{\log|q|}. \tag{15}$$

Hence, from (14) and (15), we complete the proof of Theorem 1.8.

## 4 The proof of Theorem 1.9

Since $f$ is a transcendental meromorphic solution of (5), and $\varphi_3(z)$ is a small function with respect to $f$, similar to the proof of (12), and by Lemma 2.2, we have

$$T\big(r,f(p(z))\big) \leq \frac{s+sj+1}{n}T\big(r,f(z)\big) + S(r,f) = \left(\frac{s+sj+1}{n} + o(1)\right)T(r,f).$$

Then, by Lemma 2.5, for any $\beta > 1$ and for all $r > r_0$, we have

$$T\bigl(r,f\bigl(p(z)\bigr)\bigr) \leq \left(\frac{s+sj+1}{n}+o(1)\right)T(\beta r,f). \tag{16}$$

Since $p(z)$ is a polynomial with $\deg_z p(z) = k \geq 2$, by Lemma 2.4, for given $0 < \delta < |p_k|$, let $\mu = |p_k| - \delta$, for given $\varepsilon > 0$ and for sufficiently large $r$, it follows for (16) that

$$(1-\varepsilon)T\bigl(\mu r^k,f\bigr) \leq \left(\frac{s+sj+1}{n}+o(1)\right)T(\beta r,f).$$

Set $R = \beta r$, then we have

$$(1-\varepsilon)T\bigl(\mu\beta^{-k}R^k,f\bigr) \leq \left(\frac{s+sj+1}{n}+o(1)\right)T(R,f). \tag{17}$$

Since $n < s + sj + 1$ and $\beta > 1$, $\mu > 0$, we have $\frac{s+sj+1}{n} > 1$ and $\mu\beta^{-k} > 0$. Thus, by Lemma 2.6, letting $\varepsilon \to 0$ and $\beta \to 1$, we have

$$T(r,f) = O\bigl((\log r)^{\alpha}\bigr), \quad \alpha = \frac{\log(sj+s+1)-\log n}{\log k}.$$

Thus, this completes the proof of Theorem 1.9.

## 5 The proof of Theorem 1.11

Suppose that $f$ is a transcendental meromorphic solution of (6). Since $f$ has infinitely many poles, we can take a pole $z_0$ of $f$ of multiplicity $\tau \geq 1$. Since $d \geq 4$, we see that the right side of (6) has a pole of multiplicity $d\tau$ at $z_0$. Then it follows that $qz_0$ is a pole of $f$ of multiplicity $\tau_1 = d\tau - \tau - 1$. Since $d \geq 4$ and $\tau \geq 1$, we have $\tau_1 \geq 1$. Replacing $z$ by $qz_0$ in (6), we have

$$f\bigl(q^2z_0\bigr)f'(qz_0) = R\bigl(qz_0,f(qz_0)\bigr). \tag{18}$$

Thus the right side of (18) has a pole of multiplicity $d\tau_1$ at $qz_0$. Then we see that $q^2z_0$ is a pole of $f$ of multiplicity $\tau_2 = d\tau_1 - \tau_1 - 1 = (d-1)^2\tau - (d-1) - 1$.

We proceed to follow the steps above. Since $f$ has infinitely many poles, we may construct poles $\zeta_k = q^kz_0$, $k \in N_+$ of $f$ of multiplicity $\tau_k$ satisfying

$$\begin{aligned}\tau_k &= d\tau_{k-1} - \tau_{k-1} - 1 = (d-1)^k\tau - (d-1)^{k-1} - \cdots - (d-1) - 1\\ &= (d-1)^k\tau - \frac{(d-1)^k-1}{d-2} > (d-1)^k\left(\tau - \frac{1}{d-2}\right).\end{aligned} \tag{19}$$

Since $\tau \geq 1$ and $d \geq 4$, $\tau - \frac{1}{d-2} > 0$. Thus, since $|\zeta_k| \to \infty$ as $k \to \infty$, for sufficiently large $k$, we have

$$(d-1)^k\left(\tau - \frac{1}{d-2}\right) < \tau_k \leq \tau_1 + \tau_2 + \cdots + \tau_k \leq n\bigl(|\zeta_k|,f\bigr) \leq n\bigl(|q|^k|z_0|,f\bigr). \tag{20}$$

Thus, for each sufficiently large $r$, there exists a $k \in \mathbb{N}_+$ such that $r \in [|q|^k|z_0|, |q|^{k+1}|z_0|)$. By using the same method as in the proof of Theorem 1.7, from (20), we have

$$\begin{aligned} n(r,f) &\geq (d-1)^k\left(\tau - \frac{1}{d-2}\right) \geq (d-1)^{\frac{\log r - \log|z_0| - \log|q|}{\log|q|}}\left(\tau - \frac{1}{d-2}\right) \\ &\geq K_2(d-1)^{\frac{\log r}{\log|q|}}, \end{aligned} \tag{21}$$

where

$$K_2 = \left(\tau - \frac{1}{d-2}\right)(d-1)^{\frac{-\log|z_0| - \log|q|}{\log|q|}}.$$

Since for all $r \geq r_0$, we have

$$K_2(d-1)^{\frac{\log r}{\log|q|}} \leq n(r,f) \leq \frac{1}{\log 2}N(2r,f) \leq \frac{1}{\log 2}T(2r,f).$$

Thus, it follows that

$$\rho(f) \geq \mu(f) \geq \frac{\log(d-1)}{\log|q|}.$$

Thus, this completes the proof of Theorem 1.11.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
HYX completed the main part of this article, HYX, LZY, and HW corrected the main theorems. All authors read and approved the final manuscript.

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