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Existence of a solution for a three-point boundary value problem for a second-order differential equation at resonance

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Abstract

We present a new existence result for a second-order nonlinear ordinary differential equation with a three-point boundary value problem when the linear part is noninvertible.

MSC: Primary 34B10; secondary 34B15

Keywords: nonlinear ordinary differential equation; three-point boundary value problem; problem at resonance; existence of solution

1 Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations goes back to the method of separation of variables [1]. Also, some questions in the theory of elastic stability are related to multi-point problems [2]. In 1987, Il'in and Moiseev [3, 4] studied some nonlocal boundary value problems. Then, for example, Gupta [5] considered a three-point nonlinear boundary value problem. For some recent works on nonlocal boundary value problems, we refer, for example, to [6–15] and references therein.

As indicated in [16], there has been enormous interest in nonlinear perturbations of linear equations at resonance since the seminal paper of Landesman and Lazer [17]; see [18] for further details.

Here we study the following nonlinear ordinary differential equation of second order subject to the three-point boundary condition:

$$\begin{aligned} &-u''(t) = f\big(t, u(t)\big), \quad t \in [0, T], \\ &u(0) = 0, \qquad \alpha u(\eta) = u(T), \end{aligned} \tag{1}$$

where $T > 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function $\alpha \in \mathbb{R}$ and $\eta \in (0, T)$.

In this paper we consider the resonance case $\alpha\eta = T$ to obtain a new existence result. Although this situation has already been considered in the literature [19], we point out that our approach and methodology is different.

2 Linear problem

Consider the linear second-order three-point boundary value problem

$$\begin{aligned} &-u''(t) = \sigma(t), \quad t \in [0,T], \\ &u(0) = 0, \quad \alpha u(\eta) = u(T) \end{aligned} \tag{2}$$

for a given function $\sigma \in C[0,T]$.

The general solution is

$$u(t) = c_1 + c_2 t - \int_0^t (t-s)\sigma(s)\,ds$$

with c_1, c_2 arbitrary constants.

From $u(0) = 0$, we get $c_1 = 0$. From the second boundary condition, we have

$$(T - \alpha\eta)c_2 = \int_0^T (T-s)\sigma(s)\,ds - \alpha\int_0^\eta (\eta - s)\sigma(s)\,ds. \tag{3}$$

2.1 Nonresonance case

If $\alpha\eta \neq T$, then

$$c_2 = \frac{1}{T-\alpha\eta}\left[\int_0^T (T-s)\sigma(s)\,ds - \alpha\int_0^\eta (\eta-s)\sigma(s)\,ds\right],$$

and the linear problem (2) has a unique solution for any $\sigma \in C[0,T]$. In this case, we say that (2) is a nonresonant problem since the homogeneous problem has only the trivial solution as a solution, *i.e.*, when $\sigma = 0$, $c_1 = c_2 = 0$ and $u = 0$. Note that the solution is given by

$$u(t) = \int_0^T g(t,s)\sigma(s)\,ds \tag{4}$$

with

$$g(t,s) = \begin{cases} \frac{t(T-s)}{T-\alpha\eta} - \frac{t\alpha(\eta-s)}{T-\alpha\eta} - (t-s), & 0 \le s < \min(\eta,t), \\ \frac{t(T-s)}{T-\alpha\eta} - \frac{t\alpha(\eta-s)}{T-\alpha\eta}, & 0 \le t < s < \eta < T, \\ \frac{t(T-s)}{T-\alpha\eta} - (t-s), & 0 \le \eta < s < t \le T, \\ \frac{t(T-s)}{T-\alpha\eta}, & \max(\eta,t) < s \le T. \end{cases}$$

For $T = 1$ this is precisely the function given in Lemma 2.3 of [20] or in Remark 12 of [21].

2.2 Resonance case

If $T = \alpha\eta$, then (3) is solvable if and only if

$$\int_0^T (T-s)\sigma(s)\,ds = \alpha\int_0^\eta (\eta-s)\sigma(s)\,ds, \tag{5}$$

and then (2) has a solution if and only if (5) holds. In such a case, (2) has an infinite number of solutions given by

$$u(t) = ct - \int_0^t (t-s)\sigma(s)\,ds, \qquad c \in \mathbb{R}.$$

In particular ct, $c \in \mathbb{R}$ is a solution of the homogeneous linear equation

$$-u''(t) = 0, \qquad t \in [0,T]$$

satisfying the boundary conditions

$$u(0) = 0, \qquad \alpha u(\eta) = u(T).$$

Note that

$$u(T) - u(\eta) = c_2 T - \int_0^T (T-s)\sigma(s)\,ds - c_2\eta + \int_0^\eta (\eta - s)\sigma(s)\,ds,$$

and then

$$c_2 = \frac{1}{T-\eta}\left[u(T) - u(\eta) + \int_0^T (T-s)\sigma(s)\,ds - \int_0^\eta (\eta-s)\sigma(s)\,ds\right].$$

We now use that $u(T) = \frac{T}{\eta}u(\eta)$ to get

$$\frac{1}{T-\eta}\big[u(T) - u(\eta)\big] = \frac{1}{T}u(T)$$

and

$$c_2 = \frac{1}{T-\eta}\left[\int_0^T (T-s)\sigma(s)\,ds - \int_0^\eta (\eta-s)\sigma(s)\,ds\right] + \frac{1}{T}u(T).$$

Hence the solution of (2) is given, implicitly, as

$$u(t) = \int_0^T \frac{t(T-s)}{T-\eta}\sigma(s)\,ds - \int_0^\eta \frac{t(\eta-s)}{T-\eta}\sigma(s)\,ds - \int_0^t (t-s)\sigma(s)\,ds + \frac{t}{T}u(T)$$

or, equivalently,

$$u(t) = \int_0^T k(t,s)\sigma(s)\,ds + \frac{t}{T}u(T), \tag{6}$$

where

$$k(t,s) = \begin{cases} s, & 0 \le s < \min(\eta,t), \\ t, & 0 \le t < s < \eta \le T, \\ \frac{t(T-s)}{T-\eta} - (t-s), & 0 \le \eta < s < t \le T, \\ \frac{t(T-s)}{T-\eta}, & \max(\eta,t) < s \le T. \end{cases}$$

We note that $k \in C([0,T] \times [0,T], \mathbb{R})$ and $k(t,s) \ge 0$ for every $(t,s) \in [0,T] \times [0,T]$.

3 Nonlinear problem

Defining the operators:

$$
\begin{aligned}
&F : C[0,T] \to C[0,T],\\
&[Fu](t) = f\big(t,u(t)\big), \quad u \in C[0,T], t \in [0,T],\\
&K : C[0,T] \to C[0,T],\\
&[K\sigma](t) = \int_0^T k(t,s)\sigma(s)\,ds, \quad \sigma \in C[0,T], t \in [0,T],\\
&L : C[0,T] \to C[0,T],\\
&[Lu](t) = \frac{t}{T}u(T), \quad u \in C[0,T], t \in [0,T],
\end{aligned}
$$

the nonlinear problem is equivalent to

$$u = Nu,$$

where $N = K \circ F + L$.

We note that (6) can be written as

$$u(t) - \frac{t}{T}u(T) = \int_0^T k(t,s)\sigma(s)\,ds$$

and the nonlinear problem (1) as

$$u(t) - \frac{t}{T}u(T) = \int_0^T k(t,s)f\big(s,u(s)\big)\,ds.$$

This suggests to introduce the new function $v(t) = u(t) - \frac{t}{T}u(T)$. To find a solution u, we have to find v and $u(T)$.

For every constant $c \in \mathbb{R}$, we solve

$$v(t) = \int_0^T k(t,s)f\left(s, v(s) + \frac{s}{T}c\right)ds \tag{7}$$

and let $\varphi(c)$ be the set of solutions of (7). This set may be empty (no solution), a singleton (unique solution) or with more than one element (multiple solutions). For every $v_c \in \varphi(c)$, we consider

$$u_c(t) = v_c(t) + \frac{t}{T}c,$$

and hence

$$u_c(t) = \int_0^T k(t,s)f\big(s,u_c(s)\big)\,ds + \frac{t}{T}c.$$

If $c = u_c(T)$, then u_c is a solution of the nonlinear problem (1). We then look for fixed points of the map

$$c \in \mathbb{R} \longrightarrow u_c(T) \in \mathbb{R}.$$

For $c \in \mathbb{R}$ fixed, we try to solve the integral equation (7).

Assume that there exist $a, b \in C[0,T]$ and $\alpha \in [0,1)$ such that

$$\bigl|f(t,u)\bigr| \le a(t) + b(t)|u|^{\alpha} \tag{8}$$

for every $t \in [0,T]$, $u \in \mathbb{R}$.

For $v \in C[0,T]$, define $F_c v \in C[0,T]$ as

$$[F_c v](t) = f\left(t, v(t) + \frac{t}{T}c\right).$$

Thus, a solution of (7) is precisely a fixed point of $K \circ F_c = K_c$. Note that K_c is a compact operator. For $v \in C[0,T]$, let $\|v\| = \sup_{t\in[0,T]} |v(t)|$.

For $\lambda \in (0,1)$, if $v = \lambda K_c(v)$ we have

$$v(t) = \lambda \int_0^T k(t,s) f\left(s, v(s) + \frac{s}{T}c\right)\,ds,$$

and

$$\bigl|v(t)\bigr| \le \|k\| \int_0^T f\left(s, v(s) + \frac{s}{T}c\right)\,ds \le \|k\| \cdot T\bigl[\|a\| + \|b\|\bigl(\|v\| + c\bigr)^{\alpha}\bigr].$$

Hence there exist constants a_0, b_0 such that

$$\|v\| \le a_0 + b_0\bigl(\|v\| + c\bigr)^{\alpha} \tag{9}$$

for any $v \in C[0,T]$ and $\lambda \in (0,1)$ solution of $v = \lambda K_c(v)$. This implies that v is bounded independently of $\lambda \in (0,1)$, and hence by Schaefer's fixed point theorem (Theorem 4.3.2 of [22]), K_c has at least a fixed point, *i.e.*, for given c, equation (7) is solvable.

Now suppose f is Lipschitz continuous.

Then there exists $l > 0$ such that

$$\bigl|f(t,x) - f(t,y)\bigr| \le l|x - y| \tag{10}$$

for every $t \in [0,T]$ and $x, y \in \mathbb{R}$.

Then, for $v, w \in C[0,T]$, we have

$$\bigl|[K_c v](t) - [K_c w](t)\bigr| \le \int_0^T k(t,s) l \bigl|v(s) - w(s)\bigr|\,ds$$

and

$$\|K_c v - K_c w\| \le \|k\| \cdot l \cdot T\|v - w\|.$$

Thus, for $l > 0$ small, equation (7) has a unique solution in view of the classical Banach contraction fixed point theorem.

Now, under conditions (8) and (10), set

$$c \in \mathbb{R} \longrightarrow v_c \in C[0,T],$$

where v_c is the unique solution of (7), and as a consequence of the contraction principle, this map is continuous.

Define the map

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R},$$
$$\varphi(c) = v_c(T).$$

If there exists $c \in \mathbb{R}$ such that $\varphi(c) = 0$, then for that c we have $v_c(T)$, and the function

$$u_c(t) = v_c(t) + \frac{t}{T}c$$

is such that $u_c(T) = c$, and therefore u_c is a solution of the original nonlinear problem (1).

Now, assume that

$$\lim_{u\to\pm\infty} f(t,u) = \pm\infty \tag{11}$$

uniformly on $t \in [0,T]$.

Then the growth of $\|v\|$ is sublinear in view of estimate (9). However, c growths linearly. Hence the norm of the function

$$v_c(s) + \frac{s}{T}c$$

growths asymptotically as c.

This implies that $\lim_{c\to\pm\infty} \varphi(c) = \pm\infty$, and there exists $c \in \mathbb{R}$ with $\varphi(c) = 0$.

We have the following result.

Theorem 3.1 *Suppose that f satisfies the growth conditions* (8) *and* (10). *If* (11) *holds, then* (1) *is solvable for l sufficiently small.*

Note that condition (11) is crucial since for $f(t,u) = \sigma(t)$ and, in view of (5), the problem (1) may have no solution.

Competing interests
The author declares that he has no competing interests.

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