Zhang and Khalique *Advances in Difference Equations* (2015) 2015:195
DOI 10.1186/s13662-015-0510-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Exact solitary wave and quasi-periodic wave solutions of the KdV-Sawada-Kotera-Ramani equation

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Abstract
In this paper we derive new exact solitary wave solutions and quasi-periodic traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation by using a method which we introduce here for the first time. Firstly, we reduce the associated fourth-order nonlinear ordinary differential equation (ODE) into a solvable first-order nonlinear ODE to obtain new exact traveling wave solutions, including the solitary wave and periodic solutions. Furthermore, using the new method we derive the quasi-periodic wave solutions of this equation by assuming that the solutions of the corresponding higher-order ODE are the sum of the solutions of two solvable first-order nonlinear ODEs. This new method can be used to investigate the exact traveling wave solutions and quasi-periodic wave solutions of a general class of higher-order wave equations.

Keywords: nonlinear wave equations; traveling wave solutions; sub-equation method; quasi-periodic wave solutions

1 Introduction

In this paper we study the KdV-Sawada-Kotera-Ramani equation [1–16]

$$u_t + a\left(3u^2 + u_{xx}\right)_x + b\left(15u^3 + 15uu_{xx} + u_{xxxx}\right)_x = 0, \tag{1.1}$$

which was used to theoretically study the resonances of solitons in a one-dimensional space by Hirota and Ito [13]. Equation (1.1) is reduced to the KdV equation when $b = 0$ and to the Sawada-Kotera equation when $a = 0$; thus, it is a linear combination of the KdV equation and the Sawada-Kotera equation. The existence of conservation laws for this equation was proved by Konno [14]. Some traveling wave solutions were derived in [16] by the (G'/G)-expansion method. In [15], the traveling wave solutions of (1.1) were studied by using the generalized auxiliary equation method. Unfortunately, too many undetermined coefficients were involved in this method and some conditions on these coefficients were ignored, and thus some wrong results were given in [15], which can be checked by Maple.

We aim to investigate the traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation (1.1) in the form $u(x,t) = y(\xi) = y(x - ct)$, where c is the wave speed. Under the traveling wave coordinates, (1.1) is transformed to the nonlinear ordinary differential equation of the independent variable ξ. By integrating the transformed ODE once with respect to ξ, we obtain

$$\frac{d^4y}{d\xi^4} + \left(15y + \frac{a}{b}\right)\frac{d^2y}{d\xi^2} + 15y^3 + \frac{3a}{b}y^2 - \frac{c}{b}y + g = 0, \tag{1.2}$$

where g is a constant of integration. Clearly, $u(x,t) = y(\xi) = y(x - ct)$ is a traveling wave solution of (1.1) if and only if $y(\xi)$ satisfies (1.2) with the wave speed c and any arbitrary constant g. Since the reduced ODE (1.2) is a fourth-order nonlinear equation which is equivalent to a four-dimensional system, it is very difficult to investigate (1.2) from the point of view of dynamical systems.

There are two classes of solitary waves, namely, embedded solitons and gas solitons that have been studied by many researches in the fields of nonlinear optics and water wave theory [2, 17–20]. In fact, soliton solutions are typically presented by homoclinic solutions to saddle-center equilibrium and saddle-saddle equilibrium, respectively, of the associated ODEs which describe traveling waves of the model PDEs. By using the method of dynamical systems and Congrove's results [21], Li and Zhang [1] investigated the exact explicit gap soliton, embedded soliton, periodic, and quasi-periodic wave solutions of the KdV-Sawada-Kotera-Ramani equation.

Notice that (1.2) contains $\frac{d^4y}{d\xi^4}$, $\frac{d^2y}{d\xi^2}$, and a polynomial of y. Clearly, the general form of (1.2) is given by

$$\frac{d^4y}{d\xi^4} + (Ay + B)\frac{d^2y}{d\xi^2} + Dy^3 + Ey^2 + Fy + G = 0, \tag{1.3}$$

which is actually a special case of the equation

$$\frac{d^4y}{d\xi^4} + (Ay + B)\frac{d^2y}{d\xi^2} + C\left(\frac{dy}{d\xi}\right)^2 + Dy^3 + Ey^2 + Fy + G = 0 \tag{1.4}$$

with $C = 0$. In fact, there exist a lot of nonlinear wave equations and some time-fractional nonlinear wave equations whose corresponding ODEs or their reduced ODEs are special cases of (1.4). See for example [1, 3, 9, 22, 23].

Recently by using the sub-equation method and dynamical system analysis, the bifurcation and exact solutions of (1.4) were studied in [22, 23]. Following the idea and the results in [22], the bifurcations and exact traveling wave solutions to the KdV-Sawada-Kotera-Ramani equation are obtained in Section 1. The sub-equation method has been proposed and well applied in studying the exact solutions of nonlinear differential equations [6–11]. The main idea of the sub-equation method is to assume that the solutions to higher-order ODEs are polynomials of some functions satisfying a simpler equation. However, we observe that a family of solutions to (1.3) can be the sum of two solutions to a second-order ODE which can be reduced to a first-order nonlinear ODE. By using the exact solutions and bifurcations of this sub-equation which was derived in [23], some new traveling wave solutions and quasi-periodic traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation are derived in Section 2.

2 A family of exact traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation

2.1 Preliminaries

Equation (1.3) is a special case of (1.4), which is (1.6) in [22] with $C = 0$. Thus, from Theorem 2.1 and Theorem 2.2 in [22], we have the corresponding theorems regarding (1.3).

Theorem 2.1 *Suppose that* $D \leq 3A^2/40$. *The function* $y = y(\xi)$ *solves the fourth-order differential* (1.3) *if it solves equation*

$$\left(\frac{dy}{d\xi}\right)^2 = a_3y^3 + a_2y^2 + a_1y + a_0, \tag{2.1}$$

where

$$\begin{aligned} a_3 &= \frac{-3A \pm \sqrt{9A^2 - 120D}}{30}, \\ a_2 &= -\frac{3Ba_3 + 2E}{15a_3 + 2A}, \\ a_1 &= -\frac{2(Ba_2 + a_2^2 + F)}{9a_3 + A}, \\ a_0 &= -\frac{Ba_1 + a_1a_2 + 2G}{6a_3}. \end{aligned} \tag{2.2}$$

Note that all the denominators in (2.2) are assumed to be nonzero. If the denominator of a_i in (2.2) is zero, then a_i can be arbitrary constant provided the numerator is also zero.

Theorem 2.2 *Let* $h_\pm = \frac{2\Delta(-a_2 \pm \sqrt{\Delta}) + 3a_1a_2a_3}{54a_3^2}$ *and* $y_e^\pm = \frac{-a_2 \pm \sqrt{\Delta}}{3a_3}$, *where* $\Delta = a_2^2 - 3a_1a_3 > 0$, *then the following conclusions hold*:

(1) *For* $a_0 = 2h_+$, (2.1) *has a bounded solution approaching* y_e^+ *as* ξ *goes to infinity, which can be expressed as*

$$u = \frac{-a_2 + \sqrt{\Delta}}{3a_3} - \frac{\sqrt{\Delta}}{a_3}\operatorname{sech}^2\left[\frac{1}{2}\Delta^{\frac{1}{4}}(\xi - \xi_0)\right], \tag{2.3}$$

a constant solution

$$y = \frac{-a_2 + \sqrt{\Delta}}{3a_3}, \tag{2.4}$$

and an unbounded solution

$$u = \frac{-a_2 + \sqrt{\Delta}}{3a_3} + \frac{\sqrt{\Delta}}{a_3}\operatorname{csch}^2\left[\frac{1}{2}\Delta^{\frac{1}{4}}(\xi - \xi_0)\right], \tag{2.5}$$

where ξ_0 *is an arbitrary constant.*

(2) *For* $a_0 \in (2h_-, 2h_+)$, *if* $a_3 > 0$, *then for any* $y_3 \in (\frac{-a_2 - 2\sqrt{\Delta}}{3a_3}, \frac{-a_2 - \sqrt{\Delta}}{3a_3})$,

$$y = y_3 - \frac{1}{2}\left(3y_3 + \frac{a_2}{a_3} + \sqrt{\Delta_+}\right)\operatorname{sn}^2\left(\Omega_+(\xi - \xi_0), k_+\right), \tag{2.6}$$

is a family of smooth periodic solutions of (2.1). *Here* $k_+ = 2\frac{\sqrt{3y_3^2+2\frac{a_2}{a_3}y_3+\frac{a_1}{a_3}}}{-3y_3-\frac{a_2}{a_3}+\sqrt{\Delta_+}}$, $\Omega_+ = \frac{\sqrt{2}}{4} \times \sqrt{-3a_3y_3 - a_2 + a_3\sqrt{\Delta_+}}$ *and* $\Delta_+ = (\frac{a_2}{a_3})^2 - 3y_3^2 - 2\frac{a_2}{a_3}y_3 - 4\frac{a_1}{a_3}$.

If $a_3 < 0$, *then, for any* $y_1 \in (\frac{-a_2-\sqrt{\Delta}}{3a_3}, \frac{-a_2-2\sqrt{\Delta}}{3a_3})$,

$$y = y_1 - \frac{1}{2}\left(3y_1 + \frac{a_2}{a_3} - \sqrt{\Delta_-}\right)\operatorname{sn}^2\big(\Omega_-(\xi - \xi_0), k_-\big), \tag{2.7}$$

is a family of smooth periodic solutions of (2.1). *Here* $k_- = 2\frac{\sqrt{3y_1^2+2\frac{a_2}{a_3}y_1+\frac{a_1}{a_3}}}{3y_1+\frac{a_2}{a_3}+\sqrt{\Delta_-}}$, $\Omega_- = \frac{\sqrt{2}}{4} \times \sqrt{-3a_3y_1 - a_2 - a_3\sqrt{\Delta_-}}$, *and* $\Delta_- = (\frac{a_2}{a_3})^2 - 3y_1^2 - 2\frac{a_2}{a_3}y_1 - 4\frac{a_1}{a_3}$.

(3) *For* $a_0 \in (-\infty, 2h_-] \cup (2h_+, +\infty)$, (2.1) *has no non-trivial bounded solutions. When* $a_0 = 2h_-$, *an unbounded solution is given by*

$$y = -\frac{a_2 + \sqrt{\Delta}}{3a_3} + \frac{\sqrt{\Delta}}{a_3}\sec^2\left[\frac{1}{2}\Delta^{\frac{1}{4}}(\xi - \xi_0)\right] \tag{2.8}$$

and a constant solution

$$y = -\frac{a_2 + \sqrt{\Delta}}{3a_3}. \tag{2.9}$$

2.2 A family of exact traveling wave solutions of (1.1) obtained from Theorems 2.1 and 2.2

By letting $A = 15$, $B = a/b$, $D = 15$, $E = 3a/b$, $F = -c/b$, and $G = g$, (1.3) is reduced to (1.2). Clearly $D = 15 < 3A^2/40$. Thus from Theorem 2.1, we know that $y = y(\xi)$ solves (1.2) if it solves the first-order nonlinear ODE (2.1) with $a_3 = -1$, $a_2 = -a/(5b)$, $a_1 = (4a^2 + 25bc)/(75b^2)$, and $a_0 = (375b^3g + 8a^3 + 50abc)/(1{,}125b^3)$. Note that g in (1.2) is an arbitrary constant, so a_0 is also an arbitrary constant. Consequently, we obtain the solutions of (1.2) from Theorem 2.2 provided $a_2^2 - 3a_1a_3 = (a^2 + 5bc)/(5b^2) > 0$, *i.e.*, $c > -a^2/(5b)$ for $b > 0$ or $c < -a^2/(5b)$ for $b < 0$.

Theorem 2.3 *The KdV-Sawada-Kotera-Ramani equation* (1.1) *has the following traveling wave solutions with the wave speed c satisfying* $a^2 + 5bc > 0$:

(1) *It has a bounded solitary wave solution*,

$$y = -\frac{a}{15b} - \frac{\sqrt{5(a^2 + 5bc)}}{15|b|} + \frac{\sqrt{5(a^2 + 5bc)}}{5|b|}\operatorname{sech}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right], \tag{2.10}$$

where ξ_0 *is an arbitrary constant.*

(2) *For any* $y_1 \in (-\frac{a}{15b} + \frac{\sqrt{5(a^2+5bc)}}{15|b|}, -\frac{a}{15b} + 2\frac{\sqrt{5(a^2+5bc)}}{15|b|})$,

$$y = y_1 - \frac{1}{2}\left(3y_1 + \frac{a}{5b} - \sqrt{\Delta_1}\right)\operatorname{sn}^2\big(\Omega_1(x - ct - \xi_0), k_1\big) \tag{2.11}$$

is a family of smooth periodic traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation (1.1). *Here* $\Omega_1 = \frac{\sqrt{2}}{4}\sqrt{3y_1 + \frac{a}{5b} + \sqrt{\Delta_1}}$, $k_1 = \frac{\sqrt{12y_1^2+8\frac{a}{5b}y_1-4\frac{4a^2+25bc}{75b^2}}}{3y_3+\frac{a}{5b}+\sqrt{\Delta_1}}$, *and* $\Delta_1 = -3y_1^2 - \frac{2a}{5b}y_1 + \frac{3a^2c+16a^2+100bc}{75b^2}$.

(3) *It has two classes of unbounded solutions,*

$$y=-\frac{a}{15b}-\frac{\sqrt{5(a^2+5bc)}}{15|b|}-\frac{\sqrt{5(a^2+5bc)}}{5|b|}\operatorname{csch}^2\left[\frac{1}{2}\left(\frac{a^2+5bc}{5b^2}\right)^{\frac{1}{4}}(x-ct-\xi_0)\right] \tag{2.12}$$

and

$$y=-\frac{a}{15b}+\frac{\sqrt{5(a^2+5bc)}}{15|b|}-\frac{\sqrt{5(a^2+5bc)}}{5|b|}\sec^2\left[\frac{1}{2}\left(\frac{a^2+5bc}{5b^2}\right)^{\frac{1}{4}}(x-ct-\xi_0)\right]. \tag{2.13}$$

3 Exact quasi-periodic traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation

In this section, we obtain a new family of exact traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation (1.1), which includes the quasi-periodic solutions.

3.1 A new family of exact solutions and quasi-periodic solutions of (1.3)

Let $y = U + V$, then (1.3) is reduced to

$$\begin{aligned}&\left(\frac{d^4U}{d\xi^4}+(AU+B)\frac{d^2U}{d\xi^2}+DU^3+EU^2+F_1U+g_1\right)\\&\quad+\left(\frac{d^4V}{d\xi^4}+(AV+B)\frac{d^2V}{d\xi^2}+DV^3+EV^2+F_2V+g_2\right)\\&\quad+V\left(A\frac{d^2U}{d\xi^2}+3DU^2+EU+(F-F_2)\right)\\&\quad+U\left(A\frac{d^2V}{d\xi^2}+3DV^2+EV+(F-F_1)\right)=0,\end{aligned} \tag{3.1}$$

where F_1 and F_2 are some constants to be determined later, g_1 and g_2 are arbitrary constants. Clearly, $y = U + V$ solves (1.3) if there exist some values of F_1 and F_2 such that U and V satisfy

$$\begin{cases}A\frac{d^2U}{d\xi^2}+3DU^2+EU+(F-F_2)=0,\\ \frac{d^4U}{d\xi^4}+(AU+B)\frac{d^2U}{d\xi^2}+DU^3+EU^2+F_1U+g_1=0,\end{cases} \tag{3.2}$$

and

$$\begin{cases}A\frac{d^2V}{d\xi^2}+3DV^2+EV+(F-F_1)=0,\\ \frac{d^4V}{d\xi^4}+(AV+B)\frac{d^2V}{d\xi^2}+DV^3+EV^2+F_2V+g_2=0,\end{cases} \tag{3.3}$$

respectively.

Suppose U satisfies the first equation of system (3.2). Multiplying it by $\frac{dU}{d\xi}$ and integrating once, we obtain

$$\left(\frac{dU}{d\xi}\right)^2=-\frac{2D}{A}U^3-\frac{E}{A}U^2-2\frac{F-F_2}{A}U+G_1, \tag{3.4}$$

where G_1 is an integral constant. By letting $a_3 = -2D/A$, $a_2 = -E/A$, and $a_1 = -2(F - F_2)/A$, from Theorem 2.1, we know that U solves system (3.2) if and only if

$$\begin{aligned} -\frac{2D}{A} &= \frac{-3A \pm \sqrt{9A^2 - 120D}}{30}, \\ -\frac{E}{A} &= -\frac{3Ba_3 + 2E}{15a_3 + 2A}, \\ -2\frac{F - F_2}{A} &= -\frac{2(Ba_2 + a_2^2 + F_1)}{9a_3 + A}. \end{aligned} \tag{3.5}$$

Solving the first equation of system (3.5) for D gives $D = A^2/15$ and thus $a_3 = -2A/15$. By substituting the value of a_3 into the second equation and solving for E, we have $E = AB/5$ and $a_2 = -B/5$. Then substituting the values of a_3 and a_2 into the third equation of (3.5) gives $F - F_2 + 5F_1 = 4B^2/5$. In the same way, from system (3.3), we can obtain $F - F_1 + 5F_2 = 4B^2/5$. Consequently, we can determine the two undetermined constants F_1 and F_2 as $F_1 = F_2 = B^2/5 - F/4$. We thus have the following theorem.

Theorem 3.1 *Suppose that the coefficients of* (1.3) *satisfy* $D = A^2/15$ *and* $E = AB/15$. *Then* $y(\xi) = U(\xi) + V(\xi)$ *solves* (1.3) *if* U *and* V *satisfy equation*

$$\frac{d^2\phi}{d\xi^2} = -\frac{1}{5}A\phi^2 - \frac{1}{5}B\phi - \left(\frac{5}{4A}F - \frac{1}{5A}B^2\right). \tag{3.6}$$

Obviously, (3.6) is equivalent to the first-order ODE

$$\left(\frac{d\phi}{d\xi}\right)^2 = -\frac{2}{15}A\phi^3 - \frac{1}{5}B\phi^2 - \left(\frac{5}{8A}F - \frac{1}{10A}B^2\right)\phi + G \tag{3.7}$$

if $d\phi/d\xi \neq 0$ except for the case when $\phi = (-B \pm \sqrt{5B^2 - 25F})/(2A)$. Here G is an arbitrary integral constant. From (3.7), we can obtain the solutions of (1.3).

3.2 Exact solutions and quasi-periodic solutions of the KdV-Sawada-Kotera-Ramani equation (1.1)

It is easy to see that the associated ODE (1.2) of the KdV-Sawada-Kotera-Ramani equation is the fourth-order ODE (1.3) with $A = 15$, $B = a/b$, $D = 15$, $E = 3a/b$, $F = -c/b$, and $G = g$. Clearly, the coefficients of (1.2) satisfy the conditions of Theorem 3.1, that is, $D = A^2/15$ and $E = AB/5$. Thus Theorem 3.1 implies that (1.2) admits the solutions $y = U + V$, where U and V are determined by

$$\frac{d^2\phi}{d\xi^2} = -3\phi^2 - \frac{a}{5b}\phi - \frac{4a^2 + 25bc}{75b^2} \tag{3.8}$$

or

$$\left(\frac{d\phi}{d\xi}\right)^2 = -2\phi^3 - \frac{a}{5b}\phi^2 - \frac{4a^2 + 25bc}{150b^2}\phi + G. \tag{3.9}$$

Note that G is an arbitrary constant, U and V are not constants except $-a \pm \sqrt{5a^2 + 25bc}/(30b)$. Consequently, we know that when the wave speed c satisfies $a^2 + 5bc > 0$, *i.e.*,

$c > -a^2/(5b)$ for $b > 0$ or $c < -a^2/(5b)$ for $b < 0$, (3.8) admits the following six classes of solutions for different values of G_1 and G_2:

$$\phi_1(\xi) = \frac{-a + \sqrt{5a^2 + 25bc}}{30b}; \tag{3.10}$$

$$\phi_2(\xi) = \frac{-a - \sqrt{5a^2 + 25bc}}{30b}; \tag{3.11}$$

$$\begin{aligned} \phi_3(\xi) = {} & -\frac{\operatorname{sgn}(b)a + \sqrt{5a^2 + 25bc}}{30|b|} \\ & + \frac{\sqrt{5a^2 + 25bc}}{10|b|} \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}} (\xi - \xi_0)\right]; \end{aligned} \tag{3.12}$$

$$\begin{aligned} \phi_4(\xi) = {} & -\frac{\operatorname{sgn}(b)a + \sqrt{5a^2 + 25bc}}{30|b|} \\ & - \frac{\sqrt{5a^2 + 25bc}}{10|b|} \operatorname{csch}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}} (\xi - \xi_0)\right]; \end{aligned} \tag{3.13}$$

$$\begin{aligned} \phi_5(\xi) = {} & -\frac{\operatorname{sgn}(b)a - \sqrt{5a^2 + 25bc}}{30|b|} \\ & - \frac{\sqrt{5a^2 + 25bc}}{10|b|} \sec^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}} (\xi - \xi_0)\right]; \end{aligned} \tag{3.14}$$

For any $\theta \in (\frac{-\operatorname{sgn}(b)a+\sqrt{5a^2+25bc}}{30|b|}, \frac{-\operatorname{sgn}(b)a+2\sqrt{5a^2+25bc}}{30|b|})$,

$$\phi_6(\xi, \theta) = \theta - \frac{1}{2}\left(3\theta + \frac{a}{10b} - \sqrt{\Delta_2}\right) \operatorname{sn}^2\left(\Omega_2(\xi - \xi_0), k_2\right), \tag{3.15}$$

where $\Omega_2(\theta, c) = \frac{\sqrt{2}}{4}\sqrt{6\theta + \frac{a}{5b} + 2\sqrt{\Delta_2}}$, $k_2(\theta, c) = \frac{\sqrt{12\theta^2+4\frac{a}{5b}\theta-\frac{4a^2+25bc}{75b^2}}}{3\theta+\frac{a}{10b}+\sqrt{\Delta_2}}$, and $\Delta_2(\theta, c) = -3\theta^2 - \frac{a}{5b}\theta + \frac{19a^2+100bc}{300b^2}$.

According to the above analysis and Theorem 3.1, we obtain the solutions of (1.2) and consequently the exact traveling wave solutions of the KdV-Sawada-Kotera-Ramani equation (1.1). We have the following theorem.

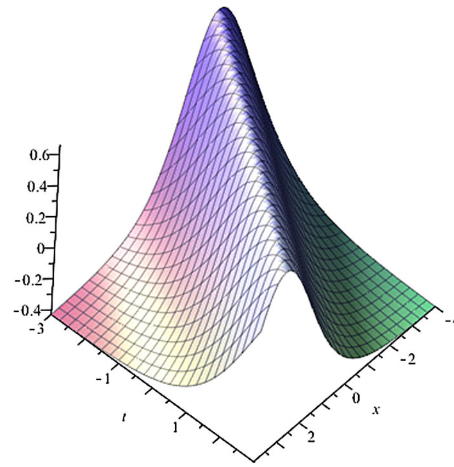
Theorem 3.2 *The KdV-Sawada-Kotera-Ramani equation* (1.1) *admits the following traveling wave solutions*: $y_{ij}(x,t) = \phi_i(\xi, c) + \phi_j(\xi, c)$, $i, j \in \{1, 2, 3, 4, 5, 6\}$, *with* $\xi = x - ct$. *Here* ϕ_i*s are determined by* (3.10)-(3.15), *respectively, and the wave speed c satisfies* $a^2 + 5bc > 0$.

In fact, from Theorem 3.2, we can obtain the following three classes of solitary wave solutions; two families of periodic wave solutions, a family of quasi-periodic wave solutions, and some unbounded solutions.

(1) For any constant c satisfying $a^2 + 5bc > 0$,

$$\begin{aligned} y_{33}(x,t) = 2\phi_3(\xi) = {} & -\frac{\operatorname{sgn}(b)a + \sqrt{5a^2 + 25bc}}{15|b|} \\ & + \frac{\sqrt{5a^2 + 25bc}}{5|b|} \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}} (x - ct - \xi_0)\right], \end{aligned} \tag{3.16}$$

Figure 1 The solitary wave solution y_{33} of the KdV-Sawada-Kotera-Ramani equation (1.1) with $a = 1$, $b = 1$, and $c = 1$.



$$y_{13}(x,t) = \phi_1(\xi) + \phi_3(\xi) = -\frac{\operatorname{sgn}(b)a + \sqrt{5a^2 + 25bc}}{15|b|} + \frac{\sqrt{5a^2 + 25bc}}{10|b|}\operatorname{sech}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right], \tag{3.17}$$

and

$$y_{23}(x,t) = \phi_2(\xi) + \phi_3(\xi) = -\frac{a}{15b} + \frac{\sqrt{5a^2 + 25bc}}{10|b|}\operatorname{sech}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right] \tag{3.18}$$

are three classes of solitary wave solutions of the KdV-Sawada-Kotera-Ramani equation. The phase portrait of solution (3.16) with $a = 1$, $b = 1$, and $c = 1$ is shown in Figure 1. Note that solution (3.16) is the same as solution (2.10) obtained earlier in Section 2, solution (3.17) is a new solitary wave solution and solution (3.18) is the solitary wave solution (76) obtained in [24].

(2) For any $\theta \in (\frac{-\operatorname{sgn}(b)a+\sqrt{5a^2+25bc}}{30|b|}, \frac{-\operatorname{sgn}(b)a+2\sqrt{5a^2+25bc}}{30|b|})$, and any constant c satisfying $a^2 + 5bc > 0$,

$$y_{16}(x,t) = \phi_1(\xi) + \phi_6(\xi,\theta) = \frac{-a + \sqrt{5a^2 + 25bc}}{30b} + \theta - \frac{1}{2}\left(3\theta + \frac{a}{10b} - \sqrt{\Delta_2}\right)\operatorname{sn}^2\bigl(\Omega_2(x - ct - \xi_0), k_2\bigr) \tag{3.19}$$

and

$$y_{26}(x,t) = \phi_2(\xi) + \phi_6(\xi,\theta) = \frac{-a - \sqrt{5a^2 + 25bc}}{30b} + \theta - \frac{1}{2}\left(3\theta + \frac{a}{10b} - \sqrt{\Delta_2}\right)\operatorname{sn}^2\bigl(\Omega_2(x - ct - \xi_0), k_2\bigr) \tag{3.20}$$

are two families of periodic wave solutions of the KdV-Sawada-Kotera-Ramani equation. Here $\Omega_2 = \frac{\sqrt{2}}{4}\sqrt{6\theta + \frac{a}{5b} + 2\sqrt{\Delta_2}}$, $k_2 = \frac{\sqrt{12\theta^2 + 4\frac{a}{5b}\theta - \frac{4a^2+25bc}{75b^2}}}{3\theta + \frac{a}{10b} + \sqrt{\Delta_2}}$, and $\Delta_2 = -3\theta^2 - \frac{a}{5b}\theta + \frac{19a^2+100bc}{300b^2}$.

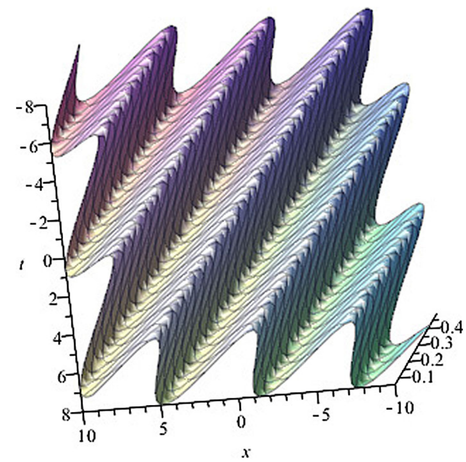


Figure 2 The quasi-periodic wave solution y_{66} of (1.1) with $a = 1$, $b = 1$, $c = 1$, $\theta_1 = -1/30 + \sqrt{30}/20$, and $\theta_2 = -\sqrt{110}(1/5 - 3\sqrt{30}/10)/60$.

(3) For any constant c satisfying $a^2 + 5bc > 0$ and two arbitrary constants θ_1 and θ_2 satisfying $\theta_i \in (\frac{-\operatorname{sgn}(b)a+\sqrt{5a^2+25bc}}{30|b|}, \frac{-\operatorname{sgn}(b)a+2\sqrt{5a^2+25bc}}{30|b|})$, $i = 1, 2$, we have

$$\begin{aligned} y_{66}(x,t) &= \phi_6(\xi,\theta_1) + \phi_6(\xi,\theta_2) \\ &= \theta_1 - \frac{1}{2}\left(3\theta_1 + \frac{a}{10b} - \sqrt{\Delta_2(\theta_1)}\right) \operatorname{sn}^2\big(\Omega_2(\theta_1)(x - ct - \xi_0), k_2(\theta_1)\big) \\ &\quad + \theta_2 - \frac{1}{2}\left(3\theta_2 + \frac{a}{10b} - \sqrt{\Delta_2(\theta_2)}\right) \operatorname{sn}^2\big(\Omega_2(\theta_2)(x - ct - \xi_0), k_2(\theta_2)\big), \end{aligned} \tag{3.21}$$

where $\Omega_2(\theta_i) = \frac{\sqrt{2}}{4}\sqrt{6\theta_i + \frac{a}{5b} + 2\sqrt{\Delta_2(\theta_i)}}$, $k_2(\theta_i) = \frac{\sqrt{12\theta_i^2+4\frac{a}{5b}\theta_i-\frac{4a^2+25bc}{75b^2}}}{3\theta_i+\frac{a}{10b}+\sqrt{\Delta_2(\theta_i)}}$, and $\Delta_2(\theta_i) = -3\theta_i^2 - \frac{a}{5b}\theta_i + \frac{19a^2+100bc}{300b^2}$, $i = 1, 2$. Note that (3.21) with $\theta_1 = \theta_2$ is a class of periodic traveling wave solutions (2.11), which were obtained in Section 2. However, (3.21) is a family of quasi-periodic traveling wave solutions when $\theta_1 \neq \theta_2$ and $\Omega_2(\theta_1)/\Omega_2(\theta_2)$ is irrational.

The phase portrait of (3.21) with $a = 1$, $b = 1$, $c = 1$, $\theta_1 = -1/30 + \sqrt{30}/20$, and $\theta_2 = -\sqrt{110}(1/5 - 3\sqrt{30}/10)/60$ is shown in Figure 2.

(4) For any $\theta \in (\frac{-\operatorname{sgn}(b)a+\sqrt{5a^2+25bc}}{30|b|}, \frac{-\operatorname{sgn}(b)a+2\sqrt{5a^2+25bc}}{30|b|})$, and any constant c satisfying $a^2 + 5bc > 0$,

$$\begin{aligned} y_{36}(x,t) &= \phi_3(\xi) + \phi_6(\xi,\theta) = -\frac{\operatorname{sgn}(b)a + \sqrt{5a^2 + 25bc}}{30|b|} + \theta \\ &\quad + \frac{\sqrt{5a^2 + 25bc}}{10|b|} \operatorname{sech}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(\xi - \xi_0)\right] \\ &\quad - \frac{1}{2}\left(3\theta + \frac{a}{10b} - \sqrt{\Delta_2}\right) \operatorname{sn}^2\big(\Omega_2(x - ct - \xi_0), k_2\big) \end{aligned} \tag{3.22}$$

is a family of bounded wave solutions of the KdV-Sawada-Kotera-Ramani equation (see Figure 3). Here $\Omega_2 = \frac{\sqrt{2}}{4}\sqrt{6\theta + \frac{a}{5b} + 2\sqrt{\Delta_2}}$, $k_2 = \frac{\sqrt{12\theta^2+4\frac{a}{5b}\theta-\frac{4a^2+25bc}{75b^2}}}{3\theta+\frac{a}{10b}+\sqrt{\Delta_2}}$, and $\Delta_2 = -3\theta^2 - \frac{a}{5b}\theta + \frac{19a^2+100bc}{300b^2}$.

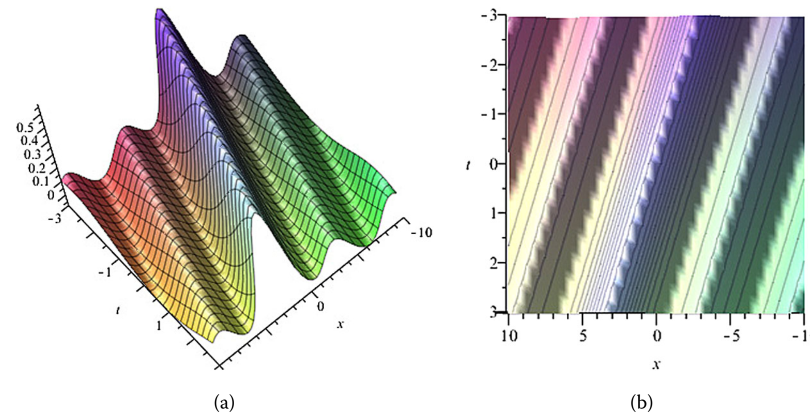


Figure 3 The bounded wave solution y_{36} of (1.1) with $a = 1$, $b = 1$, and $c = 1$. (a) Three-dimensional portrait; **(b)** overhead view with contour plot.

(5) For any constant c satisfying $a^2 + 5bc > 0$,

$$y_{14}(x,t) = \phi_1(\xi) + \phi_4(\xi) = -\frac{\operatorname{sgn}(b)a - \sqrt{5a^2 + 25bc}}{15|b|} - \frac{\sqrt{5a^2 + 25bc}}{10|b|} \operatorname{csch}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right]; \tag{3.23}$$

$$y_{24}(x,t) = \phi_1(\xi) + \phi_4(\xi) = -\frac{\operatorname{sgn}(b)a + \sqrt{5a^2 + 25bc}}{15|b|} - \frac{\sqrt{5a^2 + 25bc}}{10|b|} \operatorname{csch}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right]; \tag{3.24}$$

$$y_{15}(x,t) = \phi_1(\xi) + \phi_5(\xi) = -\frac{\operatorname{sgn}(b)a - \sqrt{5a^2 + 25bc}}{15|b|} - \frac{\sqrt{5a^2 + 25bc}}{10|b|} \sec^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right]; \tag{3.25}$$

$$y_{25}(x,t) = \phi_2(\xi) + \phi_5(\xi) = -\frac{\operatorname{sgn}(b)a}{15|b|} - \frac{\sqrt{5a^2 + 25bc}}{10|b|} \sec^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(x - ct - \xi_0)\right]; \tag{3.26}$$

$$y_{44}(x,t) = 2\phi_4(\xi) = -\frac{\operatorname{sgn}(b)a + \sqrt{5a^2 + 25bc}}{15|b|} - \frac{\sqrt{5a^2 + 25bc}}{5|b|} \operatorname{csch}^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(\xi - \xi_0)\right]; \tag{3.27}$$

$$y_{55}(x,t) = 2\phi_5(\xi) = -\frac{\operatorname{sgn}(b)a - \sqrt{5a^2 + 25bc}}{15|b|} - \frac{\sqrt{5a^2 + 25bc}}{5|b|} \sec^2\left[\frac{1}{2}\left(\frac{a^2 + 5bc}{5b^2}\right)^{\frac{1}{4}}(\xi - \xi_0)\right]; \tag{3.28}$$

$$y_{45}(x,t) = \phi_4(\xi) + \phi_5(\xi) = -\frac{\operatorname{sgn}(b)a}{15|b|} - \frac{\sqrt{5a^2+25bc}}{10|b|}\left(\operatorname{csch}^2 + \sec^2\right)\left[\frac{1}{2}\left(\frac{a^2+5bc}{5b^2}\right)^{\frac{1}{4}}(\xi - \xi_0)\right] \tag{3.29}$$

are unbounded solutions of the KdV-Sawada-Kotera-Ramani equation. Note that solutions (3.24) and (3.25) are the solutions (2.12) and (2.13), respectively.

4 Conclusion and discussion

In this paper, we studied the exact traveling wave solutions to the KdV-Sawada-Kotera-Ramani equation (1.1) via the sub-equation in the form $(dy/d\xi)^2 = a_3y^3 + a_2y^2 + a_1y + a_0$. The sub-equation of similar form, namely $(dy/d\xi)^2 = P_m(y)$, where $P_m(y)$ is a polynomial of y, has been applied to investigate some nonlinear wave equations [4–11]. In all these papers, the solutions to the original equations are usually the polynomial functions of the solutions to the sub-equations. However, by using the new method introduced in this paper (the sum of two solutions to sub-equations), we obtained many new exact traveling wave solutions to the KdV-Sawada-Kotera-Ramani equation (1.1). Especially, some quasi-periodic wave solutions were derived by using this new method. Furthermore, we obtained a very general class of exact solutions of the KdV-Sawada-Kotera-Ramani equation (1.1), which included the solitary wave solutions, periodic and quasi-periodic traveling wave solutions and some unbounded traveling solutions as well. Our results are more general than those obtained previously in the literature. For example, the solutions (12) and (14) in [16] actually can be rewritten as our solutions (3.16) and (3.27), respectively. Unfortunately, (18) and (20) in [16] do not satisfy (1.1) and hence are not the solutions of (1.1).

It is well known that not only the exact solutions but also the bifurcations of the dynamical systems can be investigated by using the dynamical system theorem [24, 25]. The planar dynamical system method has been well applied in studying the traveling wave solutions of various nonlinear wave solutions [1, 22, 23, 26–31]. However, it is usually very difficult to study the systems in a higher-dimensional space unless they can be reduced to a two-dimensional space. Normally, the higher-order differential equations can be reduced to a lower-dimensional space provided that their first integrals can be derived [1, 21, 28]. Unfortunately, it is usually intractable to derive their first integrals. In this paper, we reduced the higher-order ODE into planar dynamical system by finding its lower-order sub-equation. Whether there are any other kinds of sub-equations possessed by this class of equations is still an open problem.

The method proposed in this paper can be applied to other nonlinear wave equations, especially to higher-order nonlinear wave equations. This might pave the way to the study of the exact traveling wave solutions of higher-order nonlinear wave equations. However, whether and how this method can be used to investigate the multiple-wave solutions of higher-order nonlinear wave equations will be the topic of our future study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on deriving the results and approved the final manuscript.

Acknowledgements
L Zhang thanks the North-West University for the post-doctoral fellowship. This work is supported by the Nature Science Foundation of China (No. 11101371, No. 11422214).

Received: 30 December 2014 Accepted: 17 May 2015 Published online: 26 June 2015

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