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Existence and multiplicity of weak quasi-periodic solutions for second order Hamiltonian system

Xingyong Zhang*

*Correspondence:
zhangxingyong1@163.com
Department of Mathematics, Faculty of Science, Kunming University of Science and Technology, Kunming, Yunnan 650500, P.R. China

Abstract

In this paper we investigate the existence and multiplicity of weak quasi-periodic solutions for the second order Hamiltonian system $\frac{d[P(t)\dot{u}(t)]}{dt} + \nabla F(t,u(t)) = 0$, $t \in \mathbb{R}$, where $P(t) = (p_{ij}(t))_{N\times N}$ is a symmetric and continuous $N \times N$ matrix-value function on $\mathbb{R}$ and $F(t,x)$ is almost periodic in t uniformly for $x \in \mathbb{R}^N$. When F has superquadratic growth, we see that the system has at least one nonconstant weak quasi-periodic solution and when the assumption $F(t,-x) = F(t,x)$ is also made, we see that the system has infinitely many weak quasi-periodic solutions by variational method.

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1 Introduction and main results

In this paper, we are concerned with the existence and multiplicity of weak-quasi-periodic solutions for the second order Hamiltonian system

$$\frac{d[P(t)\dot{u}(t)]}{dt} + \nabla F\bigl(t,u(t)\bigr) = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where $u(t) = (u_1(t),\dots,u_N(t))^\tau$, $N > 1$ is an integer, $F \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, $\nabla F(t,x) = (\partial F/\partial x_1, \dots, \partial F/\partial x_N)^\tau$, $P(t) = (p_{ij}(t))_{N\times N}$ is a symmetric and continuous $N \times N$ matrix-value function on $\mathbb{R}$, the symbol $(\cdot)^\tau$ stands for the transpose of a vector or a matrix.

It is well known that the variational method is a very effective tool for investigating the existence and multiplicity of various solutions of Hamiltonian system. Lots of mathematicians have constructed many important results on existence and multiplicity of periodic solutions, subharmonic solutions and homoclinic solutions (for example, see [1–24]). However, there are less studies on almost periodic solutions of Hamiltonian systems. We refer the reader to [25–37] for some known results. Very recently, in [35], Kuang investigated the following second order Hamiltonian system:

$$\ddot{u}(t) = \nabla F\bigl(t,u(t)\bigr), \quad t \in \mathbb{R}, \tag{1.2}$$

and obtained two existence results of weak quasi-periodic solutions for system (1.2) by making use of the least action principle and the saddle point theorem, respectively.

Next, we recall some definitions.

Definition 1.1 (see [38]) A function $u(t)$ is said to be Bohr almost periodic, if for any $\varepsilon > 0$, there is a constant $l_\varepsilon > 0$, such that in any interval of length l_ε, there exists τ such that the inequality $|u(t+\tau) - u(t)| < \varepsilon$ is satisfied for all $t \in \mathbb{R}$.

Definition 1.2 (see [39]) $n \times m$ matrix-value function $M(t) = (m_{ij}(t))_{n\times m}$ is almost periodic on $\mathbb{R}$ if $m_{ij}(t)$ $(i = 1, 2, \dots, n, j = 1, 2, \dots, m)$ is Bohr almost periodic on $\mathbb{R}$.

Definition 1.3 (see [39]) $u \in C^0(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^N)$ is so called almost periodic in t uniformly for $x \in \mathbb{R}^m$ when, for each compact subset K in $\mathbb{R}^m$, for each $\varepsilon > 0$, there exists $l > 0$, and for each $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l]$ such that

$$\sup_{t\in\mathbb{R}} \sup_{x\in K} \bigl\| u(t+\tau, x) - u(t, x) \bigr\|_{\mathbb{R}^N} < \varepsilon.$$

Definition 1.4 (see [40]) $u : \mathbb{R} \to \mathbb{R}^n$ is said to be quasi-periodic with m basic frequencies if there exists a function $v \to \Phi(v) \in \mathbb{R}^n$ which is Lipschitz continuous for $v \in \mathbb{R}^m$ and periodic of period 1 in each of its arguments, and m real numbers $\omega_1, \dots, \omega_m$ linearly independent over the rationals, such that

$$u(t) = \Phi(\omega_1 t, \dots, \omega_m t).$$

Any such choice of $\omega_1, \dots, \omega_m$ will be called a set of basic frequencies for $u(t)$.

Remark 1.1 If $u \in C(\mathbb{R}, \mathbb{R}^n)$ is a periodic function, then u is quasi-periodic and if $u \in C(\mathbb{R}, \mathbb{R}^n)$ is quasi-periodic, then u is Bohr almost periodic. Moreover, if $u(t)$, $w(t)$ are Bohr almost periodic and $a, b \in \mathbb{R}$, then $au(t)$, $u(t+b)$, $u(bt)$, $u(t) + w(t)$, and $u(t)w(t)$ are Bohr almost periodic. Furthermore, if $\inf_{t\in\mathbb{R}} |w(t)| > 0$, then $\frac{u(t)}{w(t)}$ is also Bohr almost periodic (see [39]).

Remark 1.2 Let $p > 1$ and $N > 1$ be positive integers and $\{T_j\}_{j=1}^p$ be rationally independent positive real constants. Assume that $u_j(t) \in C(\mathbb{R}, \mathbb{R}^N)$ $(j = 1, 2, \dots, p)$ is T_j-periodic and Lipschitz continuous on $\mathbb{R}$. Define

$$u(t) := \sum_{j=1}^{p} u_j(t). \tag{1.3}$$

Define $\Phi : \mathbb{R}^p \to \mathbb{R}^N$ by $\Phi(v) = \sum_{j=1}^p u_j(T_j v_j)$, where $v = (v_1, \dots, v_p)^\tau$. Let $\omega_j = \frac{1}{T_j}$ $(j = 1, \dots, p)$. Then it is easy to verify that u is quasi-periodic with basic frequencies $\frac{1}{T_j}$ $(j = 1, \dots, p)$. Obviously, u is also Bohr almost periodic by Remark 1.1.

Define

$$AP^0(\mathbb{R}^N) = \bigl\{ u : \mathbb{R} \to \mathbb{R}^N | u \text{ is Bohr almost periodic} \bigr\}$$

endowed with the norm $\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)|$. Then $(AP^0(\mathbb{R}^N), \|\cdot\|_\infty)$ is a Banach space.

Define

$$AP^1(\mathbb{R}^N) = \{u \in AP^0(\mathbb{R}^N) \cap C^1(\mathbb{R}, \mathbb{R}^N) | u'(t) \in AP^0(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{AP^1(\mathbb{R}^N)} = \|u\|_\infty + \|u'\|_\infty.$$

Then $(AP^1(\mathbb{R}^N), \|\cdot\|_{AP^1(\mathbb{R}^N)})$ is also a Banach space.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, that is, f is locally Lebesgue-integrable from $\mathbb{R}$ to $\mathbb{R}^N$. Then the mean value of f is the limit (when it exists)

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt.$$

A fundamental property of almost periodic functions is that such functions have convergent means, that is, the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)\,dt$$

exists.

Let $p \in \mathbb{Z}^+$. $B^p(\mathbb{R}^N)$ is the completion of $AP^0(\mathbb{R}^N)$ into $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ with respect to the norm

$$\|u\|_p = \left\{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |u(t)|^p\,dt\right\}^{1/p}.$$

The elements of the space $B^p(\mathbb{R}^N)$ are called Besicovitch almost periodic functions.

For $u \in B^p(\mathbb{R}^N)$, if

$$\lim_{r\to 0} \frac{u(t+r) - u(t)}{r}$$

exists, then define

$$\nabla u = \lim_{r\to 0} \frac{u(t+r) - u(t)}{r}.$$

For $u, v \in B^p(\mathbb{R}^N)$, if $\|u - v\|_p = 0$, then we say that u, v belong to a class of equivalence. We will identify the equivalence class u with its continuous representant

$$u(t) = \int_0^t \nabla u(t)\,dt + c.$$

When $p = 2$, $B^2(\mathbb{R}^N)$ is a Hilbert space with its norm $\|\cdot\|_2$ and the inner product

$$\langle u, v\rangle_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \big(u(t), v(t)\big)\,dt.$$

When $u \in B^2(\mathbb{R}^N)$, define

$$a(u,\lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} u(t)\,dt,$$

which is a complex vector and is called a Fourier-Bohr coefficient of u. Let $\Lambda(u) = \{\lambda \in \mathbb{R} | a(u,\lambda) \neq 0\}$.

Define

$$B^{1,2}(\mathbb{R}^N) = \{u \in B^2(\mathbb{R}^N) | \nabla u \text{ exists and } \nabla u \in B^2(\mathbb{R}^N)\}$$

endowed with the inner product

$$\begin{aligned}\langle u,v\rangle_{B^{1,2}(\mathbb{R}^N)} &= \langle u,v\rangle_2 + \langle \nabla u, \nabla v\rangle_2 \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \big(u(t),v(t)\big)\,dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \big(\nabla u(t), \nabla v(t)\big)\,dt \end{aligned} \tag{1.4}$$

and the corresponding norm

$$\|u\|_{B^{1,2}(\mathbb{R}^N)} = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |u(t)|^2\,dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\nabla u(t)|^2\,dt\right)^{1/2}$$

(see [27, 35, 38] and [39]).

Let $p > 1$ be a positive integer and $\{T_j\}_{j=1}^p$ be rationally independent positive real constants. Define

$$\mathcal{V} = \{u \in B^{1,2}(\mathbb{R}^N) | \Lambda(u) \subset \Lambda\}, \tag{1.5}$$

where

$$\Lambda = \bigcup_{j=1}^{p} \Lambda_j = \bigcup_{j=1}^{p} \left\{\frac{2m\pi}{T_j} \Big| m \in \mathbb{Z}\right\}, \qquad \Lambda_j = \left\{\frac{2m\pi}{T_j} \Big| m \in \mathbb{Z}\right\}, \quad j = 1,\ldots,p.$$

Then $\mathcal{V}$ is a linear subspace of $B^{1,2}(\mathbb{R}^N)$ and $(\mathcal{V}, \langle\cdot,\cdot\rangle_{B^{1,2}(\mathbb{R}^N)})$ is a Hilbert space.

In [35], Kuang made the following assumptions:

(f_1) $F(t,\cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $F(t,\cdot)$ is almost periodic in t uniformly for $x \in \mathbb{R}^N$;
(f_2) $\nabla F(t,\cdot)$ is almost periodic in t uniformly for $x \in \mathbb{R}^N$;
(f_3) for any $\lambda \in \mathbb{R}/\Lambda$, $u \in \mathcal{V}$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \nabla F(t,u) e^{-i\lambda t}\,dt = 0;$$

(f_4) there exists $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, for a.e. $t \in \mathbb{R}$ and all $x \in \mathbb{R}^N$, such that

$$|\nabla F(t,x)| \le g(t);$$

(f_5) $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t,x)\,dt \to +\infty$ as $|x| \to \infty$;
(f_6) $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t,x)\,dt \to -\infty$ as $|x| \to \infty$.

Then when (f_1)-(f_5) (or (f_1)-(f_4) and (f_6)) hold, system (1.2) has at least one quasi-periodic solution.

In [36], we generalize and improve Kuang's results. We first obtain three inequalities and two of them, in some sense, generalize Sobolev's inequality and Wirtinger's inequality from the periodic case to the quasi-periodic case, respectively. Then by using the least action principle and the saddle point theorem, we obtain two existence results of weak quasi-periodic solutions for the second order Hamiltonian system with a forcing term:

$$\frac{d[P(t)\dot{u}(t)]}{dt} = \nabla F\bigl(t, u(t)\bigr) + e(t),$$

when (f_1)-(f_3) and the following assumptions hold:

$(\mathcal{P})$ $p_{ij}(t)$, $i,j = 1,2,\ldots,N$, are Bohr almost periodic and there exists $m > \frac{1}{2}$ such that

$$\bigl(P(t)x, x\bigr) > m|x|^2, \quad \text{for all } (t,x) \in \mathbb{R} \times \bigl\{\mathbb{R}^N/\{0\}\bigr\};$$

$(\mathcal{E})$ e is Bohr almost periodic and

$$\lim_{T\to\infty} \int_{-T}^{T} e(t)\,dt = 0;$$

$(\mathcal{W})$ *there exist constants* $c_0 > 0$, $k_1 > 0$, $k_2 > 0$, $\alpha \in [0,1)$, *and a nonnegative function* $w \in C([0,+\infty),[0,+\infty))$ *with the properties*:

(i) $w(s) \le w(t)$, $\forall s \le t$, $s,t \in [0,+\infty)$,

(ii) $w(s+t) \le c_0(w(s)+w(t))$, $\forall s,t \in [0,+\infty)$,

(iii) $0 \le w(t) \le k_1 t^{\alpha} + k_2$, $\forall t \in [0,+\infty)$,

(iv) $w(t) \to +\infty$, *as* $t \to \infty$;

$(f_4)'$ there exist $g, h \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^+)$ such that

$$\bigl|\nabla F(t,x)\bigr| \le g(t) w\bigl(|x|\bigr) + h(t), \quad \text{for a.e. } t \in \mathbb{R};$$

$(f_5)'$

$$\begin{aligned} &\frac{1}{w^2(|x|)} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t,x)\,dt \\ &\quad > \frac{c_0^2 \sum_{j=1}^{p} \frac{T_j^2}{12}}{2m} \biggl(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)\,dt\biggr)^2 \quad \text{as } |x| \to \infty; \end{aligned}$$

$(f_5)''$

$$\begin{aligned} &\frac{1}{w^2(|x|)} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t,x)\,dt \\ &\quad < -\frac{c_0^2(\|P\| + 2m) \sum_{j=1}^{p} \frac{T_j^2}{12}}{2(2m-1)} \biggl(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)\,dt\biggr)^2 \quad \text{as } |x| \to \infty, \end{aligned}$$

where

$$\|P\| = \sup_{t\in\mathbb{R}} \max_{|x|=1, x\in\mathbb{R}^N} \bigl|P(t)x\bigr| = \sup_{t\in\mathbb{R}} \max\bigl\{\sqrt{\lambda(t)} : \lambda(t) \text{ is the eigenvalue of } P^{\tau}(t)P(t)\bigr\}.$$

Moreover, when the assumptions $F(t,x) = F(t,-x)$ and $e(t) \equiv 0$ are also made, we obtain two results on infinitely many weak quasi-periodic solutions for the second order Hamiltonian system under the subquadratic case.

Inspired by [12, 35, 36] and [41], in this paper, we investigate the case that F has superquadratic growth and we obtain the following results.

Theorem 1.1 *Suppose that* (f_1)-(f_3) *and the following conditions hold*:

$(\mathcal{P})'$ $p_{ij}(t)$, $i,j = 1,2,\dots,N$, *are Bohr almost periodic and there exists* $m > 0$ *such that*

$$\bigl(P(t)x,x\bigr) > m|x|^2, \quad \textit{for all } (t,x) \in \mathbb{R} \times \bigl\{\mathbb{R}^N/\{0\}\bigr\}$$

and for any $\lambda \in \mathbb{R}/\Lambda$, $u \in \mathcal{V}$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} P(t)\nabla u(t) e^{-i\lambda t}\,dt = 0; \tag{1.6}$$

$(\mathcal{H}1)$

$$\limsup_{|x|\to 0} \frac{F(t,x)}{|x|^2} < \frac{m}{2[\max\{\frac{T_j}{2\pi}|j = 1,\dots,p\}]^{1/2}} \quad \textit{uniformly for all } t \in \mathbb{R};$$

$(\mathcal{H}2)$

$$\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} > \|P\| \sum_{j=1}^{p} \frac{2\pi^2}{T_j^2} \quad \textit{uniformly for all } t \in \mathbb{R};$$

$(\mathcal{H}3)$ $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(t,x)\,dt \ge 0$;

$(\mathcal{H}4)$ *there exist constants* $L > 0$, $\zeta > 0$, $\eta > 0$, *and* $\nu \in [0,2)$ *such that*

$$\left(2 + \frac{1}{\zeta + \eta|x|^\nu}\right) F(t,x) \le \bigl(\nabla F(t,x),x\bigr), \quad \textit{for all } x \in \mathbb{R}^N, |x| > L, t \in \mathbb{R}.$$

Then system (1.1) *has at least one nonconstant weak quasi-periodic solution in* $\mathcal{V}$.

Theorem 1.2 *Suppose that* (f_1)-(f_3), $(\mathcal{P})'$, $(\mathcal{H}2)$, $(\mathcal{H}3)$, *and* $(\mathcal{H}4)$, *and the following condition hold*:

$(\mathcal{H}1)'$ *there exist* $l > 0$ *and* α *which is Bohr almost periodic and* $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \alpha(t)\,dt < \frac{6ml^2}{\sum_{j=1}^{p} T_j^2}$ *such that*

$$F(t,x) \le \alpha(t), \quad \textit{for all } x \in \mathbb{R}^N, |x| \le l, t \in \mathbb{R}.$$

Then system (1.1) *has at least one nonconstant weak quasi-periodic solution in* $\mathcal{V}$.

Theorem 1.3 *Suppose that* (f_1)-(f_3), $(\mathcal{P})'$, $(\mathcal{H}1)$, $(\mathcal{H}4)$ *and the following conditions hold*:

$(\mathcal{H}2)'$ $\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} = +\infty$ *uniformly for all* $t \in \mathbb{R}$;

$(\mathcal{H}5)$ $F(t,x)$ *is even in* x *and* $F(t,0) \equiv 0$ *for all* $t \in \mathbb{R}$.

Then system (1.1) *has infinitely many weak quasi-periodic solutions* $\{u_n\}$ *which possesses high energy in* $\mathcal{V}$*, that is,*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{1}{2}\big(P(t)\nabla u_n(t),\nabla u_n(t)\big)-F\big(t,u_n(t)\big)\right]dt\to+\infty\quad \textit{as } n\to\infty.$$

Theorem 1.4 *Suppose that* (f_1)-(f_3), $(\mathcal{P})'$, $(\mathcal{H}1)'$, $(\mathcal{H}2)'$, $(\mathcal{H}4)$, *and* $(\mathcal{H}5)$ *hold. Then system* (1.1) *has infinitely many weak quasi-periodic solutions* $\{u_n\}$ *which possesses high energy in* $\mathcal{V}$.

Remark 1.3 In [36], it is remarkable that we did not require the condition (1.6). However, the condition (1.6) is necessary when we prove that the critical points of variational functional coincide with the solutions of system. Hence, we have to make a correction to our previous paper [36]. To be precise, the restriction (1.6) must be added to $(\mathcal{P})$ in [36].

Remark 1.4 In order to study the existence of periodic solutions of Hamiltonian systems, the following well-known (AR)-condition was introduced in [9]:

(AR) *there exist constants* $\mu > 2$ *and* $r > 0$ *such that*

$$\big(\nabla F(t,x),x\big)\geq \mu F(t,x),\quad \forall |x|\geq r, \textit{a.e. } t\in[0,T].$$

The (AR)-condition has been extensively used in much of the literature. In 2004, Tao and Tang [21] presented the following new condition:

$(H4)'$ there exist $\vartheta > 2$ and $\mu > \vartheta - 2$ such that

$$\limsup_{|x|\to\infty}\frac{F(t,x)}{|x|^{\vartheta}}<\infty\quad \text{uniformly for a.e. } t\in[0,T],$$
$$\liminf_{|x|\to\infty}\frac{(\nabla F(t,x),x)-2F(t,x)}{|x|^{\mu}}>0\quad \text{uniformly for a.e. } t\in[0,T].$$

In 2012, to investigate subharmonic solutions of a class of second order Hamilton system, the author and Tang [22] presented the following conditions:

(H4) there exist constants $L > 0$, $\zeta > 0$, $\eta > 0$, and $\nu \in [0,2)$ such that

$$\left(2+\frac{1}{\zeta+\eta|x|^{\nu}}\right)F(t,x)\leq\big(\nabla F(t,x),x\big),\quad \text{for all } x\in\mathbb{R}^N, |x|>L, \text{a.e. } t\in[0,T],$$

which is motivated by an earlier version due to Ding [1].

In [22], the author and Tang have proved that the (AR)-condition and $(H4)'$ imply (H4) (see Remark 1.1 in [22]). Similarly, we can prove that the following:

$(\mathcal{AR})$ there exist constants $\mu > 2$ and $r > 0$ such that

$$\big(\nabla F(t,x),x\big)\geq\mu F(t,x),\quad \forall|x|\geq r, t\in\mathbb{R}$$

and

$(\mathcal{H}4)'$ there exist $\vartheta > 2$ and $\mu > \vartheta - 2$ such that

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^\vartheta} < \infty \quad \text{uniformly for all } t \in \mathbb{R},$$
$$\liminf_{|x|\to\infty} \frac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^\mu} > 0 \quad \text{uniformly for all } t \in \mathbb{R},$$

combining with $(\mathcal{H}2)$ (or $(\mathcal{H}2)'$), imply $(\mathcal{H}4)$. Hence, Theorem 1.1-Theorem 1.4 still hold on replacing $(\mathcal{H}4)$ with $(\mathcal{H}4)'$ or replacing $(\mathcal{H}4)$ and $(\mathcal{H}2)$ (or $(\mathcal{H}2)'$) with $(\mathcal{AR})$.

Remark 1.5 When $P(t) \equiv I_{N\times N}$, $\mathcal{V}$ only contains a frequency $2\pi/T$ and $F(t,x)$ is periodic in t with period T, Theorem 1.1 and Theorem 1.2 reduce to Theorem 1.4 and Theorem 1.3 with $p = 2$ in [41], respectively. In other words, we generalize Theorem 1.3 and Theorem 1.4 with $p = 2$ in [41] from the periodic case to the quasi-periodic case.

2 Preliminaries

In [36], we presented the following two lemmas.

Lemma 2.1 (see [36], Lemma 2.1) *If* $u \in \mathcal{V}$, *then*

$$u(t) = \sum_{j=1}^{p} u_j(t) \in AP^0(\mathbb{R}^N),$$

where

$$u_j(t) = \sum_{m=-\infty}^{+\infty} a(u, \lambda_m^{(j)}) e^{i\lambda_m^{(j)} t}, \quad \lambda_m^{(j)} := \frac{2m\pi}{T_j} \in \Lambda_j$$

and

$$\|u\|_\infty \le \sqrt{p^2 + \sum_{j=1}^{p} \frac{T_j^2}{12}} \|u\|_{B^{1,2}(\mathbb{R}^N)}. \tag{2.1}$$

Lemma 2.2 (see [36], Lemma 2.2) *If* $u \in \mathcal{V}$ *and*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, dt = 0, \tag{2.2}$$

then

$$\|u\|_\infty \le \sqrt{\sum_{j=1}^{p} \frac{T_j^2}{12}} \|\nabla u\|_2 \tag{2.3}$$

and

$$\|u\|_2 \le \max\left\{ \frac{T_j}{2\pi} \middle| j = 1, \ldots, p \right\} \|\nabla u\|_2. \tag{2.4}$$

Next we denote

$$C^* = \max\left\{ \frac{T_j}{2\pi} \,\middle|\, j = 1, \ldots, p \right\}, \qquad C_* = \sqrt{\sum_{j=1}^{p} \frac{T_j^2}{12}}, \qquad C_{**} = \sqrt{p^2 + \sum_{j=1}^{p} \frac{T_j^2}{12}}.$$

Define

$$\tilde{\mathcal{V}} = \left\{ u \in \mathcal{V} \,\middle|\, \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, dt = 0 \right\}$$

and

$$\bar{\mathcal{V}} = \{ u | u \in \mathcal{V} \cap \mathbb{R}^N \}.$$

Then $\mathcal{V} = \tilde{\mathcal{V}} \oplus \bar{\mathcal{V}}$. For $u \in \mathcal{V}$, u can be written as $u = \bar{u} + \tilde{u}$, where

$$\bar{u} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, dt \in \bar{\mathcal{V}}.$$

It is easy to verify that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \tilde{u}(t)\, dt = 0.$$

Then $\tilde{u} \in \tilde{\mathcal{V}}$. On $\mathcal{V}$, we define the norm

$$\|u\| := \left(\|\bar{u}\|_2^2 + \|\nabla u\|_2^2 \right)^{1/2} = \left(|\bar{u}|^2 + \|\nabla u\|_2^2 \right)^{1/2}.$$

Lemma 2.3 *On $\mathcal{V}$, the norm $\|u\|_{B^{1,2}(\mathbb{R}^N)}$ is equivalent to the norm $\|u\|$.*

Proof Note that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \big(\bar{u}, \tilde{u}(t)\big)\, dt = 0.$$

Then it follows from (2.4) that

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|u(t)\right|^2 dt &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\bar{u} + \tilde{u}(t)\right|^2 dt \\
&= |\bar{u}|^2 + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\tilde{u}(t)\right|^2 dt \\
&\le |\bar{u}|^2 + \left(C^*\right)^2 \|\nabla u\|_2^2.
\end{aligned}$$

Thus by the Hölder inequality, we have

$$\begin{aligned}
\|u\|_{B^{1,2}(\mathbb{R}^N)} &= \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|u(t)\right|^2 dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\nabla u(t)\right|^2 dt \right\}^{1/2} \\
&\le \left\{ |\bar{u}|^2 + \left(C^*\right)^2 \|\nabla u\|_2^2 + \|\nabla u\|_2^2 \right\}^{1/2}
\end{aligned}$$

$$\begin{aligned}&\leq \left[\left(C^*\right)^2+1\right]^{1/2}\left\{|\bar{u}|^2+\|\nabla u\|_2^2\right\}^{1/2}\\&= \left[\left(C^*\right)^2+1\right]^{1/2}\|u\|.\end{aligned}$$

Moreover, by the Hölder inequality, we also have

$$\begin{aligned}|\bar{u}| &= \left|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}u(t)\,dt\right|\\&\leq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|u(t)\right|dt\\&\leq \lim_{T\to\infty}\frac{1}{\sqrt{2T}}\left(\int_{-T}^{T}\left|u(t)\right|^2dt\right)^{1/2}\\&= \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|u(t)\right|^2dt\right)^{1/2}.\end{aligned}$$

Hence, we have

$$\|u\| = \left(|\bar{u}|^2+\|\nabla u\|_2^2\right)^{1/2} \leq \|u\|_{B^{1,2}(\mathbb{R}^N)}.$$

Thus we complete the proof. □

Lemma 2.4 (see [35], Lemma 3.2) *For any $\{u_n\}\subset\mathcal{V}$, if the sequence $\{u_n\}$ converges weakly to u, then $\{u_n\}$ converges uniformly to u on any compact subset of $\mathbb{R}$.*

Lemma 2.5 (see [36], Lemma 2.5) *Suppose $(\mathcal{P})'$ holds and F satisfies* (f_1)-(f_3). *Then the functional $\varphi:\mathcal{V}\to\mathbb{R}$, defined by*

$$\varphi(u) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{1}{2}\big(P(t)\nabla u(t),\nabla u(t)\big)-F\big(t,u(t)\big)\right]dt \tag{2.5}$$

is continuously differentiable on $\mathcal{V}$, and $\varphi'(u)$ is defined by

$$\left\langle\varphi'(u),v\right\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\big(P(t)\nabla u(t),\nabla v(t)\big)-\big(\nabla F\big(t,u(t)\big),v(t)\big)\right]dt, \tag{2.6}$$

for $v\in\mathcal{V}$. Moreover, if u is a critical point of φ in $\mathcal{V}$, then

$$\nabla\big(P(t)\nabla u(t)\big)+\nabla F\big(t,u(t)\big)=0. \tag{2.7}$$

Definition 2.1 When u satisfies (2.7), we say that u is a weak solution of system (1.1).

We shall use one linking method to obtain the critical points of φ (the details can be found in [12]). Let $(E,\|\cdot\|)$ be a Banach space and let Φ be the set of all continuous maps $\Gamma=\Gamma(t)$ from $E\times[0,1]$ to E such that:

(1) $\Gamma(0)=I$, the identity map.
(2) For each $t\in[0,1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t)\in C(E\times[0,1),E)$.

(3) $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$.

(4) For each $t_0 \in [0,1)$ and each bounded set $A \subset E$,

$$\sup_{0 \le t \le t_0, u \in A} \{ \| \Gamma(t)u \| + \| \Gamma^{-1}(t)u \| \} < \infty.$$

Definition 2.2 (see [12], Definition 3.2) We say that A links B [hm] if A and B are subsets of E such that $A \cap B = \emptyset$, and for each $\Gamma(t) \in \Phi$, there is a $t' \in (0,1]$ such that $\Gamma(t')A \cap B \neq \emptyset$.

The following lemma will be used to prove our Theorem 1.1 and Theorem 1.2.

Lemma 2.6 (see [12], Theorem 3.4 and Theorem 2.12) *Let E be a Banach space, $\varphi \in C^1(E,\mathbb{R})$ and A and B two subsets of E such that A links B [hm]. Assume that*

$$\sup_A \varphi \le \inf_B \varphi$$

and

$$c := \inf_{\Gamma \in \Phi} \sup_{s \in [0,1], u \in A} \varphi\bigl(\Gamma(s)u\bigr) < \infty.$$

Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0,\infty)$ satisfying $\int_0^\infty \psi(r)\,dr = \infty$. Then there exists a sequence $\{u_n\} \subset E$ such that $\varphi(u_n) \to c$ and $\varphi'(u_n)/\psi(\|u_n\|) \to 0$, as $n \to \infty$. Moreover, if $c = \sup_A \varphi$, then there is a sequence $\{u_n\} \subset E$ satisfying $\varphi(u_n) \to c$, $\varphi'(u_n) \to 0$, and $d(u_n, B) \to 0$, as $n \to \infty$.

Remark 2.1 Since A links B [hm], by Definition 2.1, it is easy to know that $c \ge \inf_B \varphi$. By [12], if we let $\psi(r) \equiv 1$, the sequence $\{u_n\}$ coincides with (PS) sequence, that is, $\{u_n\}$ satisfies

$$\varphi(u_n) \to c, \qquad \varphi'(u_n) \to 0 \quad \text{as } n \to \infty.$$

If we let $\psi(r) = \frac{1}{1+r}$, the sequence $\{u_n\}$ is the Cerami sequence, that is, $\{u_n\}$ satisfies

$$\varphi(u_n) \to c, \qquad \bigl(1 + \|u_n\|\bigr) \bigl\| \varphi'(u_n) \bigr\| \to 0 \quad \text{as } n \to \infty.$$

We will use the symmetric mountain pass theorem (see [10], Theorem 9.12) to prove Theorem 1.3 and Theorem 1.4.

Remark 2.2 As shown in [42], a deformation lemma can be proved with replacing the usual (PS)-condition with the (C)-condition, and it turns out that the symmetric mountain pass theorem in [10] is true under the (C)-condition. We say that φ satisfies the (C)-condition, *i.e.* for every sequence $\{u_n\} \subset E$, $\{u_n\}$ has a convergent subsequence if $\varphi(u_n)$ is bounded and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$.

3 Proofs

Lemma 3.1 *Assume that $(\mathcal{P})'$ and $(\mathcal{H}1)$ hold. Then there exist $\varrho > 0$ and $b_1 > 0$ such that $\inf_{u \in \mathcal{B}} \varphi(u) \ge b_1 > 0$, where $\mathcal{B} = \tilde{\mathcal{V}} \cap \partial\mathcal{B}_\varrho$ and $\partial\mathcal{B}_\varrho = \{u \in \mathcal{V} \mid \|u\| = \varrho\}$.*

Proof It follows from $(\mathcal{H}1)$ that there exist $0 < \varepsilon_0 < \frac{m}{2(C^*)^2}$ and $r > 0$ such that

$$F(t,x) \le \varepsilon_0 |x|^2, \quad \text{for all } |x| < r, t \in \mathbb{R}. \tag{3.1}$$

Note that in $\tilde{\mathcal{V}}$, $\|u\| = \|\nabla u\|_2$. Let $\varrho = r/C_*$. Then it follows from (2.3) that, for all $u \in \tilde{\mathcal{V}} \cap \partial\mathcal{B}_\varrho$, we have $\|u\|_\infty \le r$. Hence, by $(\mathcal{P})'$, (2.4), and (3.1), we obtain, for all $u \in \tilde{\mathcal{V}} \cap \partial\mathcal{B}_\varrho$,

$$\begin{aligned}
\varphi(u) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\frac{1}{2} \big(P(t)\nabla u(t), \nabla u(t)\big) - F\big(t, u(t)\big) \right] dt \\
&\ge \frac{m}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\nabla u(t)\right|^2 dt - \varepsilon_0 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|u(t)\right|^2 dt \\
&\ge \frac{m}{2} \|\nabla u\|_2^2 - \varepsilon_0 \big(C^*\big)^2 \|\nabla u\|_2^2 \\
&= \left(\frac{m}{2} - \varepsilon_0 \big(C^*\big)^2 \right) \varrho^2 := b_1 > 0.
\end{aligned}$$

The proof is complete. □

Lemma 3.2 *Assume that* $(\mathcal{P})'$ *and* $(\mathcal{H}1)'$ *hold. Then there exist* $\varrho > 0$ *and* $b_1 > 0$ *such that* $\inf_{u\in\mathcal{B}} \varphi(u) \ge b_1 > 0$, *where* $\mathcal{B} = \tilde{\mathcal{V}} \cap \partial\mathcal{B}_\varrho$ *and* $\partial\mathcal{B}_\varrho = \{u \in \mathcal{V} | \|u\| = \varrho\}$.

Proof Note that in $\tilde{\mathcal{V}}$, $\|u\| = \|\nabla u\|_2$. Let $\varrho = l/C_*$. Then it follows from (2.3) that, for all $u \in \tilde{\mathcal{V}} \cap \partial\mathcal{B}_\varrho$, we have $\|u\|_\infty \le l$. Hence, by $(\mathcal{P})'$ and $(\mathcal{H}1)'$, we see that, for all $u \in \tilde{\mathcal{V}} \cap \partial\mathcal{B}_\varrho$,

$$\begin{aligned}
\varphi(u) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\frac{1}{2} \big(P(t)\nabla u(t), \nabla u(t)\big) - F\big(t, u(t)\big) \right] dt \\
&\ge \frac{m}{2} \|\nabla u\|_2^2 - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \alpha(t)\, dt \\
&= \frac{ml^2}{2C_*^2} - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \alpha(t)\, dt := b_1 > 0.
\end{aligned}$$

The proof is complete. □

Lemma 3.3 *Assume that* $(\mathcal{P})'$, $(\mathcal{H}2)$, *and* $(\mathcal{H}3)$ *hold. Then there exists a sufficiently large positive constant* Θ *such that* $\sup_A \varphi \le 0$, *where*

$$A = \big\{ x \in \mathbb{R}^N : \|x\| \le \Theta \big\} \cup \big\{ sw_0 + x : x \in \mathbb{R}^N, s \ge 0, w_0 \in \tilde{\mathcal{V}}, \|sw_0 + x\| = \Theta \big\}.$$

Proof At first, by $(\mathcal{H}3)$, it is easy to obtain

$$\varphi(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t,x)\, dt \le 0. \tag{3.2}$$

Let

$$w_0 = \left(\sum_{j=1}^{p} \sin \frac{2\pi t}{T_j}, 0, \ldots, 0 \right)^{\tau}.$$

Then $\nabla w_0 = (\sum_{j=1}^{p} \frac{2\pi}{T_j}\cos\frac{2\pi t}{T_j}, 0, \ldots, 0)^\tau$,

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|w_0(t)\right|^2\,dt \\
&\quad= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\sum_{j=1}^{p}\sin^2\frac{2\pi t}{T_j} + \sum_{i,j=1,\ldots,p,i\neq j}\sin\frac{2\pi t}{T_i}\sin\frac{2\pi t}{T_j}\right]dt \\
&\quad= \sum_{j=1}^{p}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin^2\frac{2\pi t}{T_j}\,dt + \sum_{i,j=1,\ldots,p,i\neq j}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin\frac{2\pi t}{T_i}\sin\frac{2\pi t}{T_j}\,dt \\
&\quad= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\nabla w_0(t)\right|^2\,dt \\
&\quad= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\sum_{j=1}^{p}\frac{4\pi^2}{T_j^2}\cos^2\frac{2\pi t}{T_j} + \sum_{i,j=1,\ldots,p,i\neq j}\frac{4\pi}{T_iT_j}\cos\frac{2\pi t}{T_i}\cos\frac{2\pi t}{T_j}\right]dt \\
&\quad= \sum_{j=1}^{p}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{4\pi^2}{T_j^2}\cos^2\frac{2\pi t}{T_j}\,dt \\
&\qquad+ \sum_{i,j=1,\ldots,p,i\neq j}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{4\pi}{T_iT_j}\cos\frac{2\pi t}{T_i}\cos\frac{2\pi t}{T_j}\,dt \\
&\quad= \sum_{j=1}^{p}\frac{2\pi^2}{T_j^2}.
\end{aligned}$$

Obviously, $w_0 \in B^{1,2}(\mathbb{R}^N)$. Note that, for all $j \in \{1, \ldots, p\}$, we have

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos\lambda t\sin\frac{2\pi t}{T_j}\,dt \\
&\quad= \lim_{T\to\infty}\frac{1}{4T}\left[\int_{-T}^{T}\cos\left(\lambda+\frac{2\pi}{T_j}\right)t\,dt - \int_{-T}^{T}\sin\left(\lambda-\frac{2\pi}{T_j}\right)t\,dt\right] \\
&\quad= \begin{cases} 0 & \text{as } \lambda \neq -\frac{2\pi}{T_j}, \\ \frac{1}{2} & \text{as } \lambda = -\frac{2\pi}{T_j} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin\lambda t\sin\frac{2\pi t}{T_j}\,dt \\
&\quad= -\lim_{T\to\infty}\frac{1}{4T}\left[\int_{-T}^{T}\cos\left(\lambda+\frac{2\pi}{T_j}\right)t\,dt - \int_{-T}^{T}\cos\left(\lambda-\frac{2\pi}{T_j}\right)t\,dt\right] \\
&\quad= \begin{cases} 0 & \text{as } \lambda \neq \pm\frac{2\pi}{T_j}, \\ \frac{1}{2} & \text{as } \lambda = \frac{2\pi}{T_j}, \\ -\frac{1}{2} & \text{as } \lambda = -\frac{2\pi}{T_j}. \end{cases}
\end{aligned}$$

Hence, for all $j \in \{1, \ldots, p\}$, we have

$$\begin{aligned}
a(w_0, \lambda) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} w_0(t)\,dt \\
&= \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{j=1}^{p} e^{-i\lambda t} \sin \frac{2\pi t}{T_j} \right) dt, 0, \ldots, 0 \right)^{\tau} \\
&= \left(\sum_{j=1}^{p} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\cos \lambda t - i \sin \lambda t) \sin \frac{2\pi t}{T_j}\, dt, 0, \ldots, 0 \right)^{\tau} \\
&= \begin{cases} 0 & \text{as } \lambda \neq \pm \frac{2\pi}{T_j}, \\ (-\frac{ip}{2}, 0, \ldots, 0)^{\tau} & \text{as } \lambda = \frac{2\pi}{T_j}, \\ (\frac{p}{2} + \frac{ip}{2}, 0, \ldots, 0)^{\tau} & \text{as } \lambda = -\frac{2\pi}{T_j}, \end{cases}
\end{aligned}$$

which implies that $\Lambda(w_0) = \{\pm \frac{2\pi}{T_j} | j = 1, \ldots, p\}$. Obviously, $\Lambda(w_0) \subset \Lambda$. So $w_0 \in \mathcal{V}$. Moreover, the equality

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} w_0(t)\,dt = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_{j=1}^{p} \sin \frac{2\pi t}{T_j} \right) dt, 0, \ldots, 0 \right)^{\tau} = 0$$

implies that $w_0 \in \tilde{\mathcal{V}}$. By $(\mathcal{H}2)$, there exist $\beta > \|P\| \sum_{j=1}^{p} \frac{2\pi^2}{T_j^2}$ and $r_0 > 0$ such that

$$F(t,x) \geq \beta |x|^2, \quad \forall |x| \geq r_0. \tag{3.3}$$

Since $F(t, \cdot)$ is almost periodic in t uniformly for $x \in \mathbb{R}^N$, there exists $M_0 > 0$ such that $|F(t,x)| \leq M_0$ for all $t \in \mathbb{R}$ and $|x| \leq r_0$. Then by (3.3), we have

$$F(t,x) \geq \beta |x|^2 - \beta r_0^2 - M_0. \tag{3.4}$$

It follows from (3.4) and $(\mathcal{P})'$ that

$$\begin{aligned}
&\varphi(x + s w_0) \\
&\quad = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\frac{s^2}{2} \big(P(t) \nabla w_0(t), \nabla w_0(t) \big) - F\big(t, x + s w_0(t)\big) \right] dt \\
&\quad \leq \frac{\|P\| s^2}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \nabla w_0(t) \right|^2 dt \\
&\qquad - \beta \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| x + s w_0(t) \right|^2 dt + \beta r_0^2 + M_0 \\
&\quad = \frac{\|P\| s^2}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \nabla w_0(t) \right|^2 dt - \beta \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x|^2\, dt \\
&\qquad - \beta s^2 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| w_0(t) \right|^2 dt + \beta r_0^2 + M_0 \\
&\quad \leq \frac{\|P\| s^2}{2} \sum_{j=1}^{p} \frac{2\pi^2}{T_j^2} - \frac{\beta}{2} s^2 - \beta |x|^2 + \beta r_0^2 + M_0,
\end{aligned}$$

which, together with $\beta > \|P\| \sum_{j=1}^{p} \frac{2\pi^2}{T_j^2}$, implies that

$$\varphi(x + sw_0) \to -\infty \quad \text{as } s^2 + |x|^2 \to +\infty. \tag{3.5}$$

Thus (3.2) and (3.5) implies our conclusion. □

Lemma 3.4 *Assume that* $(\mathcal{P})'$, $(\mathcal{H}2)$, *and* $(\mathcal{H}4)$ *hold. Then any Cerami sequence* $\{u_n\}$ *has a convergent subsequence in* $\mathcal{V}$.

Proof Assume that there exists a positive constant C_1 such that

$$\left|\varphi(u_n)\right| \le C_1, \qquad \left(1 + \|u_n\|\right)\left\|\varphi'(u_n)\right\| \le C_1, \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

By $(\mathcal{H}4)$, we have

$$\left[\left(\nabla F(t,x),x\right) - 2F(t,x)\right]\left(\zeta + \eta|x|^{\nu}\right) \ge F(t,x), \quad \forall x \in \mathbb{R}^N, |x| > L, t \in \mathbb{R}. \tag{3.7}$$

Then by (f_1), (f_2), and (3.7), there exists a constant $C_2 > 0$ such that

$$\left[\left(\nabla F(t,x),x\right) - 2F(t,x)\right]\left(\zeta + \eta|x|^{\nu}\right) \ge F(t,x) - C_2, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{3.8}$$

It follows from (3.4) and (3.8) that there exist $C_3 > 0$ and $C_4 > 0$ such that

$$\begin{aligned}\left(\nabla F(t,x),x\right) - 2F(t,x) &\ge \frac{F(t,x) - C_2}{\zeta + \eta|x|^{\nu}} \\ &\ge \frac{\beta|x|^2 - \beta r_0^2 - M_0 - C_2}{\zeta + \eta|x|^{\nu}} \\ &\ge C_3|x|^{2-\nu} - C_4, \quad \forall x \in \mathbb{R}^N.\end{aligned} \tag{3.9}$$

Hence, by (3.9), we have

$$\begin{aligned}3C_1 &\ge 2\varphi(u_n) - \left\langle\varphi'(u_n), u_n\right\rangle \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[\left(\nabla F\left(t,u_n(t)\right), u_n(t)\right) - 2F\left(t,u_n(t)\right)\right] dt \\ &\ge C_3 \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left|u_n(t)\right|^{2-\nu} dt - C_4.\end{aligned} \tag{3.10}$$

This shows that $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |u_n(t)|^{2-\nu}\, dt$ is bounded. It follows from (3.3) and (3.7) that

$$\left[\left(\nabla F(t,x),x\right) - 2F(t,x)\right]\left(\zeta + \eta|x|^{\nu}\right) \ge F(t,x) \ge \beta|x|^2 > 0, \quad \forall |x| > L + r_0, t \in \mathbb{R}. \tag{3.11}$$

By $(\mathcal{P})'$, (f_1), (f_2), (3.6), (3.8), and (3.11), we have

$$\begin{aligned}&\frac{\min\{m,1\}}{2}\|u_n\|^2_{B^{1,2}(\mathbb{R}^N)} \\ &\quad \le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{1}{2}\left(P(t)\nabla u(t), \nabla u(t)\right) dt + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{1}{2}\left|u_n(t)\right|^2 dt\end{aligned}$$

$$
\begin{aligned}
&= \varphi(u_n) + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F\bigl(t,u_n(t)\bigr)\,dt + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{1}{2}\bigl|u_n(t)\bigr|^2\,dt \\
&\le C_1 + \lim_{T\to\infty}\frac{1}{4T}\int_{-T}^{T}\bigl|u_n(t)\bigr|^2\,dt + C_2 \\
&\quad + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\bigl(\zeta+\eta\bigl|u_n(t)\bigr|^{\nu}\bigr)\bigl[\bigl(\nabla F\bigl(t,u_n(t)\bigr),u_n(t)\bigr) - 2F\bigl(t,u_n(t)\bigr)\bigr]\,dt \\
&\le C_1 + \lim_{T\to\infty}\frac{1}{4T}\int_{-T}^{T}\bigl|u_n(t)\bigr|^2\,dt + C_2 \\
&\quad + \lim_{T\to\infty}\frac{1}{2T}\int_{\{t\in[-T,T]:|u_n(t)|\le L+r_0\}}\bigl(\zeta+\eta\bigl|u_n(t)\bigr|^{\nu}\bigr) \\
&\quad \times\bigl[\bigl(\nabla F\bigl(t,u_n(t)\bigr),u_n(t)\bigr) - 2F\bigl(t,u_n(t)\bigr)\bigr]\,dt \\
&\quad + \lim_{T\to\infty}\frac{1}{2T}\int_{\{t\in[-T,T]:|u_n(t)|> L+r_0\}}\bigl(\zeta+\eta\bigl|u_n(t)\bigr|^{\nu}\bigr) \\
&\quad \times\bigl[\bigl(\nabla F\bigl(t,u_n(t)\bigr),u_n(t)\bigr) - 2F\bigl(t,u_n(t)\bigr)\bigr]\,dt \\
&\le C_1 + \lim_{T\to\infty}\frac{1}{4T}\int_{-T}^{T}\bigl|u_n(t)\bigr|^2\,dt + C_2 + C_5 + \bigl(\zeta+\eta\|u_n\|_\infty^{\nu}\bigr) \\
&\quad \times\lim_{T\to\infty}\frac{1}{2T}\int_{\{t\in[-T,T]:|u_n(t)|> L+r_0\}}\bigl[\bigl(\nabla F\bigl(t,u_n(t)\bigr),u_n(t)\bigr) - 2F\bigl(t,u_n(t)\bigr)\bigr]\,dt \\
&= C_1 + \lim_{T\to\infty}\frac{1}{4T}\int_{-T}^{T}\bigl|u_n(t)\bigr|^2\,dt + C_2 + C_5 + \bigl(\zeta+\eta\|u_n\|_\infty^{\nu}\bigr) \\
&\quad \times\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\bigl[\bigl(\nabla F\bigl(t,u_n(t)\bigr),u_n(t)\bigr) - 2F\bigl(t,u_n(t)\bigr)\bigr]\,dt - \bigl(\zeta+\eta\|u_n\|_\infty^{\nu}\bigr) \\
&\quad \times\lim_{T\to\infty}\frac{1}{2T}\int_{\{t\in[-T,T]:|u_n(t)|\le L+r_0\}}\bigl[\bigl(\nabla F\bigl(t,u_n(t)\bigr),u_n(t)\bigr) - 2F\bigl(t,u_n(t)\bigr)\bigr]\,dt \\
&\le C_1 + \|u_n\|_\infty^{\nu}\lim_{T\to\infty}\frac{1}{4T}\int_{-T}^{T}\bigl|u_n(t)\bigr|^{2-\nu}\,dt + C_2 + C_5 \\
&\quad + C_6\bigl(\zeta+\eta\|u_n\|_\infty^{\nu}\bigr) + 3C_1\bigl(\zeta+\eta\|u_n\|_\infty^{\nu}\bigr) \\
&\le C_1 + \frac{C_{**}^{\nu}}{2}\|u_n\|_{B^{1,2}(\mathbb{R}^N)}^{\nu}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\bigl|u_n(t)\bigr|^{2-\nu}\,dt + C_2 + C_5 \\
&\quad + C_6\bigl(\zeta+\eta C_{**}^{\nu}\|u_n\|_{B^{1,2}(\mathbb{R}^N)}^{\nu}\bigr) + 3C_1\bigl(\zeta+\eta C_{**}^{\nu}\|u_n\|_{B^{1,2}(\mathbb{R}^N)}^{\nu}\bigr),
\end{aligned}
\tag{3.12}
$$

where C_5 and C_6 are positive constants. $\nu < 2$ and $m > 0$, (3.12), and the boundedness of $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|u_n(t)|^{2-\nu}\,dt$ imply that $\|u_n\|$ is bounded. Similar to the argument in [35] and [36], we see that there exists a subsequence, still denoted by $\{u_n\}$, and $u^* \in \mathcal{V}$ such that $\|u_n - u^*\|_{B^{1,2}(\mathbb{R}^N)} \to 0$ in $\mathcal{V}$. By Lemma 2.3, we have $\|u_n - u^*\| \to 0$ in $\mathcal{V}$. □

Proof of Theorem 1.1 Lemma 3.1, Lemma 3.3 and Example 3 of Section 3.5 in [12] imply that A links B [hm]. Lemma 2.6 and Remark 2.1 imply that there is a Cerami sequence $\{u_n\}$, that is, there is a sequence $\{u_n\}$ satisfying

$$
\varphi(u_n) \to c, \qquad \bigl(1 + \|u_n\|\bigr)\bigl\|\varphi'(u_n)\bigr\| \to 0 \quad \text{as } n \to \infty.
$$

Then by Lemma 3.4, we know that there exists a subsequence, still denoted by $\{u_n\}$, and $u^* \in \mathcal{V}$ such that $\|u_n - u^*\| \to 0$ in $\mathcal{V}$. Since $\|\varphi'(u_n)\| \to 0$ and $\varphi'(\cdot)$ is continuous, $\varphi'(u^*) = 0$. Hence u^* is a solution of system (1.1) and by the continuity of $\varphi(\cdot)$, Remark 2.1 and Lemma 3.1, we know that $\varphi(u^*) = c \geq \inf_{\mathcal{B}} \varphi > 0$. Obviously, by $(\mathcal{H}3)$, it is easy to see that $u^* \notin \mathbb{R}^N$. Thus we complete the proof. □

Proof of Theorem 1.2 The proof is easy to complete by replacing Lemma 3.1 with Lemma 3.2 in the proof of Theorem 1.1. □

Lemma 3.5 *Assume that* $(\mathcal{P})'$ *and* $(\mathcal{H}2)'$ *hold. Then for each finite dimensional space* $\hat{\mathcal{V}} \subset \mathcal{V}$, *there exists* $R > 0$ *such that* $\varphi(u) \leq 0$ *on* $\hat{\mathcal{V}}/\mathcal{B}_R$.

Proof In fact, since $\hat{\mathcal{V}}$ is finite dimensional, all norms on $\hat{\mathcal{V}}$ are equivalent. Hence, there exist $d_1, d_2 > 0$ such that

$$d_1\|u\|^2 \leq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \bigl|u(t)\bigr|^2\,dt \leq d_2\|u\|^2. \tag{3.13}$$

By $(\mathcal{H}2)'$, we know that there exist constants $\beta > \frac{\|P\|}{2d_1}$ and $r_1 > 0$ such that

$$F(t,x) \geq \beta|x|^2, \quad \forall |x| \geq r_1, t \in \mathbb{R}. \tag{3.14}$$

It follows from (f_1) and (3.14) that there exists $C_7 > 0$ such that

$$F(t,x) \geq \beta|x|^2 - C_7, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}. \tag{3.15}$$

Then by $(\mathcal{P})'$, (3.13), and (3.15), we have

$$\begin{aligned}
\varphi(u) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[\frac{1}{2}\bigl(P(t)\nabla u(t), \nabla u(t)\bigr) - F\bigl(t, u(t)\bigr)\right] dt \\
&\leq \frac{\|P\|}{2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \bigl|\nabla u(t)\bigr|^2\,dt - \beta \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \bigl|u(t)\bigr|^2\,dt + C_7 \\
&\leq \frac{\|P\|}{2}\|u\|^2 - \beta d_1\|u\|^2 + C_7.
\end{aligned}$$

Note that $\beta > \frac{\|P\|}{2d_1}$. So $\varphi(u) \to -\infty$, as $\|u\| \to \infty$. Thus we complete the proof. □

Proof of Theorem 1.3 (*and Theorem* 1.4) By $(\mathcal{H}5)$, we know that φ is even and $\varphi(0) = 0$. Since $(\mathcal{H}2)'$ implies that $(\mathcal{H}2)$, Lemma 3.4 on replacing $(\mathcal{H}2)$ with $(\mathcal{H}2)'$ still holds. Then by Lemma 3.1 (Lemma 3.2 corresponding to Theorem 1.4), Lemma 3.4, Lemma 3.5, Remark 2.2, and the symmetric mountain pass theorem, we see that φ possesses an unbounded sequence of critical values. Thus we complete the proof. □

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author read and approved the final manuscript.

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