Kılıç and Gülbahar *Journal of Inequalities and Applications* (2016) 2016:57
DOI 10.1186/s13660-016-1004-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the sectional curvature of lightlike submanifolds

Erol Kılıç[1] and Mehmet Gülbahar[2*]

*Correspondence: mehmetgulbahar85@gmail.com
[2]Department of Mathematics, Faculty of Arts and Sciences, Siirt University, Siirt, 56100, Turkey
Full list of author information is available at the end of the article

Abstract
The main purpose of this paper is to show how to obtain rigidity theorems with the help of curvature invariants in submanifolds of a semi-Riemannian manifold. For this purpose, the bounded sectional curvature is introduced and some special submanifolds of r-lightlike submanifolds of a semi-Riemannian manifold are investigated.

MSC: 53C40; 53C42; 53C50

Keywords: curvature; lightlike submanifold; semi-Riemannian manifold

1 Introduction

In 1979, Kulkarni [1] proved that the sectional curvature of a semi-Riemannian manifold M is unbounded from above and below at each point unless the manifold has constant sectional curvature. Later, Nomizu [2] showed that if there exists a real number d such that at any point $p \in M$, the sectional curvature $K(\Pi)$ of a 2-plane section Π satisfies

$$K(\Pi) \leq d, \tag{1}$$

then M is of constant sectional curvature.

In [3], Dajczer and Nomizu, and in [4], Harris remarked that if the absolute value of the sectional curvature $|K(\Pi)|$ is bounded for all timelike 2-planes Π (or for all spacelike 2-planes Π) at $p \in M$, then M is of constant sectional curvature.

In Riemannian geometry, there are various relations between the intrinsic and extrinsic curvature invariants of a submanifold, known as Chen inequalities, in the literature [5–44]. But different from the Riemannian context, from the Kulkarni result, it is too restrictive to relate the intrinsic invariant of a submanifold with the extrinsic ones for a submanifold of a semi-Riemannian manifold. This reveals the necessity to re-investigate or modify the domain of sectional curvature map in semi-Riemannian geometry and lightlike geometry. For this purpose, the authors showed in [45, 46] that the domain of the sectional curvature map in a Lorentzian manifold is not a linear subspace as it was used in the literature but it is a polynomial subspace of a projective vector space which makes it possible for the sectional curvature map on any Lorentzian manifold to be bounded. This is revolutionary information which might lead one to require a revision for many studies related to the sectional curvature map in semi-Riemannian geometry and lightlike geometry.

In this paper, we extend this modified sectional curvature from Lorentzian manifolds to semi-Riemannian manifolds under the name of 'bounded sectional curvature'. We introduce some special r-lightlike submanifolds and establish some relationships involving intrinsic curvatures and extrinsic curvatures for r-lightlike submanifolds of a semi-Riemannian manifold.

2 Preliminaries

Let $(\widetilde{M},\tilde{g})$ be a real $(\widetilde{m}+\tilde{n})$-dimensional semi-Riemannian manifold, where $\tilde{n}\geq 1$, $\widetilde{m}>1$ with $\tilde{g}$ a semi-Riemannian metric on $\widetilde{M}$ of constant index $\tilde{q}\in\{1,\dots,\widetilde{m}+\tilde{n}-1\}$. Suppose (M,g) to be an $\tilde{n}$-dimensional lightlike submanifold of $(\widetilde{M},\tilde{g})$ where g denotes the restriction of $\tilde{g}$ to M which we assume be degenerate. Then there exists a smooth distribution, called *radical space* of the tangent space T_pM at $p\in M$, defined by

$$\operatorname{Rad} T_pM = T_pM\cap T_pM^{\perp}\neq\{0\},\tag{2}$$

where

$$T_pM^{\perp}=\bigl\{v_p\in T_p\widetilde{M}:\tilde{g}_p(v_p,w_p)=0,\forall w_p\in T_pM\bigr\}.\tag{3}$$

Let us consider the rank of $\operatorname{Rad} T_pM$ to be r ($r>0$), $\tilde{n}=n+r$ ($n\geq 0$), and $\widetilde{m}=n+m+2r$ ($m\geq 0$). Then there exist the following four possible cases:

Case 1. M is called a *r-lightlike submanifold* if $1\leq r<\min\{n+r,m+r\}$.
Case 2. M is called a *coisotropic submanifold* if $m=0$.
Case 3. M is called a *isotropic submanifold* if $n=0$.
Case 4. M is called a *totally lightlike submanifold* if $m=n=0$.

Let (M,g) be an $(n+r)$-dimensional lightlike submanifold of $(\widetilde{M},\tilde{g})$. Let $S(TM)$ and $S(TM^{\perp})$ be a complementary non-degenerate vector bundle of $\operatorname{Rad} TM$ in TM and $TM^{\perp}$, respectively, $\operatorname{tr}(TM)$ be a complementary vector bundle to TM in $T\widetilde{M}|_M$. Then we have

$$T\widetilde{M}|_M=\bigl(\operatorname{Rad} TM\oplus \operatorname{ltr}(TM)\bigr)\oplus_{\mathrm{orth}} S(TM)\oplus_{\mathrm{orth}} S\bigl(TM^{\perp}\bigr),\tag{4}$$

where $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum and $\oplus$ denotes the direct sum, but it is not orthogonal.

For any r-lightlike submanifold, there exists a local quasi-orthonormal frame field $\{\xi_1,\dots,\xi_r,e_1,\dots,e_n,N_1,\dots,N_r,u_1,\dots,u_m\}$ on a local coordinate neighborhood of $\mathcal{U}$ of M such that this basis satisfies the following relation:

$$\tilde{g}(N_i,\xi_j)=\delta_{ij},\qquad \tilde{g}(N_i,N_j)=\tilde{g}(N_i,u_j)=\tilde{g}(\xi_i,u_j)=0,\quad \forall i,j\in\{1,\dots,r\},\tag{5}$$

where δ_{ij} is the Kronecker delta function and

$$\Gamma(\operatorname{Rad} TM|_{\mathcal{U}})=\operatorname{Span}\{\xi_1,\dots,\xi_r\},\qquad \Gamma\bigl(\operatorname{ltr}(TM)|_{\mathcal{U}}\bigr)=\operatorname{Span}\{N_1,\dots,N_r\},$$

$$\Gamma\bigl(S(TM)|_{\mathcal{U}}\bigr)=\operatorname{Span}\{e_1,\dots,e_n\},\qquad \Gamma\bigl(S\bigl(TM^{\perp}\bigr)|_{\mathcal{U}}\bigr)=\operatorname{Span}\{u_1,\dots,u_m\}.$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P be the projection morphism of $\Gamma(TM)$ to $\Gamma(S(TM))$. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sum_{l=1}^{r} B^l(X,Y)N_l + \sum_{\alpha=1}^{m} D^\alpha(X,Y)u_\alpha, \tag{6}$$

$$\widetilde{\nabla}_X N_k = -A_{N_k}X + \sum_{l=1}^{r} \rho_{kl}(X)N_l + \sum_{\alpha=1}^{m} \rho_{\alpha k}(X)u_\alpha, \tag{7}$$

$$\widetilde{\nabla}_X u_\beta = -A_{u_\beta}X + \sum_{l=1}^{r} \varepsilon_{\beta l}(X)N_l + \sum_{\alpha=1}^{m} \varepsilon_{\beta\alpha}(X)u_\alpha, \tag{8}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{l=1}^{r} C^l(X,PY)\xi_l, \tag{9}$$

$$\nabla_X \xi_k = -A^*_{\xi_k}X - \sum_{l=1}^{r} \rho_{kl}(X)\xi_l \tag{10}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are the induced linear connection on TM and $S(TM)$, respectively; B^l and D^α are coefficients of the lightlike second fundamental form and coefficients of the screen second fundamental form of TM, respectively, C^l are the coefficients of the local second fundamental form on $S(TM)$, A_{N_l}, A_{u_α} are the shape operators on M, $A^*_{\xi_k}$ is the shape operator on $S(TM)$ and ε_l, ε_α, ρ_l, ρ_α are 1-forms on M [47].

The second fundamental form h and the local second fundamental form h^* are given by

$$h(X,Y) = \sum_{l=1}^{r} B^l(X,Y)N_l + \sum_{\alpha=1}^{m} D^\alpha(X,Y)u_\alpha \tag{11}$$

and

$$h^*(X,PY) = \sum_{l=1}^{r} C^l(X,PY)\xi_l, \tag{12}$$

respectively. The submanifold $(M, g, S(TM))$ is called *totally geodesic* if

$$h(X,Y) = 0 \tag{13}$$

for all $X, Y \in \Gamma(TM)$ and it is called *totally umbilical* [48] if there exists a smooth transversal vector field $H \in \Gamma(\mathrm{tr}(TM))$ such that

$$h(X,Y) = \tilde{g}(X,Y)H \tag{14}$$

for all $X, Y \in \Gamma(TM)$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. Consider

$$\mu_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{l=1}^{r} \varepsilon_i B^l(e_i,e_i) \quad \text{and} \quad \mu_2 = \frac{1}{n}\sum_{i=1}^{n}\sum_{\alpha=1}^{r} \varepsilon_i\varepsilon_\alpha D^\alpha(e_i,e_i), \tag{15}$$

where $\varepsilon_i = g(e_i, e_i)$, $\varepsilon_\alpha = g(e_\alpha, e_\alpha)$ for any $i \in \{1, \ldots, n\}$ and $\alpha \in \{1, \ldots, m\}$. The mean curvature vectors on TM and on $\Gamma(S(TM))$ at $p \in M$, denoted by $H(p)$ and $H^*(p)$, are given by

$$H(p) = \frac{1}{n} \operatorname{trace}|_{S(TM)} h = \sum_{l=1}^{r} \mu_1 N_l + \sum_{\alpha=1}^{m} \mu_2 u_\alpha, \tag{16}$$

$$H^*(p) = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \sum_{\ell=1}^{r} C^\ell(e_i, e_i)\xi_\ell + \frac{1}{n} \sum_{l=1}^{r} \mu_1 N_l + \frac{1}{n} \sum_{\alpha=1}^{m} \mu_2 u_\alpha, \tag{17}$$

respectively. From equation (16), we can see that the submanifold is minimal if and only if $H(p)$ vanishes identically and $D^\alpha = 0$ on $\operatorname{Rad}(TM)$ [49, 50].

Let us denote curvature tensors of the ambient manifold and the submanifold by $\widetilde{R}$ and R, respectively. Then the following relation between these tensors holds:

$$\begin{aligned} \tilde{g}(\widetilde{R}(X, Y)PZ, PW) = {} & g(R(X, Y)PZ, PW) + \sum_{l=1}^{r} B^\ell(X, PZ)C^\ell(Y, PW) \\ & - \sum_{l=1}^{r} B^l(Y, PZ)C^l(X, PW) \\ & + \sum_{\alpha=1}^{m} \varepsilon_\alpha \big[D^\alpha(X, PZ)D^\alpha(Y, PW) - D^\alpha(Y, PZ)D^\alpha(X, PW)\big] \end{aligned} \tag{18}$$

for all $X, Y, Z, U \in \Gamma(TM)$ [47].

Let $\Pi = \operatorname{Span}\{X, Y\}$ be a 2-dimensional non-degenerate plane in T_pM. Then the sectional curvature at p is expressed by

$$K(\Pi) = \frac{g(R_p(X, Y)Y, X)}{g_p(X, X)g_p(Y, Y) - g_p(X, Y)^2}. \tag{19}$$

We note that since C^l is not symmetric, the sectional curvature map does not need to be symmetric on any lightlike submanifold of a semi-Riemannian manifold [47].

Now, we recall the following result [51].

Theorem 1 *Let $(M, g, S(TM))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \tilde{g})$. Then the following assertions are equivalent*:

(i) *$S(TM)$ is integrable.*
(ii) *h^* is symmetric on $\Gamma(S(TM))$.*
(iii) *A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.*

As a consequence of Theorem 1, we obtain the following theorem.

Theorem 2 *Let $(M, g, S(TM))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \tilde{g})$. The sectional curvature map is symmetric if and only if $S(TM)$ is integrable.*

3 Bounded sectional curvature

We start by taking into consideration a quotient space given by

$$S(TM) \oplus_{\text{orth}} S(TM)/\operatorname{SL}(2, \mathbb{R}), \tag{20}$$

where $\mathrm{SL}(2,R)$ denotes the special linear transformation. For any given two vector pairs (X,Y) and (A,B) in this space, $(X,Y)\sim(A,B)$ if $A=aX+bY$ and $B=cX+dY$ with $ad-bc=1$. It is clear that the $\sim$ relation is an equivalence relation. Furthermore, for this relation in this space, we can write $(X,Y)\sim(A,B)$ if and only if $A\wedge B=X\wedge Y$, where $\wedge$ is the wedge product. Since $X\wedge Y$ is an element of the vector space of anti-symmetric contravariant two tensors $\wedge^2 S(TM)$ which is also known as the second exterior power of $S(TM)$ [52], any element of $S(TM)\oplus_{\mathrm{orth}} S(TM)/\mathrm{SL}(2,\mathbb{R})$ can be considered as an element of $\wedge^2 S(TM)$ satisfying

$$\Pi\wedge\Pi=0 \tag{21}$$

for all $\Pi\in\wedge^2 S(TM)$. We note that equation (21) holds because of the property of anti-symmetry of the wedge product. Thus, we have

$$\big(S(TM)\wedge S(TM)\big)/\mathrm{SL}(2,\mathbb{R})\cong\big\{\Pi\in\wedge^2 S(TM):\Pi\wedge\Pi=0\big\}. \tag{22}$$

Now, we consider the space of planes in $S(TM)$. It is well known that any vector pair spanned a plane section in $S(TM)$ are related by a general linear group $\mathrm{GL}(2,\mathbb{R})$. Therefore, the space of planes in $S(TM)$, denoted by the Grassmanian $G_{r_2}(S(TM))$, is given by

$$G_{r_2}\big(S(TM)\big)\equiv\big(S(TM)\oplus_{\mathrm{orth}} S(TM)\big)/\mathrm{GL}(2,\mathbb{R}). \tag{23}$$

Since the Grassmanian can be embedded into the real projective space $\mathbb{P}(\wedge^2 S(TM))$ but is not embedded into the space $\wedge^2 S(TM)$ (this embedding is also known as the Plücker embedding [52]) it can be written

$$G_{r_2}\big(S(TM)\big)=\big\{\Pi=X\wedge Y\in\mathbb{P}\big(\wedge^2 S(TM)\big):\Pi\wedge\Pi=0\big\}. \tag{24}$$

Eventually, if $S(TM)$ is semi-Riemannian, then the sectional curvature map is defined by

$$K:G_{r_2}\big(S(TM)\big)\cap\big\{\Pi=X\wedge Y:G(\Pi,\Pi)\neq 0\big\}\to\mathbb{R}, \tag{25}$$

where

$$G(\Pi,\Pi)=g(X,X)g(Y,Y)-g(X,Y)^2. \tag{26}$$

In the case of $S(TM)$ is Riemannian, then $G(\Pi,\Pi)\neq 0$ for all $\Pi\in\mathbb{P}(\wedge^2 S(TM))$ and thereby the sectional curvature map in the Riemannian context is given by

$$K:G_{r_2}\big(S(TM)\big)\to\mathbb{R}. \tag{27}$$

As a consequence of the above information, we give the following definition.

Definition 1 Let $(M,g,S(TM))$ be an $(n+r)$-dimensional r-lightlike submanifold of an $\widetilde{m}$-dimensional semi-Riemannian manifold $(\widetilde{M},\tilde{g})$ and $S(TM)$ be integrable. The map

$$K:G_{r_2}\big(S(TM)\big)\cap\big\{\Pi:G(\Pi,\Pi)\neq 0\big\}\to\mathbb{R}, \tag{28}$$

which is defined by

$$K(\Pi) = \frac{R(\Pi, \Pi)}{G(\Pi, \Pi)}, \tag{29}$$

is called *bounded sectional curvature map*.

Proposition 1 *Let (M, g) be an $(n + r)$-dimensional r-lightlike submanifold of an $\widetilde{m}$-dimensional semi-Riemannian manifold $(\widetilde{M}, \tilde{g})$ and $S(TM)$ be integrable. Then the bounded sectional curvature map is well defined, bounded, and independent of the choice of basis on Π.*

Proof Let $\{e_{a_1} \wedge e_{a_2} : a_1 < a_2\}$ be a basis $\wedge^2 S(TM)$. Suppose that $\Pi = e_{a_1} \wedge e_{a_2} = e_{a'_1} \wedge e_{a'_2}$. Then one can write

$$e_{a'_1} = ae_{a_1} + be_{a_2},$$
$$e_{a'_2} = ce_{a_1} + de_{a_2},$$

with $ad - bc \neq 0$. Here, it is clear that the area obeys

$$G(e_{a'_1} \wedge e_{a'_2}, e_{a'_1} \wedge e_{a'_2}) = (ad - bc)^2 G(e_{a_1} \wedge e_{a_2}, e_{a_1} \wedge e_{a_2}).$$

Since $S(TM)$ is integrable and R is symmetric, we have

$$R(e_{a'_1}, e_{a'_2}, e_{a'_2}, e_{a'_1}) = (ad - bc)^2 R(e_{a_1}, e_{a_2}, e_{a_2}, e_{a_1}),$$

which implies that $K(\Pi)$ is independent of the choice of basis on Π, it is well defined and both bounded from above or bounded from below. □

4 Special lightlike submanifolds

We begin this section with the following definition of [53, 54].

Definition 2 Let $(\widetilde{M}, \tilde{g})$ be an $\widetilde{m}$-dimensional semi-Riemannian manifold of index $\tilde{q}$. A distribution on $\widetilde{M}$ is called *maximally timelike* if it is timelike and has rank $\tilde{q}$. A distribution on $\widetilde{M}$ is called *maximally spacelike* if it is spacelike and has rank $(\widetilde{m} - \tilde{q})$.

Now, we recall the following theorem and proposition of Baum in [53].

Theorem 3 (Existence of maximally timelike-spacelike distributions) *Let $(\widetilde{M}, \tilde{g})$ be a semi-Riemannian manifold. Then there is a $\tilde{g}$-orthogonal decomposition such that $T\widetilde{M} = \widetilde{\mathcal{V}} \oplus_{\text{orth}} \widetilde{\mathcal{H}}$, where $\widetilde{\mathcal{V}}$ is a maximally timelike and $\widetilde{\mathcal{H}}$ is a maximally spacelike distribution on $\widetilde{M}$.*

Proposition 2 (Maximally timelike-spacelike distributions are isomorphic) *Let $(\widetilde{M}, \tilde{g})$ be a semi-Riemannian manifold. Every maximally timelike (or spacelike) distributions on $\widetilde{M}$ are isomorphic as smooth vector bundles over $\widetilde{M}$.*

Let $(M, g, S(TM))$ be an $(n+r)$-dimensional r-lightlike submanifold and $S(TM)$ be an integrable distribution of index q. Consider $\{e_1, \dots, e_q, e_{q+1}, \dots, e_n\}$ to be an orthonormal basis of $S(TM)$. Then there exists a g-orthogonal decomposition given by

$$S(TM) = \mathcal{V} \oplus_{\mathrm{orth}} \mathcal{H}, \tag{30}$$

where $\mathcal{V} = \mathrm{Span}\{e_1, \dots, e_q\}$ is the maximally timelike distribution and $\mathcal{H} = \mathrm{Span}\{e_{q+1}, \dots, e_n\}$ is the maximally spacelike distribution.

The aforementioned concepts can be constructed on the coscreen distribution $S(TM^\perp)$. Let $\widetilde{\mathcal{V}}$ be a maximally timelike and $\widetilde{\mathcal{V}}$ be a maximally spacelike distributions on $S(TM^\perp)$. Then there exists also a $\tilde{g}$-orthogonal decomposition of $S(TM^\perp)$ given by

$$S(TM^\perp) = \widetilde{\mathcal{V}} \oplus_{\mathrm{orth}} \widetilde{\mathcal{H}}, \tag{31}$$

where $\widetilde{\mathcal{V}} = \mathrm{Span}\{\tilde{e}_1, \dots, \tilde{e}_{\tilde{q}}\}$, $\widetilde{\mathcal{H}} = \mathrm{Span}\{\tilde{e}_{\tilde{q}+1}, \dots, \tilde{e}_m\}$.

From (11), we can write

$$h(X,Y) = \sum_{l=1}^{r} B^l(X,Y)N_l + h^{\widetilde{\mathcal{V}}}(X,Y) + h^{\widetilde{\mathcal{H}}}(X,Y), \tag{32}$$

where

$$h^{\widetilde{\mathcal{V}}}(X,Y) = \sum_{\alpha=1}^{\tilde{q}} D^\alpha(X,Y)\tilde{e}_\alpha \quad \text{and} \quad h^{\widetilde{\mathcal{H}}}(X,Y) \sum_{\alpha=\tilde{q}+1}^{m} D^\alpha(X,Y)\tilde{e}_\alpha \tag{33}$$

for all $X, Y \in TM$.

Now, we shall state some special r-lightlike submanifolds definitions.

Definition 3 Let $(M, g, S(TM))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \tilde{g})$ of index $(q+\tilde{q})$ and $S(TM)$ be an integrable distribution of index q. The submanifold will be called:

1. *Timelike $\mathcal{V}$-geodesic* if $h^s|^{\widetilde{\mathcal{V}}}_{\mathcal{V}\times\mathcal{V}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X, Y \in \mathcal{V}$ and $\alpha \in \{1, \dots, \tilde{q}\}$.
2. *Timelike $\mathcal{H}$-geodesic* if $h^s|^{\widetilde{\mathcal{V}}}_{\mathcal{H}\times\mathcal{H}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X, Y \in \mathcal{H}$ and $\alpha \in \{1, \dots, \tilde{q}\}$.
3. *Timelike mixed geodesic* if $h^s|^{\widetilde{\mathcal{V}}}_{\mathcal{V}\times\mathcal{H}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X \in \mathcal{V}$, $Y \in \mathcal{H}$ and $\alpha \in \{1, \dots, \tilde{q}\}$.
4. *Timelike geodesic* if $h^{\widetilde{\mathcal{V}}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X, Y \in TM$ and $\alpha \in \{1, \dots, \tilde{q}\}$.
5. *Timelike screen geodesic* if $h^*|_{\mathcal{V}} = 0$, *i.e.*, $C^l(X,Y) = 0$ for all $X, Y \in \mathcal{V}$ and $l \in \{1, \dots, r\}$.
6. *Spacelike $\mathcal{V}$-geodesic* if $h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{V}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X, Y \in \mathcal{V}$ and $\alpha \in \{\tilde{q}+1, \dots, m\}$.
7. *Spacelike $\mathcal{H}$-geodesic* if $h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{H}\times\mathcal{H}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X, Y \in \mathcal{H}$ and $\alpha \in \{\tilde{q}+1, \dots, m\}$.
8. *Spacelike mixed geodesic* if $h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{H}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X \in \mathcal{V}$, $Y \in \mathcal{H}$, and $\alpha \in \{\tilde{q}+1, \dots, m\}$.

9. *Spacelike geodesic* if $h^{\widetilde{\mathcal{H}}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X, Y \in TM$ and $\alpha \in \{\tilde{q}+1,\dots,m\}$.
10. *Spacelike screen geodesic* if $h^*|_{\mathcal{H}} = 0$, *i.e.*, $C^l(X,Y) = 0$ for all $X, Y \in \mathcal{H}$ and $l \in \{1,\dots,r\}$.
11. *Mixed geodesic* if $h^s|_{\mathcal{V}\times\mathcal{H}} = 0$, *i.e.*, $D^\alpha(X,Y) = 0$ for all $X \in \mathcal{V}$, $Y \in \mathcal{H}$ and $\alpha \in \{1,\dots,m\}$.
12. *Mixed screen geodesic* if $h^*|_{\mathcal{V}\times\mathcal{H}} = 0$, *i.e.*, $C^l(X,Y) = 0$ for all $X, Y \in TM$ and $l \in \{1,\dots,r\}$.

We also note that the submanifold is:
1. timelike geodesic if and only if $h^s|^{\widetilde{\mathcal{V}}}_{\mathcal{V}\times\mathcal{V}} = h^s|^{\widetilde{\mathcal{V}}}_{\mathcal{H}\times\mathcal{H}} = h^s|^{\widetilde{\mathcal{V}}}_{\mathcal{V}\times\mathcal{H}} = 0$,
2. spacelike geodesic if and only if $h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{V}} = h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{H}\times\mathcal{H}} = h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{H}} = 0$,
3. mixed geodesic if and only if $h^s|^{\widetilde{\mathcal{V}}}_{\mathcal{V}\times\mathcal{H}} = h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{H}} = 0$.

In view of Definition 3, we give the following proposition.

Proposition 3 *Let $(M, g, S(TM))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \tilde{g})$ of index $(q+\tilde{q})$ and $S(TM)$ be an integrable distribution of index q. Then the following statements are true*:

(a) *The submanifold is timelike $\mathcal{V}$-geodesic and timelike $\mathcal{H}$-geodesic, then the mean curvature vector on $\Gamma(S(TM))$ is spacelike.*

(b) *The submanifold is spacelike $\mathcal{V}$-geodesic and spacelike $\mathcal{H}$-geodesic, then the mean curvature vector on $\Gamma(S(TM))$ is timelike.*

Example 1 Let us consider the submanifold M of the semi-Euclidean space $\mathbb{R}^8_4$ with the signature $(-,-,-,-,+,+,+,+)$ given by

$$\phi(x_1,x_2,x_3,x_4) = \left(\frac{1}{\sqrt{2}}x_1, \cos x_2, \sin x_2, \sinh x_3, \cosh x_3, \frac{1}{\sqrt{2}}x_1, \frac{1}{\sqrt{2}}x_4, \frac{1}{\sqrt{2}}x_4\right)$$

for all $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$. Then we have

$$\begin{aligned}
\xi_1 &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_6}, & e_1 &= -\sin x_2\frac{\partial}{\partial x_2} + \cos x_2\frac{\partial}{\partial x_3}, \\
e_2 &= \cosh x_3\frac{\partial}{\partial x_4} + \sinh x_3\frac{\partial}{\partial x_5}, & e_3 &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_7} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_8}, \\
N_1 &= -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_6}, & e_4 &= \cos x_2\frac{\partial}{\partial x_2} + \sin x_2\frac{\partial}{\partial x_3}, \\
e_5 &= \sinh x_3\frac{\partial}{\partial x_4} + \cosh x_3\frac{\partial}{\partial x_5}, & e_6 &= \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_7} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_8}.
\end{aligned}$$

It is easy to see that e_1, e_2, e_4 are timelike unit vectors, e_3, e_5, e_6 are spacelike unit vectors and M is a 1-lightlike submanifold with screen distribution $S(TM) = \mathrm{Span}\{e_1,e_2,e_3\}$, $\mathrm{Rad}(TM) = \mathrm{Span}\{\xi_1\}$, $\mathrm{ltr}(TM) = \mathrm{Span}\{N_1\}$, and $S(TM^\perp) = \mathrm{Span}\{e_4,e_5,e_6\}$.

Here, we have also

$$S(TM) = \mathcal{V} \oplus_{\mathrm{orth}} \mathcal{H},$$

where $\mathcal{V} = \text{Span}\{e_1, e_2\}$ and $\mathcal{H} = \text{Span}\{e_3\}$ and

$$S(TM^{\perp}) = \widetilde{\mathcal{V}} \oplus_{\text{orth}} \widetilde{\mathcal{H}},$$

where $\widetilde{\mathcal{V}} = \text{Span}\{e_4\}$ and $\widetilde{\mathcal{H}} = \text{Span}\{e_5, e_6\}$. By a straightforward computation, we have $B = C = 0$ and

$$\begin{aligned}
&D_{11}^4 = 1, \qquad D_{11}^5 = D_{11}^6 = 0, \\
&D_{12}^4 = D_{12}^5 = D_{12}^6 = D_{13}^4 = D_{13}^5 = D_{13}^6 = 0, \\
&D_{22}^5 = 1, \qquad D_{22}^4 = D_{22}^6 = D_{23}^4 = D_{23}^5 = D_{23}^6 = 0, \\
&D_{33}^4 = D_{33}^5 = D_{33}^6 = 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left|h|_{\mathcal{V}\times\mathcal{V}}^{\widetilde{\mathcal{V}}}\right|^2 \neq 0, \qquad \left|h|_{\mathcal{H}\times\mathcal{H}}^{\widetilde{\mathcal{V}}}\right|^2 = 0, \qquad \left|h|_{\mathcal{V}\times\mathcal{H}}^{\widetilde{\mathcal{V}}}\right|^2 = 0, \\
&\left|h|_{\mathcal{V}\times\mathcal{V}}^{\widetilde{\mathcal{H}}}\right|^2 \neq 0, \qquad \left|h|_{\mathcal{H}\times\mathcal{H}}^{\widetilde{\mathcal{H}}}\right|^2 = 0, \qquad \left|h|_{\mathcal{V}\times\mathcal{H}}^{\widetilde{\mathcal{H}}}\right|^2 = 0,
\end{aligned}$$

which shows that the submanifold is not timelike $\mathcal{V}$-geodesic and spacelike $\mathcal{V}$-geodesic but it is timelike $\mathcal{H}$-geodesic, spacelike $\mathcal{H}$-geodesic, and mixed geodesic.

Similarly, examples for the other cases can be given.

5 Some relations for *r*-lightlike submanifolds

We begin this section with the following definition.

Definition 4 Let $(M, g, S(TM))$ be an $(n + r)$-dimensional r-lightlike submanifold of a semi-Riemannian manifold and $S(TM)$ be an integrable distribution of index q. *The bounded screen Ricci tensor*, denoted by $\text{Ric}_{S(TM)}$, is defined by

$$\text{Ric}_{S(TM)}(X, Y) = \text{tr}\{Z \to R(X, Z)Y\} \tag{34}$$

for any $X, Y \in \Gamma(S(TM))$.

Suppose $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. *The bounded screen Ricci curvature* at a unit vector $e_i \in \Gamma(S(TM))$, denoted by $\text{Ric}_{S(TM)}(e_i)$, is given by

$$\text{Ric}_{S(TM)}(e_i) = \sum_{j\neq i=1}^{n} R(e_i, e_j, e_j, e_i) = \sum_{j\neq i=1}^{n} K_{ij}. \tag{35}$$

We note that:

(a) If $n = 1$, then the bounded screen Ricci curvature vanishes identically.

(b) If $n = 2$, then the bounded screen Ricci curvature becomes the bounded sectional curvature.

Remark 1 We note that the screen Ricci curvature is bounded when the screen distribution of a lightlike submanifold is Riemannian. This map was first of all introduced by

Duggal in [55] and named by the authors in [56, 57] in the case of a lightlike hypersurface of a Lorentzian manifold in which we know that $S(TM)$ is Riemannian.

Theorem 4 *Let $(M, g, S(TM))$ be an $(r+3)$-dimensional r-lightlike submanifold of a semi-Riemannian manifold and $S(TM)$ be an integrable distribution. The bounded screen Ricci curvature is constant at every unit vector on $\Gamma(S(TM))$ if and only if the bounded sectional curvature is constant.*

Proof Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\Gamma(S(TM))$. If $\mathrm{Ric}_{S(TM)}$ is constant, then we can write

$$\begin{aligned}
&\mathrm{Ric}_{S(TM)}(e_1) = K_{12} + K_{13} = \lambda, \\
&\mathrm{Ric}_{S(TM)}(e_2) = K_{21} + K_{23} = \lambda, \\
&\mathrm{Ric}_{S(TM)}(e_3) = K_{31} + K_{32} = \lambda,
\end{aligned}$$

where λ is a constant. Thus, we have

$$K_{12} = \frac{1}{2}\left[\mathrm{Ric}_{S(TM)}(e_1) + \mathrm{Ric}_{S(TM)}(e_2) - \mathrm{Ric}_{S(TM)}(e_3)\right] = \frac{1}{2}\lambda,$$

which shows that K_{12} is constant. The converse part of this theorem is straightforward. □

Taking the trace in (18) with respect to $S(TM)$ and putting (35) in it, we have the following result.

Lemma 1 *Let $(M, g, S(TM))$ be an $(n+r)$-dimensional r-lightlike submanifold of an $\widetilde{m}$-dimensional semi-Riemannian manifold of index $(q+\tilde{q})$ and $S(TM)$ be an integrable distribution. Suppose $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. For any unit vector $X \in \Gamma(S(TM))$, we have*

$$\mathrm{Ric}_{S(TM)}(X) = \widetilde{\mathrm{Ric}}_{S(TM)}(X) + S(X), \tag{36}$$

where

$$\widetilde{\mathrm{Ric}}_{S(TM)}(X) = \varepsilon \sum_{j=1}^{n} \varepsilon_j \widetilde{R}(X, e_j, e_j, X), \qquad g(X, X) = \varepsilon = \mp 1 \tag{37}$$

and

$$\begin{aligned}
S(X) = \varepsilon \Bigg[&\sum_{j=1}^{n} \varepsilon_j \Bigg[\sum_{l=1}^{r} B^l(e_j, e_j) C^l(X, X) - \sum_{l=1}^{r} B^\ell(X, e_j) C^\ell(e_j, X) \Bigg] \\
&- \sum_{j=1}^{n} \varepsilon_j \Bigg[\sum_{\alpha=1}^{m} \varepsilon_\alpha D^\alpha(X, e_j) D^\alpha(e_j, X) - D^\alpha(e_j, e_j) D^\alpha(X, X) \Bigg] \Bigg].
\end{aligned} \tag{38}$$

Here, $\widetilde{\mathrm{Ric}}_{S(TM)}$ is the Ricci curvature of n-plane section (screen distribution) of $\widetilde{M}$ given in [21].

Theorem 5 *Let $(M, g, S(TM))$ be an $(n+r)$-dimensional minimal r-lightlike submanifold of an $\widetilde{m}$-dimensional semi-Riemannian space form $\widetilde{M}(c)$ and $S(TM)$ be an integrable distribution. For any spacelike unit vector $X \in \Gamma(S(TM))$, we have*:

(a)

$$\mathrm{Ric}_{S(TM)}(X) \leq (n-1)c + \left|h^{\ell}|_{\mathcal{H}_1\times\mathcal{V}}\right|\left|h^{*}|_{\mathcal{H}_1\times\mathcal{V}}\right| - \left|h^{\ell}|_{\mathcal{H}_1\times\mathcal{H}}\right|\left|h^{*}|_{\mathcal{H}_1\times\mathcal{H}}\right| + \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{H}_1\times\mathcal{V}}\right|^2 + \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{H}_1\times\mathcal{H}}\right|^2 \tag{39}$$

and

$$\mathrm{Ric}_{S(TM)}(X) \geq (n-1)c + \left|h^{\ell}|_{\mathcal{H}_1\times\mathcal{V}}\right|\left|h^{*}|_{\mathcal{H}_1\times\mathcal{V}}\right| - \left|h^{\ell}|_{\mathcal{H}_1\times\mathcal{H}}\right|\left|h^{*}|_{\mathcal{H}_1\times\mathcal{H}}\right| - \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{H}_1\times\mathcal{V}}\right|^2 - \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{H}_1\times\mathcal{H}}\right|^2, \tag{40}$$

where $\mathcal{H}_1 = \mathrm{Span}\{X\}$.

(b) *The equality cases of both the inequalities* (39) *and* (40) *are true simultaneously for all spacelike vector $X \in \Gamma(S(TM))$ if and only if D vanishes on $S(TM)$.*

Proof (a) From (36) and (38) we get

$$\begin{aligned}\mathrm{Ric}_{S(TM)}(e_i) = \sum_{j=1}^{n}\varepsilon_i\varepsilon_j\Bigg[&\sum_{l=1}^{r}B^{l}(e_j,e_j)C^{l}(e_i,e_i) - \sum_{l=1}^{r}B^{\ell}(e_i,e_j)C^{\ell}(e_j,e_i)\\ &+\sum_{\alpha=1}^{m}\varepsilon_\alpha D^{\alpha}(e_j,e_j)D^{\alpha}(e_i,e_i) - D^{\alpha}(e_i,e_j)D^{\alpha}(e_j,e_i)\Bigg]\\ &+(n-1)c.\end{aligned} \tag{41}$$

Since M is minimal we obtain

$$\begin{aligned}\mathrm{Ric}_{S(TM)}(X) = (n-1)c &+ \left|h^{\ell}|_{\mathcal{H}_1\times\mathcal{V}}\right|\left|h^{*}|_{\mathcal{H}_1\times\mathcal{V}}\right| - \left|h^{\ell}|_{\mathcal{H}_1\times\mathcal{H}}\right|\left|h^{*}|_{\mathcal{H}_1\times\mathcal{H}}\right|\\ &+ \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{H}_1\times\mathcal{V}}\right|^2 + \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{H}_1\times\mathcal{H}}\right|^2 - \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{H}_1\times\mathcal{V}}\right|^2 - \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{H}_1\times\mathcal{H}}\right|^2.\end{aligned} \tag{42}$$

Taking into consideration (42), we have both the inequalities (39) and (40).

(b) The equality cases of both (39) and (40) inequalities are true simultaneously for all spacelike vector $X \in \Gamma(S(TM))$ if and only if

$$\left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{H}_1\times\mathcal{V}}\right| = \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{H}_1\times\mathcal{H}}\right| = \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{H}_1\times\mathcal{V}}\right| = \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{H}_1\times\mathcal{H}}\right| = 0, \tag{43}$$

which implies that D vanishes on $S(TM)$. □

With similar arguments as in the proof of Theorem 5, we obtain the following theorem.

Theorem 6 *Let $(M, g, S(TM))$ be an $(n+r)$-dimensional minimal r-lightlike submanifold of an $\widetilde{m}$-dimensional semi-Riemannian space form $\widetilde{M}(c)$ and $S(TM)$ be an integrable distribution. For any timelike unit vector $Y \in \Gamma(S(TM))$, we have*:

(a)

$$\begin{aligned}\mathrm{Ric}_{S(TM)}(Y) &\leq (n-1)c - \left|h^{\ell}|_{\mathcal{V}_1\times\mathcal{V}}\right|\left|h^{*}|_{\mathcal{V}_1\times\mathcal{V}}\right| + \left|h^{\ell}|_{\mathcal{V}_1\times\mathcal{H}}\right|\left|h^{*}|_{\mathcal{V}_1\times\mathcal{H}}\right| \\ &\quad + \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{V}_1\times\mathcal{V}}\right|^2 + \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{V}_1\times\mathcal{H}}\right|^2 \end{aligned} \tag{44}$$

and

$$\begin{aligned}\mathrm{Ric}_{S(TM)}(Y) &\geq (n-1)c - \left|h^{\ell}|_{\mathcal{V}_1\times\mathcal{V}}\right|\left|h^{*}|_{\mathcal{V}_1\times\mathcal{V}}\right| + \left|h^{\ell}|_{\mathcal{V}_1\times\mathcal{H}}\right|\left|h^{*}|_{\mathcal{V}_1\times\mathcal{H}}\right| \\ &\quad - \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{V}_1\times\mathcal{V}}\right|^2 - \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{V}_1\times\mathcal{H}}\right|^2, \end{aligned} \tag{45}$$

where $\mathcal{V}_1 = \mathrm{Span}\{Y\}$.

(b) *The equality cases of both the inequalities* (44) *and* (45) *are true simultaneously for all timelike vector* $X \in \Gamma(S(TM))$ *if and only if* D *vanishes on* $S(TM)$.

Now, we give the following definition.

Definition 5 Let $(M, g, S(TM))$ be an $(n + r)$-dimensional r-lightlike submanifold of semi-Riemannian manifold and $S(TM)$ be an integrable distribution of index q. Suppose $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. *The bounded screen scalar curvature* at a point $p \in M$, denoted by $r_{S(TM)}(p)$, is given by

$$r_{S(TM)}(p) = \frac{1}{2}\sum_{i,j=1}^{n} K_{ij}. \tag{46}$$

With similar arguments to the proof of Theorem 4.7 in [56], we have the following proposition immediately.

Proposition 4 *Let* $(M, g, S(TM))$ *be a* $(2n + r)$*-dimensional* r*-lightlike submanifold and* $S(TM)$ *be an integrable distribution. Then the bounded screen Ricci curvature is constant if and only if*

$$r_{S(TM)}(\pi_n) = r_{S(TM)}(\pi_n^{\perp}), \tag{47}$$

where π_n *is an* n*-dimensional non-degenerate sub-plane section of* $\Gamma(S(TM))$ *and* $\pi_n^{\perp}$ *is complementary vector bundle of* π_n *in* $\Gamma(S(TM))$.

Taking the trace in equation (36), we have the following result.

Lemma 2 *Let* $(M, g, S(TM))$ *be an* $(n + r)$*-dimensional* r*-lightlike submanifold and* $S(TM)$ *be an integrable distribution. Then we have*

$$\begin{aligned} 2r_{S(TM)}(p) &= 2\tilde{r}_{S(TM)}(p) + n\mu_1 \sum_{\ell=1}^{r}(\mathrm{trace}\, A_{N_\ell}) + n\mu_2 \sum_{\alpha=1}^{m}(\mathrm{trace}\, A_{u_\alpha}) \\ &\quad + 2\left|h^{\ell}|_{\mathcal{V}\times\mathcal{H}}\right|\left|h^{*}|_{\mathcal{V}\times\mathcal{H}}\right| - \left|h^{\ell}|_{\mathcal{V}\times\mathcal{V}}\right|\left|h^{*}|_{\mathcal{V}\times\mathcal{V}}\right| \\ &\quad - \left|h^{\ell}|_{\mathcal{H}\times\mathcal{H}}\right|\left|h^{*}|_{\mathcal{H}\times\mathcal{H}}\right| + \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{V}\times\mathcal{V}}\right|^2 + \left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{H}\times\mathcal{H}}\right|^2 \\ &\quad - \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{V}}\right|^2 - \left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{H}\times\mathcal{H}}\right|^2 + 2\left|h^{s}|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{H}}\right|^2 - 2\left|h^{s}|^{\widetilde{\mathcal{V}}}_{\mathcal{V}\times\mathcal{H}}\right|^2, \end{aligned} \tag{48}$$

where

$$\tilde{r}_{S(TM)}(p) = \frac{1}{2}\sum_{i,j=1}^{n}\widetilde{K}_{ij}. \tag{49}$$

Here, $\tilde{r}_{S(TM)}(e_i)$ is the scalar curvature of n-plane section (screen distribution) of $\widetilde{M}$ given in [21].

Theorem 7 *Let $(M, g, S(TM))$ be an $(n + r)$-dimensional r-lightlike submanifold of a semi-Riemannian space form $\widetilde{M}(c)$ and $S(TM)$ be an integrable distribution. Then we have*:

(a)

$$\begin{aligned} 2r_{S(TM)}(p) \le{}& n(n-1)c + n\mu_1\sum_{\ell=1}^{r}(\operatorname{trace} A_{N_\ell}) + n\mu_2\sum_{\alpha=1}^{m}(\operatorname{trace} A_{u_\alpha}) \\ &+ 2\big|h^\ell|_{\mathcal{V}\times\mathcal{H}}\big|\big|h^*|_{\mathcal{V}\times\mathcal{H}}\big| - \big|h^\ell|_{\mathcal{V}\times\mathcal{V}}\big|\big|h^*|_{\mathcal{V}\times\mathcal{V}}\big| \\ &- \big|h^\ell|_{\mathcal{H}\times\mathcal{H}}\big|\big|h^*|_{\mathcal{H}\times\mathcal{H}}\big| + \big|h^s|_{\mathcal{V}\times\mathcal{V}}^{\widetilde{\mathcal{V}}}\big|^2 + \big|h^s|_{\mathcal{H}\times\mathcal{H}}^{\widetilde{\mathcal{V}}}\big|^2 \\ &+ 2\big|h^s|_{\mathcal{V}\times\mathcal{H}}^{\widetilde{\mathcal{H}}}\big|^2. \end{aligned} \tag{50}$$

The equality case of (50) *is true for all $p \in M$ if and only if M is spacelike $\mathcal{V}$-geodesic, spacelike $\mathcal{H}$-geodesic and timelike mixed geodesic.*

(b)

$$\begin{aligned} 2r_{S(TM)}(p) \ge{}& n(n-1)c + n\mu_1\sum_{\ell=1}^{r}(\operatorname{trace} A_{N_\ell}) + n\mu_2\sum_{\alpha=1}^{m}(\operatorname{trace} A_{u_\alpha}) \\ &+ 2\big|h^\ell|_{\mathcal{V}\times\mathcal{H}}\big|\big|h^*|_{\mathcal{V}\times\mathcal{H}}\big| - \big|h^\ell|_{\mathcal{V}\times\mathcal{V}}\big|\big|h^*|_{\mathcal{V}\times\mathcal{V}}\big| \\ &- \big|h^\ell|_{\mathcal{H}\times\mathcal{H}}\big|\big|h^*|_{\mathcal{H}\times\mathcal{H}}\big| - \big|h^s|_{\mathcal{V}\times\mathcal{V}}^{\widetilde{\mathcal{H}}}\big|^2 - \big|h^s|_{\mathcal{H}\times\mathcal{H}}^{\widetilde{\mathcal{H}}}\big|^2 \\ &- 2\big|h^s|_{\mathcal{V}\times\mathcal{H}}^{\widetilde{\mathcal{V}}}\big|^2. \end{aligned} \tag{51}$$

The equality case of (51) *is true for all $p \in M$ if and only if M is timelike $\mathcal{V}$-geodesic, timelike $\mathcal{H}$-geodesic and spacelike mixed geodesic.*

Now, we recall a class of r-lightlike submanifolds of a semi-Riemannian manifold of an arbitrary signature which admits an integrable unique screen distribution as follows.

Definition 6 [47] An r-lightlike submanifold is called a *screen locally conformal* if

$$C^\ell(X,Y) = \varphi_\ell B^\ell(X,Y), \quad \forall X,Y \in \Gamma(TM|_{\mathcal{U}}), \ell \in \{1,\dots,r\}, \tag{52}$$

where each φ_ℓ is a conformal smooth function on a neighborhood $\mathcal{U}$ in M. If each φ_ℓ is a non-zero constant, then the submanifold is called *screen homothetic*.

Lemma 3 [39] *If $a_1,\dots,a_n$ are n-real numbers $(n > 1)$, then*

$$\frac{1}{n}\left(\sum_{i=1}^{n}a_i\right)^2 \le \sum_{i=1}^{n}a_i^2, \tag{53}$$

with equality if and only if $a_1 = \cdots = a_n$.

Theorem 8 *Let $(M, g, S(TM))$ be an $(n + r)$-dimensional screen conformal $(\varphi_\ell > 0)$ r-lightlike submanifold of an $\widetilde{m}$-dimensional semi-Riemannian space form $\widetilde{M}(c)$, $S(TM)$ be an integrable distribution of index q and $S(TM^\perp)$ be Riemannian. Then we have*

$$2r_{S(TM)}(p) \leq n(n-1)c + \sum_{\ell=1}^{r} n^2\varphi_\ell\mu_1^2 + n\mu_2\sum_{\alpha=1}^{m}(\operatorname{trace}A_{u_\alpha}) - q\mu_1^2|_{\mathcal{V}} - (n-q)\mu_1^2|_{\mathcal{H}} + 2\varphi_\ell\left|h^\ell|_{\mathcal{V}\times\mathcal{H}}\right|^2 + 2\left|h^s|^{\widetilde{\mathcal{H}}}_{\mathcal{V}\times\mathcal{H}}\right|^2. \tag{54}$$

The equality case of (54) is true for all $p \in M$ if and only if $h^\ell(X, X) = h^\ell(Y, Y)$ and $h^s(X, Y) = 0$ for all two timelike or spacelike vectors $X, Y \in \Gamma(S(TM))$.

Proof Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. If $S(TM^\perp)$ is a Riemannian distribution, then we have $\widetilde{\mathcal{V}} = 0$. From equation (52), it follows that

$$h^*(X, Y) = \varphi_\ell h^\ell(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{55}$$

Taking into account Lemma 3 and equation (55), we get

$$\mu_1\sum_{\ell=1}^{n}\operatorname{trace}A_{N_\ell} = \sum_{\ell=1}^{n} n^2\varphi_\ell\mu_1^2, \tag{56}$$

$$q\mu_1^2|_{\mathcal{V}} \leq \left|h^\ell|_{\mathcal{V}\times\mathcal{V}}\right|\left|h^*|_{\mathcal{V}\times\mathcal{V}}\right|, \tag{57}$$

$$(n-q)\mu_1^2|_{\mathcal{H}} \leq \left|h^\ell|_{\mathcal{H}\times\mathcal{H}}\right|\left|h^*|_{\mathcal{H}\times\mathcal{H}}\right|, \tag{58}$$

where

$$\mu_1|_{\mathcal{V}} = -\frac{1}{q}\big(B(e_1, e_1) + \cdots + B(e_q, e_q)\big)$$

and

$$\mu_1|_{\mathcal{H}} = \frac{1}{n-q}\big(B(e_{q+1}, e_{q+1}) + \cdots + B(e_n, e_n)\big).$$

If we put (56), (57), and (58) in (48), we obtain the inequality (54).

Assuming the equality case of (54), in view of Lemma 3 in (57) and (58), for each $\ell \in \{1, \dots, r\}$, we have

$$B^\ell(e_1, e_1) = \cdots = B^\ell(e_q, e_q), B^\ell(e_{q+1}, e_{q+1}) = \cdots = B^\ell(e_n, e_n),$$

and for each $i, j \in \{1, \dots, q\}$, $a, b \in \{q+1, \dots, n\}$, $\alpha \in \{1, \dots, m\}$, we have

$$D^\ell(e_i, e_j) = D^\ell(e_a, e_b) = 0.$$

This completes the proof of the theorem. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Arts and Sciences, İnönü University, Malatya, 44280, Turkey. [2]Department of Mathematics, Faculty of Arts and Sciences, Siirt University, Siirt, 56100, Turkey.

Acknowledgements
The authors are thankful to the referees for their valuable comments toward the improvement of the paper. The first author of this study is supported by 113F388 coded project of the Scientific and Technological Research Council of Turkey (TÜBİTAK).

Received: 28 October 2015 Accepted: 1 February 2016 Published online: 12 February 2016

References

1. Kulkarni, RS: The values of sectional curvature in indefinite metric. Comment. Math. Helv. **54**, 173-176 (1979)
2. Nomizu, K: Remarks on sectional curvature of an indefinite metrics. Proc. Am. Math. Soc. **89**(3), 473-476 (1983)
3. Dajczer, M, Nomuzi, K: On the boundedness of Ricci curvature of an indefinite metric. Bol. Soc. Bras. Mat. **11**, 25-30 (1980)
4. Harris, SG: A triangle comparison theorem for Lorentz manifolds. Indiana Univ. Math. J. **31**, 289-308 (1982)
5. Aktan, N, Sarikaya, MZ, Özüsağlam, E: B.Y. Chen's inequality for semi-slant submanifolds in T-space forms. Balk. J. Geom. Appl. **13**(1), 1-10 (2008)
6. Alegre, P, Chen, B-Y, Munteanu, MI: Riemannian submersions, δ-invariants, and optimal inequality. Ann. Glob. Anal. Geom. **42**(3), 317-331 (2012)
7. Arslan, K, Ezentas, R, Mihai, I, Murathan, C, Özgür, C: Certain inequalities for submanifolds in (κ,μ)-contact space forms. Bull. Aust. Math. Soc. **64**(2), 201-212 (2001)
8. Carriazo, A, Fernández, LM, Hans-Uber, MBBY: Chen's inequality for S-space forms: applications to slant immersions. Indian J. Pure Appl. Math. **34**(9), 1287-1298 (2003)
9. Chen, B-Y: Some pinching and classification theorems for minimal submanifolds. Arch. Math. **60**, 568-578 (1993)
10. Chen, B-Y: A Riemannian invariant and its application to submanifolds theory. Results Math. **27**, 17-26 (1995)
11. Chen, B-Y: Mean curvature and shape operator of isometric immersions in real space forms. Glasg. Math. J. **38**(1), 87-97 (1996)
12. Chen, B-Y: Pseudo-Riemannian Geometry, δ-Invariants and Applications. World Scientific, Hackensack (2011)
13. Chen, B-Y, Dillen, F, Verstraelen, L: δ-Invariants and their applications to centroaffine geometry. Differ. Geom. Appl. **22**(3), 341-354 (2005)
14. Chen, B-Y, Dillen, F, Van der Veken, J, Vrancken, L: Curvature inequalities for Lagrangian submanifolds: the final solution. Differ. Geom. Appl. **31**(6), 808-819 (2013)
15. Costache, S, Zamfir, I: An improved Chen-Ricci inequality for special slant submanifolds in Kenmotsu space forms. Ann. Pol. Math. **110**, 81-89 (2014)
16. De Smet, PJ, Dillen, F, Verstraelen, L, Vrancken, L: A pointwise inequality in submanifold theory. Arch. Math. **35**, 115-128 (1999)
17. Decu, S, Haesen, S, Verstraelen, L: Optimal inequalities involving Casorati curvatures. Bull. Transilv. Univ. Brasov, Ser. B **14**(49), 85-93 (2007)
18. Deng, S: Improved Chen-Ricci inequality for Lagrangian submanifolds in quaternion space forms. Int. Electron. J. Geom. **5**(1), 163-170 (2012)
19. Dillen, F, Petrovic, M, Verstraelen, L: Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen's equality. Isr. J. Math. **100**(1), 163-169 (1997)
20. Gupta, RS, Ahmad, I, Haider, SMK: B.Y. Chen's inequality and its application to slant immersions into Kenmostu manifolds. Kyungpook Math. J. **44**, 101-110 (2004)
21. Hong, S, Matsumoto, K, Tripathi, MM: Certain basic inequalities for submanifolds of locally conformal Kaehlerian space forms. SUT J. Math. **4**(1), 75-94 (2005)
22. Kim, YH, Kim, DS: A basic inequality for submanifolds in Sasakian space forms. Houst. J. Math. **25**, 247-257 (1999)
23. Lee, CW, Yoon, DW, Lee, JW: Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections. J. Inequal. Appl. **2014**, Article ID 327 (2014)
24. Lee, JW, Vilcu, GE: Inequalities for generalized normalized δ-Casorati curvatures of slant submanifolds in quaternionic space forms. Taiwan. J. Math. **19**(3), 691-702 (2015)
25. Li, G, Wu, C: Slant immersions of complex space forms and Chen's inequality. Acta Math. Sci., Ser. B **25**(2), 223-232 (2005)
26. Liu, X: On Ricci curvature of totally real submanifolds in a quaternion projective space. Arch. Math. **38**, 297-305 (2002)
27. Matsumoto, K, Mihai, I, Tazawa, Y: Ricci tensor of slant submanifolds in complex space forms. Kodai Math. J. **26**(1), 85-94 (2003)
28. Mihai, A: B.Y. Chen inequalities for slant submanifolds in generalized complex space forms. Rad. Mat. **12**, 215-231 (2004)
29. Mihai, A, Özgür, C: Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection. Taiwan. J. Math. **14**(4), 1465-1477 (2010)
30. Mihai, I, Al-Solamy, FR, Shahid, MH: On Ricci curvature of a quaternion CR-submanifold in a quaternion space form. Rad. Mat. **12**(1), 91-98 (2003)
31. Oiagă, A, Mihai, I: B.-Y. Chen inequalities for slant submanifolds in complex space forms. Demonstr. Math. **32**(4), 835-846 (1999)
32. Oprea, T: Chen's inequality in the Lagrangian case. Colloq. Math. **108**, 163-169 (2007)
33. Özgür, C, Arslan, K: On some class of hypersurfaces in E^{n+1} satisfying Chen's equality. Turk. J. Math. **26**, 283-293 (2002)
34. Sasahara, T: On Chen invariant of CR-submanifolds in a complex hyperbolic space. Tsukuba J. Math. **26**, 119-132 (2002)

35. Suceavă, B: Some remarks on B.Y. Chen's inequality involving classical invariants. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. **45**, 405-412 (1999)
36. Slesar, V, Şahin, B, Vilcu, GE: Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms. J. Inequal. Appl. **2014**, Article ID 123 (2014)
37. Shukla, SS, Rao, PK: Ricci curvature of quaternion slant submanifolds in quaternion space forms. Acta Math. Acad. Paedagog. Nyházi. **28**(1), 69-81 (2012)
38. Tripathi, MM, Kim, JS, Kim, SB: A note on Chen's basic equality for submanifolds in a Sasakian space form. Int. J. Math. Math. Sci. **2003**(11), 711-716 (2003)
39. Tripathi, MM: Certain basic inequalities for submanifolds in (κ,μ) space. In: Recent Advances in Riemannian and Lorentzian Geometries (Baltimore, MD, 2003). Contemp. Math., vol. 337, pp. 187-202. Am. Math. Soc., Providence (2003)
40. Uddin, S, Khan, KA: An inequality for contact CR-warped product submanifolds of nearly cosymplectic manifolds. J. Inequal. Appl. **2012**, Article ID 304 (2012)
41. Vilcu, GE: Inequalities for slant submanifolds in quaternionic space forms. Turk. J. Math. **34**(1), 115-128 (2010)
42. Vilcu, GE: On Chen invariant and inequalities in quaternionic geometry. J. Inequal. Appl. **2013**, Article ID 66 (2013)
43. Yoon, DW, Cho, KS: Inequality for warped products in generalized Sasakian space forms. Int. Math. J. **5**(3), 225-235 (2004)
44. Zhang, P, Zhang, L, Song, W: Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection. Taiwan. J. Math. **18**(6), 1841-1862 (2014)
45. Punzi, R, Schuller, FP, Wohlfarth, MNR: Geometric obstruction of black holes. Ann. Phys. **322**, 1335-1372 (2007)
46. Schuller, FP, Wohlfarth, MNR: Sectional curvature bounds in gravity: regularisation of the Schwarzchild singularity. Nucl. Phys. B **698**, 319-334 (2004)
47. Duggal, KL, Şahin, B: Differential Geometry of Lightlike Submanifolds. Birkhäuser, Basel (2010)
48. Duggal, KL, Jin, DH: Totally umbilical lightlike submanifolds. Kodai Math. J. **26**, 49-68 (2003)
49. Bejan, CL, Duggal, KL: Global lightlike manifolds and harmonicity. Kodai Math. J. **28**, 131-145 (2005)
50. Duggal, KL, Şahin, B: Generalized Cauchy-Riemann lightlike submanifolds. Acta Math. Hung. **112**(1-2), 113-136 (2006)
51. Duggal, KL, Bejancu, A: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Mathematics and Its Applications, vol. 364. Kluwer Academic, Dordrecht (1996)
52. Griffiths, P, Harris, J: Principles of Algebraic Geometry. Wiley-Interscience, New York (1978)
53. Baum, H: Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten (German) [English: Spin structures and Dirac operators on pseudo-Riemannian manifolds]. Teubner Texts in Math., vol. 41. Teubner, Leipzig (1981)
54. Nardmann, M: Pseudo-Riemannian metrics with prescribed scalar curvature. arXiv:math/0409435v2
55. Duggal, KL: On scalar curvature in lightlike geometry. J. Geom. Phys. **57**(2), 473-481 (2007)
56. Gülbahar, M, Kılıç, E, Keleş, S: Chen-like inequalities on lightlike hypersurfaces of a Lorentzian manifold. J. Inequal. Appl. **2013**, Article ID 266 (2013)
57. Gülbahar, M, Kılıç, E, Keleş, S: Some inequalities on screen homothetic lightlike hypersurfaces of a Lorentzian manifold. Taiwan. J. Math. **17**, 2083-2100 (2013)