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Asymptotic behavior of solutions to a class of fourth-order nonlinear evolution equations with dispersive and dissipative terms

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Abstract

We study the long time asymptotic behavior of solutions to a class of fourth-order nonlinear evolution equations with dispersive and dissipative terms. By using the integral estimation method combined with the Gronwall inequality, we point out that the global strong solutions of the problems decay to zero exponentially with the passage of time to infinity. The proof is rigorous and only based on some relatively weak assumptions on the nonlinear term.

MSC: 35L35; 35L75; 35B40

Keywords: nonlinear evolution equation; dispersive; dissipative; asymptotic behavior

1 Introduction

Nonlinear pseudo-hyperbolic equations were proposed from biological and mechanical problems in recent years, such as nerve conduction and the longitudinal vibration of rods with viscous effects, which have important practical and theoretical backgrounds.

More and more attention is paid to nonlinear evolution equations with dispersive and dissipative terms because these problems arise widely in practical applications. In 1980, Zhu [1] investigated the communication of longitudinal deformation wave for a flexible rod, he considered the influence of fourth-order nonlinear evolution equations with nonlinear, dispersive and dissipative terms,

$$u_{tt} - C_0^2\left(1 + na_n u_x^{n-1}\right)u_{xx} - \beta u_{xxtt} = \gamma u_{xxt},$$

where $C_0, \gamma > 0, \beta > 0$, and $a_n \neq 0$ are constants. The C_0 represents the phase velocity of zero frequency linear wave. In [1–3] Zhu, Zhang, and Yang discussed the solitary wave solutions of the equations. In [4, 5] Saxton and Hrusa discussed the well-posedness of local solution and the existence of a global solution. In [6] Liu and Zhao considered the global existence of $W^{k,p}$ solutions to wave equations with a dispersive term. The main part of the equation for their research is $u_{tt} - \alpha u_{xxt} - u_{xxtt} = \sigma(u_x)_x$, the initial conditions were mentioned in [6].

In the research of elasto-plastic-microstructure models we mention the equation with a nonlinear term $\sigma(u_x^2)_x + f(u)$ [7]. In [8] the authors discussed the initial boundary value problem with nonlinear term $\sigma(u_x)_x + f(u)$. They obtained the global existence and blow-up of solutions. Therefore, these nonlinear evolution equations with different nonlinear terms such as $\sigma(u_x)_x + f(u)$ have a strong physical background. The existence of a global solution has many classical results to the problem [9–13].

It is well known that the stability of differential equation is associated with problems that are described by the development of long time behavior. When the global properties of the solutions have been well studied, researchers in various fields naturally place stability as the focus. The asymptotic behavior of solutions is an important part of the stability content, there is some work on these equations. In [14], Shang investigated the initial and boundary value problem; by using the Galerkin method and the energy estimate method, the author not only obtained the whole existence and uniqueness of strong solution but also considered the asymptotic behavior and blow-up of solutions. In [15], the authors considered the asymptotic behavior and blow-up of solutions to a nonlinear evolution equation of fourth order. For other relevant work, we refer to [16–19].

This study, based on the previous research, analyzes the global asymptotic behavior of strong solutions to problem (1)-(3):

$$u_{tt} - u_{xx} - \alpha u_{xxt} - u_{xxtt} = \sigma(u_x)_x + f(u), \quad 0 < x < 1, t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad 0 < x < 1, \tag{2}$$

$$u(0,t) = u(1,t) = 0, \quad t \geq 0, \tag{3}$$

by using the integral estimation method and the Gronwall inequality. The result is the further generalization of some existing results. Here $\alpha \geq 0$ is a constant, $\sigma(u)$ is a rational function of u, f is a given nonlinear smooth function.

Throughout this paper, let $\|\cdot\|_p$ stand for the norm $\|\cdot\|_{L^p(0,1)}$, $\|\cdot\|$ stand for $\|\cdot\|_2$, and let the inner product be $(u,v) = \int_0^1 uv\,\mathrm{d}x$.

2 Preliminaries

Lemma 2.1 (Gronwall inequality) [20] *Assume that $y(t) \in L^1(0,T)$ and there exist positive constants a and b, such that*

$$y(t) \leq a + b\int_0^T y(z)\,\mathrm{d}z, \quad 0 \leq t \leq T,$$

then we have

$$y(t) \leq ae^{bt}, \quad 0 \leq t \leq T.$$

In order to obtain the asymptotic behavior of solutions with dispersive and dissipative terms to problem (1)-(3), we first introduce the energy as follows:

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|u_x\|^2 + \|u_{xt}\|^2\right) + \int_0^1 \left(\Phi(u_x) - F(u)\right)\mathrm{d}x,$$

where

$$\Phi(s) = \int_0^s \sigma(\tau)\,\mathrm{d}\tau, \qquad F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

3 Main results

Theorem 3.1 *Assume that* $0 \le \Phi(s) \le k_1\sigma(s)s$, $0 \le -F(u) \le -k_2 f(u)u$, $\forall s \in \mathbb{R}$, *where* k_i, $i = 1,2$, *are two nonnegative constants. Let* $u(x,t)$ *be the global strong solution to problem* (1)-(3), *for positive constants* λ *and* C, *satisfying the inequality*

$$\|u_t\|^2 + \|u_x\|^2 + \|u_{xt}\|^2 + 2\int_0^1 \bigl(\Phi(u_x) - F(u)\bigr)\,\mathrm{d}x \le CE(0)e^{-\lambda t}, \quad 0 \le t < \infty. \tag{4}$$

Proof We multiply the PDE (1) by u_t and integrate over $(0,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\bigl(\|u_t\|^2 + \|u_x\|^2 + \|u_{xt}\|^2\bigr) + \int_0^1 \bigl(\Phi(u_x) - F(u)\bigr)\,\mathrm{d}x\right) + \alpha\|u_{xt}\|^2 = 0,$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \alpha\|u_{xt}\|^2 = 0. \tag{5}$$

Take $\delta > 0$, multiply equality (5) by $e^{\delta t}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(e^{\delta t}E(t)\bigr) + \alpha e^{\delta t}\|u_{xt}\|^2 = \delta e^{\delta t}E(t). \tag{6}$$

We integrate (6) on t from 0 to t, then we obtain the equality

$$\begin{aligned} &e^{\delta t}E(t) + \alpha\int_0^t e^{\delta\tau}\|u_{x\tau}\|^2\,\mathrm{d}\tau \\ &\quad= E(0) + \delta\int_0^t e^{\delta\tau}E(\tau)\,\mathrm{d}\tau \\ &\quad= E(0) + \frac{\delta}{2}\int_0^t e^{\delta\tau}\bigl(\|u_\tau\|^2 + \|u_{x\tau}\|^2\bigr)\,\mathrm{d}\tau \\ &\qquad+ \delta\int_0^t e^{\delta\tau}\left(\frac{1}{2}\|u_x\|^2 + \int_0^1 \bigl(\Phi(u_x) - F(u)\bigr)\,\mathrm{d}x\right)\mathrm{d}\tau. \end{aligned} \tag{7}$$

Since $-F(u) \ge 0$, we obtain $E(t) \ge 0$ as $0 \le t < \infty$. Because $0 \le \Phi(s) \le k_1\sigma(s)s$, $0 \le -F(u) \le -k_2 f(u)u$ combined with equality (1) we get

$$\begin{aligned} &\frac{1}{2}\|u_x\|^2 + \int_0^1 \bigl(\Phi(u_x) - F(u)\bigr)\,\mathrm{d}x \\ &\quad\le k\bigl(\|u_x\|^2 - (u_{tt} - u_{xx} - u_{xxtt} - \alpha u_{xxt}, u)\bigr) \\ &\quad= k\bigl(\|u_x\|^2 - (u_{tt}, u) + (u_{xx}, u) + (u_{xxtt}, u) + \alpha(u_{xxt}, u)\bigr) \\ &\quad= -k\left((u_{tt}, u) + (u_{xtt}, u_x) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|^2\right), \end{aligned} \tag{8}$$

where $k = \max\{1/2, k_1, k_2\}$. Hence we have

$$\begin{aligned} &\int_0^t e^{\delta\tau}\left(\frac{1}{2}\|u_x\|^2 + \int_0^1 \bigl(\Phi(u) - F(u)\bigr)\,\mathrm{d}x\right)\mathrm{d}\tau \\ &\quad\le -k\int_0^t e^{\delta\tau}\left((u_{\tau\tau}, u) + (u_{x\tau\tau}, u_x) + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|u_x\|^2\right)\mathrm{d}\tau. \end{aligned} \tag{9}$$

By integration by parts, we get

$$
\begin{aligned}
&-\int_0^t e^{\delta\tau}(u_{\tau\tau},u)\,\mathrm{d}\tau \\
&\quad= -e^{\delta t}(u_t,u) + (u_1,u_0) + \delta\int_0^t e^{\delta\tau}(u_\tau,u)\,\mathrm{d}\tau + \int_0^t e^{\delta\tau}\|u_\tau\|^2\,\mathrm{d}\tau \\
&\quad\le \frac{1}{2}e^{\delta t}\big(\|u_t\|^2 + \|u\|^2\big) + \frac{1}{2}\big(\|u_1\|^2 + \|u_0\|^2\big) \\
&\qquad + \frac{1}{2}\delta\int_0^t e^{\delta\tau}(\|u_\tau)\|^2 + \|u\|^2)\,\mathrm{d}\tau + \int_0^t e^{\delta\tau}\|u_\tau\|^2\,\mathrm{d}\tau,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&-\int_0^t e^{\delta\tau}(u_{x\tau\tau},u_x)\,\mathrm{d}\tau \\
&\quad= -e^{\delta t}(u_{xt},u_x) + (u_{x1},u_{x0}) + \delta\int_0^t e^{\delta\tau}(u_{x\tau},u_x)\,\mathrm{d}\tau + \int_0^t e^{\delta\tau}\|u_{x\tau}\|^2\,\mathrm{d}\tau \\
&\quad\le \frac{1}{2}e^{\delta t}\big(\|u_{xt}\|^2 + \|u_x\|^2\big) + \frac{1}{2}\big(\|u_{x1}\|^2 + \|u_{x0}\|^2\big) \\
&\qquad + \frac{\delta}{2}\int_0^t e^{\delta\tau}\big(\|u_{x\tau}\|^2 + \|u_x\|^2\big)\,\mathrm{d}\tau + \int_0^t e^{\delta\tau}\|u_{x\tau}\|^2\,\mathrm{d}\tau,
\end{aligned}
\tag{11}
$$

where $u_{x1} = u_x(x,1), u_{x0} = u_x(x,0)$, and

$$
\begin{aligned}
-\frac{1}{2}\int_0^t e^{\delta\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\|u_x\|^2\,\mathrm{d}\tau &= -\frac{1}{2}e^{\delta t}\|u_x\|^2 + \frac{1}{2}\|u_{x0}\|^2 + \frac{\delta}{2}\int_0^t e^{\delta\tau}\|u_x\|^2\,\mathrm{d}\tau \\
&\le \frac{1}{2}\|u_{x0}\|^2 + \frac{\delta}{2}\int_0^t e^{\delta\tau}\|u_x\|^2\,\mathrm{d}\tau.
\end{aligned}
\tag{12}
$$

Inserting (10)-(12) into (9), we have

$$
\begin{aligned}
&-k\int_0^t e^{\delta\tau}\left((u_{\tau\tau},u) + (u_{x\tau\tau},u_x) + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|u_x\|^2\right)\mathrm{d}\tau \\
&\quad= -ke^{\delta t}\left((u_t,u) + (u_{xt},u_x) + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|^2\right) \\
&\qquad - k\left((u_1,u_0) + (u_{x1},u_{x0}) + \frac{\alpha}{2}\|u_{x0}\|^2\right) - k\int_0^t e^{\delta\tau}\big(\|u_\tau\|^2 + \|u_{x\tau}\|^2\big)\,\mathrm{d}\tau \\
&\qquad - \delta k\int_0^t e^{\delta\tau}\left((u_\tau,u) + (u_{x\tau},u_x) + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|u_x\|^2\right)\mathrm{d}\tau \\
&\quad\le M,
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
M = {}& k\int_0^t e^{\delta\tau}\big(\|u_\tau\|^2 + \|u_{x\tau}\|^2\big)\,\mathrm{d}\tau \\
&+ \frac{k}{2}e^{\delta t}\big(\|u_t\|^2 + \|u\|^2 + (1+\alpha)\|u_x\|^2 + \|u_{xt}\|^2\big) \\
&+ \frac{k}{2}e^{\delta t}\big(\|u_1\|^2 + \|u_0\|^2 + (1+\alpha)\|u_{x0}\|^2 + \|u_{x1}\|^2\big) \\
&+ \frac{k}{2}\delta\int_0^t e^{\delta\tau}\big(\|u_\tau\|^2 + \|u\|^2 + (1+\alpha)\|u_x\|^2 + \|u_{x\tau}\|^2\big)\,\mathrm{d}\tau.
\end{aligned}
$$

By substituting of the inequality (13) into (7) and using the Poincaré inequality, there exist positive constants C_0 and C_1, and we find

$$\begin{aligned} &\mathrm{e}^{\delta t}E(t) + \alpha \int_0^t \mathrm{e}^{\delta\tau} \|u_{x\tau}\|^2 \,\mathrm{d}\tau \\ &\quad\le C_0E(0) + \frac{\delta}{2}\int_0^t \mathrm{e}^{\delta\tau}\bigl(\|u_\tau\|^2 + \|u_{x\tau}\|^2\bigr)\,\mathrm{d}\tau + C_1\delta\mathrm{e}^{\delta t}E(t) + C_1\delta^2\int_0^t \mathrm{e}^{\delta\tau}E(\tau)\,\mathrm{d}\tau \\ &\quad\le C_0E(0) + \frac{\delta}{2}(1+\lambda_1)\int_0^t \mathrm{e}^{\delta\tau}\|u_{x\tau}\|^2\,\mathrm{d}\tau + C_1\delta\mathrm{e}^{\delta t}E(t) + C_1\delta^2\int_0^t \mathrm{e}^{\delta\tau}E(t)\,\mathrm{d}\tau, \end{aligned} \tag{14}$$

where

$$\lambda_1 = \sup_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\|u\|^2}{\|u_x\|^2}.$$

Take δ, such that

$$0 < \delta < \min\left\{\frac{2\alpha}{1+\lambda_1}, \frac{1}{2C_1}\right\}.$$

Then combining with (14) we can obtain

$$\mathrm{e}^{\delta T}E(t) \le 2C_0E(0) + 2C_1\delta^2\int_0^t \mathrm{e}^{\delta\tau}E(\tau)\,\mathrm{d}\tau, \tag{15}$$

which according to the Gronwall inequality represented by Lemma 2.1 leads to

$$\mathrm{e}^{\delta t}E(t) \le 2C_0E(0)\mathrm{e}^{2C_1\delta^2 t}, \quad 0 \le t < \infty, \tag{16}$$

and

$$E(t) \le 2C_0E(0)\mathrm{e}^{-\lambda t}, \quad 0 \le t < \infty, \tag{17}$$

that is,

$$\|u_t\|^2 + \|u_x\|^2 + \|u_{xt}\|^2 + 2\int_0^1 \bigl(\Phi(u_x) - F(u)\bigr)\,\mathrm{d}x \le CE(0)e^{-\lambda t}, \quad 0 \le t < \infty,$$

where $\lambda = \delta(1 - 2C_1\delta) > 0, C = 2C_0$.

The proof is complete. □

Corollary 3.1 *If the conditions of Theorem* 3.1 *are satisfied, the following result for the global strong solution u of problem* (1)-(3):

$$\|u_t\|^2 + \|u_x\|^2 + \|u_{xt}\|^2 + 2\int_0^1 \Phi(u_x)\,\mathrm{d}x \le 2CE(0)e^{-\lambda t}, \quad 0 \le t < \infty,$$

still holds.

Corollary 3.2 *If* (1) *is replaced by the following*:

$$u_{tt} - u_{xx} - \alpha u_{xxt} - \beta u_{xxtt} = \sigma(u_x)_x + f(u),$$

then the conclusion of Theorem 3.1 *still holds*. *Here* α *and* β *are positive constants*.

This paper has investigated the asymptotic behavior of the global strong solution to a class of fourth-order nonlinear evolution equations with both dispersive and dissipative terms. By using the multiplier method and the integral estimate methods, we prove that the global strong solutions of the problem decay to zero exponentially as the time tends to infinity, under weaker conditions regarding the nonlinear term. It should be pointed out that the method in the present paper can also be extended to the case when the initial-boundary value problem of nonlinear evolution equations are multidimensional. For this case, a different expression for the asymptotic behavior will be employed. This work will be left for our future research.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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