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Parabolic problems with data measure

Mohammmed Kbiri Alaoui

Correspondence: mka_la@yahoo.fr
Department of Mathematics, King Khalid University, P.O. Box 9004, Abha, Kingdom of Saudi Arabia

Abstract

The article deals with the existence of solutions of some unilateral problems in the Orlicz-Sobolev spaces framework when the right-hand side is a Radon measure.

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Keywords: unilateral problem, radon measure, Orlicz-Sobolev spaces

1 Introduction

We deal with boundary value problems

$$\begin{cases} u \geq \psi & \text{a.e. in } Q = \Omega \times [0,T], \\ \dfrac{\partial u}{\partial t} + \mathcal{A}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial Q = \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \qquad (\mathcal{P})$$

where

$$\mathcal{A}(u) = -\operatorname{div}(a(.,t,u,\nabla u)),$$

$T > 0$ and Ω is a bounded domain of $\mathbf{R}^N$, with the segment property. $a : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (t, s, ξ) in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}^N$, and continuous with respect to (s, ξ) in $\mathbf{R} \times \mathbf{R}^N$ for almost every x in Ω) such that for all $\xi, \xi^* \in \mathbf{R}^N$, $\xi \neq \xi^*$,

$$a(x,t,s,\xi)\xi \geq \alpha M(|\xi|) \tag{1.1}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{1.2}$$

$$|a(x,t,s,\xi)| \leq c(x,t) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\xi|), \tag{1.3}$$

where $c\,(x,t)$ belongs to $E_{\overline{M}}(Q), c \geq 0$, P is an N-function such that $P \ll M$ and k_i ($i = 1,2,3,4$) belongs to $\mathbf{R}^+$ and α to $\mathbf{R}^+_*$.

$$\mu \in M_b^+(Q), \quad u_0 \in M_b^+(\Omega), \tag{1.4}$$

$$\psi \in L^\infty(\Omega) \cap W_0^1 E_M(\Omega). \tag{1.5}$$

There have obviously been many previous studies on nonlinear differential equations with nonsmooth coefficients and measures as data. The special case was cited in the references (see [1,2]).

It is noteworthy that the articles mentioned above differ in significant way, in the terms of the structure of the equations and data. In [1], when $f \in L^1(0,T;L^1(\Omega))$ and $u_0 \in L^1(\Omega)$. The authors have shown the existence of solutions u of the corresponding equation of the problem $(\mathcal{P}), u \in L^q(0,T;W_0^{1,q}(\Omega))$ for every q such that $q < p - \frac{N}{N+1}$ which is more restrictive than the one given in the elliptic case $\left(q < \frac{N_{(p-1)}}{N-1}\right)$.

In this article, we are interested with an obstacle parabolic problem with measure as data. We give an improved regularity result of the study of Boccardo et al. [1].

In [1], the authors have shown the existence of a weak solutions for the corresponding equation, the function $a(x, t, s, \xi)$ was assumed to satisfy a polynomial growth condition with respect to u and ∇u. When trying to relax this restriction on the function $a(., s, \xi)$, we are led to replace the space $L^p(0, T; W^{1,p}(\Omega))$ by an inhomogeneous Sobolev space $W^{1,x}L_M$ built from an Orlicz space L_M instead of L^p, where the N-function M which defines L_M is related to the actual growth of the Carathéodory's function.

For simplicity, one can suppose that there exist $\alpha > 0$, $\beta > 0$ such that

$$a(x,t,u,\nabla u) = a(x,t,u)\frac{M(|\nabla u|)}{|\nabla u|^2}\nabla u \text{ and } \alpha \le —a(x,t,s)| \le \beta.$$

2 Preliminaries

Let $M : \mathbf{R}^+ \to \mathbf{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation: $M(t) = \int_0^t a(\tau)d\tau$ where $a : \mathbf{R}^+ \to \mathbf{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \bar{a}(\tau)d\tau$, where $\bar{a} : \mathbf{R}^+ \to \mathbf{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \le t\}$ (see [3,4]).

The N-function M is said to satisfy the Δ_2 condition if, for some $k > 0$:

$$M(2t) \le kM(T) \quad \text{for all } t \ge 0, \tag{2.1}$$

when this inequality holds only for $t \ge t_0 > 0$, M is said to satisfy the Δ_2 condition near infinity.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q; i.e., for each $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon t)} \to 0 \quad \text{as } t \to \infty.$$

Let Ω be an open subset of $\mathbf{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_m(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))dx < +\infty \quad \left(\text{resp.} \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right).$$

Note that $L_M(\Omega)$ is a Banach space under the norm $||u||_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}$ and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $||\cdot||_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm $||u||_{1,M,\Omega} = \sum_{|\alpha|\leq 1} ||D^\alpha u||_{M,\Omega}$. Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of N + 1 copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \to 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$. If M satisfies the Δ_2 condition on $\mathbf{R}^+$(near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (cf. [5,6]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

For $k > 0$, $s \in \mathbf{R}$, we define the truncation at height $k, T_k(s) = [k - (k - |s|)_+]sign(s)$.

The following abstract lemmas will be applied to the truncation operators.

Lemma 2.1 [7]*Let* $F : \mathbf{R} \to \mathbf{R}$ *be uniformly lipschitzian, with* $F(0) = 0$. *Let* M *be an N-function and let* $u \in W_0^1L_M(\Omega)$*(resp.*$W_0^1E_M(\Omega)$*).*

Then $F(u) \in W_0^1L_M(\Omega)$*(resp.*$W_0^1E_M(\Omega)$ *). Moreover, if the set of discontinuity points of* F' *is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e.\ in\ \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e.\ in\ \{x \in \Omega : u(x) \in D\} \end{cases}$$

Let Ω be a bounded open subset of $\mathbf{R}^N$, $T > 0$ and set $Q = \Omega \times]0, T[$. Let $m \geq 1$ be an integer and let M be an N-function. For each $\alpha \in \mathbf{IN}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbf{R}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows $W^{m,x}L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q) \forall |\alpha| \le m\}$ $W^{m,x}E_M(Q) = \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q) \forall |\alpha| \le m\}$.

The last space is a subspace of the first one, and both are Banach spaces under the norm $||u|| = \sum_{|\alpha|\le m} ||D_x^\alpha u||_{M,Q}$. We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_m(Q)$ which have as many copies as there are α-order derivatives, $|\alpha| \le m$. We shall also consider the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$. If $u \in W^{m,x}L_M(Q)$, then the function : $t \mapsto u(t) = u(t,.)$ is defined on $[0, T]$ with values in $W^m L_M(\Omega)$. If, further, $u \in W^{m,x}E_M(Q)$ then the concerned function is a $W^m E_M(\Omega)$-valued and is strongly measurable. Furthermore, the following imbedding holds: $W^{m,x}E_M(Q) \subset L^1(0,T; W^m E_M(\Omega))$. The space $W^{m,x}L_M(Q)$ is not in general separable, if $u \in W^{m,x}L_M(Q)$, we cannot conclude that the function $u(t)$ is measurable on $[0,T]$. However, the scalar function $t \mapsto ||u(t)||_{M,\Omega}$, is in $L^1(0,T)$. The space $W_0^{m,x}E_M(Q)$ is defined as the (norm) closure in $W^{m,x}E_M(Q)$ of $\mathcal{D}(\Omega)$. We can easily show as in [6] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(\Omega)$ with respect of the weak * topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ is limit, in $W^{m,x}L_M(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\lambda > 0$ such that for all $|\alpha| \le m$,

$$\int_Q M\left(\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}\right) dx\, dt \to 0 \text{ as } i \to \infty,$$

this implies that (u_i) converges to u in $W^{m,x}L_M(Q)$ for the weak topology $\sigma(\prod L_M, \prod L_{\overline{M}})$. Consequently, $\overline{\mathcal{D}(Q)}^{\sigma(\prod L_M, \prod E_{\overline{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\prod L_M, \prod L_{\overline{M}})}$, and this space will be denoted by $W_0^{m,x}L_M(Q)$.

Furthermore, $W_0^{m,x}E_M(Q) = W_0^{m,x}L_M(Q) \cap \prod E_M$. Poincaré's inequality also holds in $W_0^{m,x}L_M(Q)$, i.e., there is a constant $C > 0$ such that for all $u \in W_0^{m,x}L_M(Q)$ one has $\sum_{|\alpha|\le m} ||D_x^\alpha u||_{M,Q} \le C \sum_{|\alpha|=m} ||D_x^\alpha u||_{M,Q}$. Thus both sides of the last inequality are equivalent norms on $W_0^{m,x}L_M(Q)$. We have then the following complementary system:

$$\begin{pmatrix} W_0^{m,x}L_M(Q) & F \\ W_0^{m,x}E_M(Q) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{m,x}E_M(Q)$. It is also, except for an isomorphism, the quotient of $\prod L_{\overline{M}}$ by the polar set $W_0^{m,x}E_M(Q)\perp$, and will be denoted by $F = W^{-m,x}L_{\overline{M}}(Q)$, and it is shown that $W^{-m,x}L_{\overline{M}}(Q) = \left\{f = \sum_{|\alpha|\le m} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q)\right\}$. This space will be equipped with the usual quotient norm $||f|| = \inf \sum_{|\alpha|\le m} ||f_\alpha||_{\overline{M},Q}$ where the infimum is taken on all possible decompositions $f = \sum_{|\alpha|\le m} D_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q)$. The space F_0 is then given by $F_0 = \left\{f = \sum_{|\alpha|\le m} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q)\right\}$ and is denoted by $F_0 = W^{-m,x}E_{\overline{M}}(Q)$.

We can easily check, using Lemma 4.4 of [6], that each uniformly lipschitzian mapping F, with $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1: $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.

3 Main results

First, we give the following results which will be used in our main result.

3.1 Useful results

Hereafter, we denote by $\mathcal{X}_N$ the real number defined by $\mathcal{X}_N = NC_N^{1/N}$, C_N is the measure of the unit ball of $\mathbf{R}^N$, and for a fixed $t \in [0, T]$, we denote $\mu(\theta) = meas\{(x,t) : |u(x, t)| > \theta\}$.

Lemma 3.1 [8]*Let* $u \in W_0^{1,x}L_M(Q)$, *and let fixed* $t \in [0, T]$, *then we have*

$$-\mu'(\theta) \geq -\frac{1}{\mathcal{X}_N \mu(\theta)^{1-\frac{1}{N}}} \mathcal{S}\left(-\frac{1}{\mathcal{X}_N \mu(\theta)^{1-\frac{1}{N}}} \frac{d}{d\theta} \int_{\{|u|>\theta\}} M(|\nabla u|)dx \right), \forall \theta > 0$$

and where $\mathcal{S}$*is defined by*

$$\frac{1}{\mathcal{S}(s)} = \sup\{t : B(t) \leq s\}, \quad B(s) = \frac{M(s)}{s}.$$

Lemma 3.2 *Under the hypotheses (1.1)-(1.3), if f,* u_0 *are regular functions and f,* $u_0 \geq 0$, *then there exists at least one positive weak solution of the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(u) = f \ \ in \quad Q, \\ u = 0 \qquad\qquad on \ \ \partial Q, \\ u(x, 0) = u_0(x) \ in \quad \Omega. \end{cases} \qquad (E)$$

such that

$$\frac{\partial u}{\partial t} \geq 0, \quad a.e. \ \ t \in (0, T).$$

Proof

Let u be a continuous function, we say that u satisfies (*) if: there exists a continuous and increasing function β such that $||u(t) - u(s)||_2 \leq \beta(||u_0||_2)|t - s|$, where $u_0(x) = u(x, 0)$.

Let $X := \left\{ u \in W_0^{1,x}L_M(Q) \cap L^2(Q) \text{ s.t. } u \text{ satisfies } (*) \text{ and} \dfrac{du}{dt} \in L^\infty(0, T, L^2(\Omega)) \right\}$.

Let us consider the set $\mathcal{C} = \{v \in X : v(t) \in C, \quad \dfrac{\partial v}{\partial t} \geq 0 \ \text{ a.e. } t \in (0, T)\}$, where C is a closed convex of $W_0^1 L_M(\Omega)$. It is easy to see that $\mathcal{C}$ is a closed convex (since all its elements satisfy (*)).

We claim that the problem

$$\begin{cases} u \in \mathcal{C} \cap L^2(Q) \\ \dfrac{\partial u}{\partial t} + \mathcal{A}(u) = f \text{ in } \ Q, \\ u = 0 \qquad\qquad \text{on } \ \partial Q, \\ u(x, 0) = u_0 \qquad \text{in } \ \Omega. \end{cases} \qquad (E')$$

has a weak solution which is unique in the sense defined in [9].

Indeed, let us consider the approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} + \mathcal{A}(u_n) + nT_n(\Phi(u_n)) = f \text{ in } \Omega, \\ u_n(.,0) = u_0 \qquad\qquad\qquad\qquad \text{in } \Omega. \end{cases} \qquad (E'')$$

where the functional Φ is defined by $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ such that

$$\Phi(v) := \begin{cases} 0 & \text{if } v \in \mathcal{C}, \\ +\infty & \text{otherwise.} \end{cases}$$

The existence of such $u_n \in X$ was ensured by Kacur et al. [10].

Following the same proof as in [9], we can prove the existence of a solution u of the problem (E') as limit of u_n (for more details see [9]).

Lemma 3.3 *Let* $v \in W_0^{1,x}L_M(Q)$ *such that* $\dfrac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ *and* $v \geq \psi$, $\psi \in L^\infty(\Omega) \cap W_0^1E_M(\Omega)$.

Then, there exists a smooth function (v_j) *such that*

$$v_j \geq \psi,$$

$v_j \to v$ *for the modular convergence in* $W_0^{1,x}L_M(Q)$,

$\dfrac{\partial v_i}{\partial t} \to \dfrac{\partial v}{\partial t}$ *for the modular convergence in* $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$.

For the proof, we use the same technique as in [11] in the parabolic case.

3.2 Existence result

Let M be a fixed N-function, we define K as the set of N-function D satisfying the following conditions:

i) $M(D^{-1}(s))$ is a convex function,

ii) $\int_0^{\cdot} DoB^{-1}\left(\dfrac{1}{r^{1-1/N}}\right) dr < +\infty, B(t) = \dfrac{M(t)}{t}$,

iii) There exists an N-function H such that $Ho\overline{M}^{-1}oM \leq D$ and $\overline{H} \leq D$ near infinity.

Theorem 3.1 *Under the hypotheses (1.1)-(1.5), The problem* (*P*) *has at least one solution u in the following sense:*

$$\begin{cases} u \geq \psi \, a.e. \text{ in } Q \\ T_k(u) \in W_0^{1,x}L_M(Q), u \in W_0^{1,x}L_D(Q) \quad \forall D \in K \\ -\int_Q u\dfrac{\partial \varphi}{\partial t} + \int_Q a(.,u,\nabla u)\nabla\varphi dxdt - \int_\Omega \varphi du_0 = \int_Q \varphi d\mu, \end{cases}$$

for all $\phi \in D(\mathbf{R}^{N+1})$ *which are zero in a neighborhood of* $(0, T) \times \partial\Omega$ *and* $\{T\} \times \Omega$.

Remark 3.1 *(1) If* $\psi = -\infty$ *in the problem (P), then the above theorem gives the same regularity as in the elliptic case.*

(2) An improved regularity is reached for all N*-function satisfying the conditions (i)-(ii)-(iii).*

For example, $u \in W_0^{1,x}L_D(Q), D(t) = \dfrac{t^q}{Log^\sigma(e+t)}$, *for all* $q < \dfrac{N(p-1)}{N-1}, \sigma > 1$.

In the sequel and throughout the article, we will omit for simplicity the dependence on x and t in the function $a(x, t, s, \xi)$ and denote $\epsilon(n, j, \mu, s, m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{s\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon(n,j,\mu,s,m)=0,$$

and this will be in the order in which the parameters we use will tend to infinity, that is, first n, then j, μ, s, and finally m. Similarly, we will write only $\epsilon(n)$, or $\epsilon(n, j)$,... to mean that the limits are made only on the specified parameters.

3.2.1 A sequence of approximating problems

Let (f_n) be a sequence in $D(Q)$ which is bounded in $L^1(Q)$ and converge to μ in $M_b(Q)$.

Let (u_0^n) be a sequence in $D(\Omega)$ which is bounded in $L^1(\Omega)$ and converge to u_0 in $M_b(\Omega)$.

We define the following problems approximating the original (P):

$$\begin{cases}\dfrac{\partial u_n}{\partial t}+\mathcal{A}(u_n)-nT_n((u_n-\psi)^-)=f_n & \text{in } Q,\\ u_n=0 & \text{on } \partial Q\\ u_n(.,0)=u_0^n & \text{in } \Omega.\end{cases}\qquad (P_n)$$

Lemma 3.4 *Under the hypotheses (1.1)-(1.3), there exists at least one solution u_n of the problem (P_n) such that $\dfrac{\partial u_n}{\partial t}\geq 0$a.e. in Q.*

For the proof see Lemma 3.2.

3.2.2 A priori estimates

Lemma 3.5 *There exists a subsequence of (u_n) (also denoted (u_n)), there exists a measurable function u such that:*

$$u\geq\psi,\, T_k(u)\in W_0^{1,x}L_M(Q)\,\textit{for all } k>0$$
$$u_n\rightharpoonup u \textit{ weakly in } W_0^{1,x}L_D(Q)\,\textit{for all } D\in K.$$

Proof:

Recall that $u_n \geq 0$ since $f_n \geq 0$.

Let $h > 0$ and consider the following test function $v = T_h(u_n - T_k(u_n))$ in (P_n), we obtain

$$\ll\frac{\partial u_n}{\partial t},v\gg+\alpha\int_{\{k<|u_n|\leq k+h\}}M(|\nabla u_n|)dxdt-n\int_Q T_n((u_n-\psi)^-)vdxdt\leq\int_Q f_n v dxdt$$

We have

$$\ll\frac{\partial u_n}{\partial t},T_h(u_n-T_k(u_n))\gg=\int_\Omega\int_0^{u_n(x,T)}T_h(s-T_k(s))-\int_\Omega\int_0^{u_0^n}T_h(s-T_k(s)).$$

So,

$$-\int_Q nT_n((u_n-\psi)^-)\frac{T_h(u_n-T_k(u_n))}{h}dx\,dt\leq C.$$

Now, let us fix $k > ||\psi||_\infty$, we deduce the fact that: $nT_n(u_n-\psi)(u_n-k)\mathcal{X}_{\{u_n\leq\psi\}}\mathcal{X}_{\{k<u_n\leq k+h\}}\geq 0$.

Let h to tend to zero, one has

$$n\int_Q T_n((u_n-\psi)^-)dxdt\leq C. \tag{3.1}$$

Let us use as test function in (P_n),$v = T_k(u_n)$, then as above, we obtain

$$\int_Q M(|\nabla T_k(u_n)|) \leq C_1 k. \tag{3.2}$$

Then $(T_k(u_n)_n)$ is bounded in $W_0^{1,x}L_M(Q)$, and then there exist some $\omega_k \in W_0^{1,x}L_M(Q)$ such that

$T_k(u_n) \rightharpoonup \omega_k$, weakly in $W_0^{1,x}L_M(Q)$ for $\sigma(\prod L_M, \prod E_{\overline{M}})$, strongly in $E_M(Q)$ and a.e in Q.

Let consider the C^2 function defined by

$$\mu_k(s) \begin{cases} s & |s| \leq k/2 \\ ksign(s) & |s| \geq k \end{cases}$$

Multiplying the approximating equation by $\eta_k'(u_n)$, we get $\dfrac{\partial \eta_k(u_n)}{\partial t} - div(a(.,u_n,\nabla u_n)\eta_k'(u_n)) + a(.,u_n \nabla u_n)\eta_k''(u_n) = f_n \eta_k'(u_n) + n(T_n((u_n - \psi)^-))\eta_k'(u_n)$ in the distributions sense. We deduce then that $\eta_k(u_n)$ being bounded in $W_0^{1,x}L_M(Q)$ and $\dfrac{\partial \eta_k(u_n)}{\partial t}$ in $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$. By Corollary 1 of [12], $\eta_k(u_n)$ is compact in L^1 (Q).

Following the same way as in [2], we obtain for every $k > 0$,

$$T_k(u_n) \rightharpoonup T_k(u), \text{ weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}}), \text{ strongly in } L^1(Q) \text{ and a.e in } Q. \tag{3.3}$$

Using now the estimation (3.1) and Fatou's lemma to obtain

$(u - \psi)^- = 0$ and so , $u \geq \psi$.

Let fixed a $t \in [0, T]$. We argue now as for the elliptic case, the problem becomes:

$$\frac{\partial u_n}{\partial t} - \text{div}(a(.,u_n,\nabla u_n)) = f_n + nT_n((u_n - \psi)^-) \quad \text{in } \Omega. \qquad (P_n')$$

We denote $g_n := nT_n((u_n - \psi)^-)$.

Let ϕ be a truncation defined by

$$\varphi(\xi) = \begin{cases} 0 & 0 \leq \xi \leq \theta \\ \dfrac{1}{h}(\xi - t) & \theta < \xi < \theta + h \\ 1 & \xi \geq \theta + h \\ -\varphi(-\xi) & \xi < 0 \end{cases} \tag{3.4}$$

for all θ, $h > 0$.

Using $v = \phi\ (u_n)$ as a test function in the approximate elliptic problem (P_n'), we obtain by using the same technique as in [8]

$$-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} M(|\nabla u_n|) dx \leq C \int_{\{|u_n| \geq \theta\}} (f_n + g_n - \frac{\partial u_n}{\partial t})\, dx. \tag{3.5}$$

here and below C denote positive constants not depending on n.

By using Lemma 3.1, we obtain (supposing $-\mu'(\theta) > 0$ which does not affect the proof) and following the same way as in [8], we have for $D \in K$,

$$-\frac{d}{d\theta}\int_{\{|u_n|>\theta\}} D(|\nabla u_n|)dx \leq (-\mu'(\theta))DoB^{-1}\left(\left(-\frac{1}{\mathcal{X}_{N\mu}(\theta)^{1-\frac{1}{N}}}\frac{d}{d\theta}\int_{\{|u_n|>\theta\}} M(|\nabla u_n|dx)\right)\right). \tag{3.6}$$

Let denote $k(t,s) := \int_0^s u_{n*}(t,\rho)d\rho$, then

$$\frac{\partial k}{\partial t}(t,s) = \int_0^s \frac{\partial u_{n*}(t,\rho)}{\partial t}d\rho, \quad \int_{u_n>\theta}\frac{\partial u_n}{\partial t}dx = \frac{\partial k}{\partial t}(t,\mu(\theta)).$$

Using Lemma 3.1, denoting $F(t,\mu(\theta)) := \int_0^{\mu(\theta)} (f_{n*}+g_{n*})(\rho)\, d\rho$ one has

$$1 \leq \frac{-\mu'(\theta)}{\mathcal{X}_N\mu(\theta)^{1-\frac{1}{N}}}B^{-1}\left(\frac{1}{\mathcal{X}_N\mu(\theta)^{1-\frac{1}{N}}}\left[F(t,\mu(\theta)) - \frac{\partial k}{\partial t}(t,\mu(\theta))\right]\right).$$

Remark also that $F(t,s) \geq \frac{\partial k}{\partial t}(t,s)$ and using Lemma 3.2, we have $\left|\frac{\partial k}{\partial t}(t,s)\right| \leq F(t,s)$.

Combining the inequalities (3.5) and (3.6) we obtain,

$$-\frac{d}{d\theta}\int_{\{|u_n|>\theta\}} D(|\nabla u_n|)dx \leq (-\mu'(\theta))DoB^{-1}\left(-\frac{1}{\mathcal{X}_N\mu(\theta)^{1-\frac{1}{N}}}\left[F(t,\mu(\theta)) - \frac{\partial k}{\partial t}(t,\mu(\theta))\right]\right). \tag{3.7}$$

and since the function $\theta \to \int_{\{|u_n|>\theta\}} D(|\nabla u_n|)\, dx$ is absolutely continuous, we get

$$\begin{aligned}\int_\Omega D(|\nabla u_n|)dx &= \int_0^{+\infty}\left(-\frac{d}{d\theta}\int_{\{|u_n|>\theta\}} D(|\nabla u_n|)dx\right)dt\\ &\leq \frac{1}{C'}\int_0^{C'|\Omega|} DoB^{-1}\left(\left(\frac{C}{s^{1-1/N}}\right)\right)ds(\text{using 3.1 and 3.7}).\end{aligned}$$

Then, the sequence (u_n) is bounded in $W_0^{1,x}L_D(Q)$ and we deduce that $u \in W_0^{1,x}L_D(Q)$ for all N-function $D \in K$.

3.3 Almost everywhere convergence of the gradients

Lemma 3.6 *The subsequence* (u_n) *obtained in Lemma 3.5 satisfies:*

$$\nabla u_n \to \nabla u \ \ a.e.\ \ in\ Q.$$

Proof:

Let $m > 0$, $k > 0$ such that $m > k$. Let ρ_m be a truncation defined by

$$\rho m(s) = \begin{cases} 1 & |s| \leq m,\\ m+1-|s| & m < |s| < m+1,\\ 0 & |s| \geq m+1.\end{cases}$$

$$R_m(s) = \int_0^s \rho_m(t)dt \text{ and } \omega_{\mu,j} = T_k(v_j)_{\mu.}$$

where $v_j \in D(Q)$ such that $v_j \geq \psi$ and $v_j \to T_k(u)$ with the modular convergence in $W_0^{1,x}L_M(Q)$ (for the existence of such function see [11] since $\psi \in L^\infty(\Omega) \cap W_0^1E_M(\Omega)$).

ω_μ is the mollifier function defined in Landes [13], the function $\omega_{\mu,j}$ have the following properties:

$$\begin{cases} \dfrac{\partial \omega_{\mu,j}}{\partial t} = \mu(T_k(v_j) - \omega_{\mu,j}), \omega_{\mu,j}(0) = 0, |\omega_{\mu,j}| \le k, \\ \omega_{\mu,j} \to T_k(u)_\mu \text{in } W_0^{1,x}L_M(Q) \quad \text{for the modular convergence with respect to } j, \\ T_k(u)_\mu \to T_k(u) \quad \text{in} \quad W_0^{1,x}L_M(Q) \text{ for the modular convergence with respect to } \mu. \end{cases}$$

Set $v = (T_k(u_n) - \omega_{\mu,j})\, \rho_m(u_n)$ as test function, we have

$$\begin{aligned} &\ll \frac{\partial u_n}{\partial t}, v \gg \\ &+ \int_Q a(., u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla \omega^i_{\mu,j})\rho_m(u_n) \end{aligned} \tag{1}$$

$$\begin{aligned} &+ \int_Q a(., u_n, \nabla u_n)\nabla u_n (T_k(u_n) - \omega_{\mu,j})\rho'_m(u_n) \\ &= \int_Q f_n v dxdt + n \int_Q T_n((u_n - \psi)^-) v dxdt \\ &=: (3) + (4). \end{aligned} \tag{2}$$

Let us recall that for $u_n \in W_0^{1,x}L_M(Q)$, there exists a smooth function $u_{n\sigma}$ (see [14]) such that

$$\begin{aligned} u_{n\sigma} &\to u_n \text{ for the modular convergence in } W_0^{1,x}L_M(Q), \\ \frac{\partial u_{n\sigma}}{\partial t} &\to \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,x}L_{\overline{M}}(Q) + L^1(Q). \end{aligned}$$

$$\begin{aligned} &\ll \frac{\partial u_n}{\partial t}, v \gg = \lim_{\sigma \to 0+} \int_Q (u_{n\sigma})'(T_k(u_{n\sigma}) - \omega_{\mu,j})\rho_m(u_{n\sigma}) \\ &= \lim_{\sigma \to 0+} \left(\int_Q (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))'(T_k(u_{n\sigma}) - \omega_{\mu,j}) dxdt + \int_Q (T_k(u_{n\sigma})'(T_k(u_{n\sigma}) - \omega_{\mu,j}) dxdt \right) \\ &= \lim_{\sigma \to 0+} \left[\int_\Omega (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j}) dx \right]_0^T \\ &- \int_Q (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j})' dxdt \\ &+ \int_Q (T_k(u_{n\sigma})'(T_k(u_{n\sigma}) - \omega_{\mu,j}) dxdt =: I_1 + I_2 + I_3. \end{aligned}$$

Remark also that,

$$\begin{aligned} &R_m(u_{n\sigma}) \ge T_k(u_{n\sigma}) \text{ if } u_{n\sigma} < k \text{ and } R_m(u_{n\sigma}) > k = T_k(u_{n\sigma}) \ge |\omega_{\mu,j}| \text{if } u_{n\sigma} \ge k. \\ &I_1 = \int_\Omega (R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T))(T_k(u_{n\sigma})(T) - \omega_{\mu,j}(T)) dx \\ &I_1 \ge \int_{un\sigma(T) \le k} (R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T))(T_k(u_{n\sigma})(T) - \omega_{\mu,j}(T)) dx \end{aligned}$$

and it is easy to see that $\limsup_{\sigma \to 0+} I_1 \ge \varepsilon(n, j, \mu)$.

Concerning I_2,

$$\begin{aligned} I_2 &= -\int_{u_{n\sigma} \le k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j})' dxdt + \int_{u_{n\sigma} > k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(\omega_{\mu,j})' dxdt \\ &=: I_2^1 + I_2^2. \end{aligned}$$

As in I_1, we obtain $I_2^1 \ge \varepsilon(n, j, \mu)$, and

$$I_2^2 = \int_{u_{n\sigma}>k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(\omega_{\mu,j})' dxdt \geq \mu \int_{u_{n\sigma}>k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(v_j) - T_k(u_{n\sigma}))' dxdt,$$

thus by using the fact that $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j})\mathcal{X}_{u_{n\sigma}>k} \geq 0$.

So, $\limsup_{\sigma\to 0^+} I_2^2 \geq \mu \int_{u_n>k} (R_m(u_n) - T_k(u_n))(T_k(v_j) - T_k(u_n))' dxdt = \varepsilon(n,j)$.

About I_3,

$$\begin{aligned} I_3 &= \int_Q (T_k(u_{n\sigma}))'(T_k(u_{n\sigma}) - \omega_{\mu,j})dxdt \\ &= \int_Q (T_k(u_{n\sigma}) - \omega_{\mu,j})'(T_k(u_{n\sigma}) - \omega_{\mu,j})dxdt + \int_Q (\omega_{\mu,j})'(T_k(u_{n\sigma}) - \omega_{\mu,j})dxdt. \end{aligned}$$

Set $\Phi(s) = s^2/2$, $\Phi \geq 0$,then

$$\begin{aligned} I_3 &= \left[\int_\Omega \Phi(T_k(u_{n\sigma}) - \omega_{\mu,j})dx\right]_0^T + \mu \int_Q (T_k(v_j) - \omega_{\mu,j})(T_k(u_{n\sigma}) - \omega_{\mu,j})dxdt \\ &\geq \varepsilon(n,j,\mu) + \mu \int_Q (T_k(v_j) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j})dxdt (\text{as } in\ I_2). \end{aligned}$$

So,

$$\begin{aligned} \limsup_{\sigma\to 0^+} I_3 &\geq \varepsilon(n,j,\mu) + \mu \int_Q (T_k(v_j) - T_k(u_n))(T_k(u_n) - \omega_{\mu,j})dxdt \\ &= \varepsilon(n,j,\mu) + \mu \int_Q (T_k(v_j) - T_k(u))(T_k(u) - \omega_{\mu,j})dxdt + \varepsilon(n), \end{aligned}$$

and easily we deduce, $\limsup_{\sigma\to 0^+} I_3 \geq \varepsilon(n,j,\mu)$.

Finally we conclude that: $\ll \frac{\partial u_n}{\partial t}, (T_k(u_n) - \omega_{\mu,j})\rho_m(u_n) \gg \geq \varepsilon(n,j,\mu)$.

We are interested now with the terms of (1)-(4).

About (1):

$$\begin{aligned} &\int_Q a(., u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_{\mu,j})\rho_m(u_n)dx\,dt \\ &= \int_{u_n\leq k} a(., u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_{\mu,j})\rho_m(u_n)dx\,dt + \int_{u_n>k} a(., u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_{\mu,j})\rho_m(u_n)dx\,dt \\ &= \int_Q a(., T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla\omega_{\mu,j})dx\,dt + \int_{u_n>k} a(., u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_{\mu,j})\rho_m(u_n)dx\,dt \end{aligned}$$

recall that $\rho_m(u_n) = 1$ on $\{|u_n| \leq k\}$.

Let $s > 0$, $Q_s = \{(x,t) \in Q : |\nabla T_k(u)| \leq s\}$, $Q_j^s = \{(x,t) \in Q : |\nabla T_k(v_j)| \leq s\}$.

$$\begin{aligned} &\int_Q a(., u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla\omega_{\mu,j})\rho_m(u_n)dx\,dt \\ &= \int_Q \Big(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s)\Big)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)dx\,dt \\ &+ \int_Q a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)dx\,dt \\ &+ \int_Q a(., T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\mathcal{X}_j^s dx\,dt \\ &- \int_Q a(., u_n, \nabla u_n)\nabla\omega_{\mu,j}\rho_m(u_n)dx\,dt \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

By using the inequality (1.3), we can deduce the existence of some measurable function h_k such that

$$a(., T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{in} (L_{\overline{M}}(Q))^N \text{for} \sigma(\prod L_M, \prod E_{\overline{M}}),$$

$$J_2 = \int_Q a(., T_k(u), \nabla T_k(v_j)\mathcal{X}_j^s)(\nabla T_k(u) - \nabla T_k(v_j)\mathcal{X}_j^s) dx\, dt + \varepsilon(n),$$

since

$$a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s) \to a(., T_k(u), \nabla T_k(v_j)\mathcal{X}_j^s) \text{ strongly in } (E_{\overline{M}}(Q))^N,$$
$$a(., T_k(u), \nabla T_k(v_j)\mathcal{X}_j^s) \to a(., T_k(u), \nabla T_k(u)\mathcal{X}_j^s) \text{ strongly in } (E_{\overline{M}}(Q))^N,$$

and $\nabla T_k(v_j)\mathcal{X}_j^s \to \nabla T_k(u)\mathcal{X}^s$ strongly in $(L_{\overline{M}}(Q))^N$.

Then,

$$J_2 = \varepsilon(n, j).$$

Following the same way as in J_2, one has

$$J_3 = \int_Q h_k \nabla T_k(u) dx\, dt + \varepsilon(n, j, \mu, s).$$

Concerning the terms J_4 :

$$\begin{aligned} J_4 &= -\int_Q a(., T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla \omega_{\mu,j}^i \rho_m(u_n)\, dx\, dt \\ &= -\int_{|u_n| \le k} a(., T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla \omega_{\mu,j} \rho_m(u_n)\, dx\, dt \\ &\quad - \int_{k<|u_n| \le m+1} a(., T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla \omega_{\mu,j} \rho_m(u_n)\, dx\, dt. \end{aligned}$$

Letting $n \to \infty$, then

$$J_4 = -\int_{k<|u| \le m+1} h_{m+1} \nabla \omega_{\mu,j} \rho_m(u)\, dx\, dt - \int_{|u| \le k} h_k \nabla \omega_{\mu,j} \rho_m(u)\, dx\, dt + \varepsilon(n).$$

Taking now the limits $j \to \infty$ and after $\mu \to \infty$ in the last equality, we obtain

$$J_4 = -\int_Q h_k \nabla T_k(u)\, dx\, dt + \varepsilon(n, j, \mu).$$

Then,

$$(1) = \int_Q \left(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s) \right) (\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s) + \varepsilon(n, j, \mu, s).$$

About (2):

$$|\int_Q a(., u_n, \nabla u_n) \nabla u_n (T_k(u_n) - \omega_{\mu,j}) \rho_m'(u_n)| dxdt \le C(k) \int_{m<|u_n| \le m+1} a(., u_n, \nabla u_n) \nabla u_n\, dxdt.$$

Since (u_n) is bounded in $W_0^{1,x} L_D(Q)$ and using (iii), we have $(a(., u_n, \nabla u_n))$ is bounded in $L_H(Q)$, then

$$|\int_{m<|u_n| \le m+1} a(., u_n, \nabla u_n) \nabla u_n\, dxdt| \le ||a(., u_n, \nabla u_n)||_{H,Q} ||\nabla u_n||_{D, m<|u_n| \le m+1} \le \varepsilon(n, m),$$

so,

$$(2) \leq \varepsilon(n, m).$$

About (4):

Since $u \geq \psi$, then $T_k(u) \geq T_k(\psi)$ and there exist a smooth function $v_j \geq T_k(\psi)$ such that $v_j \to T_k(u)$ for the modular convergence in $W_0^{1,x}L_M(Q)$.

$$(4) = n \int_Q T_n((u_n - \psi)^-)(T_k(u_n) - T_k(v_j)_\mu)\rho_m(u_n)dxdt \leq \varepsilon(n, j, \mu).$$

Taking into account now the estimation of (1), (2), (4)and (5), we obtain

$$\int_Q \Big(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s)\Big)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)\, dx\, dt \leq \varepsilon(n, j, \mu, s, m). \quad (3.8)$$

On the other hand,

$$\begin{aligned}
&\int_Q (a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(u)\mathcal{X}^s))(\nabla T_k(u_n) - \nabla T_k(u)\mathcal{X}^s)\, dx\, dt \\
&- \int_Q \Big(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s)\Big)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)\, dx\, dt \\
&= \int_Q a(., T_k(u_n), \nabla T_k(u_n))(\nabla T_k(v_j)\mathcal{X}_j^s - \nabla T_k(u)\mathcal{X}^s)\, dx\, dt \\
&- \int_Q a(., T_k(u_n), \nabla T_k(u)\mathcal{X}^s)(\nabla T_k(v_j)\mathcal{X}_j^s - \nabla T_k(u)\mathcal{X}^s)\, dx\, dt \\
&+ \int_Q a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)\, dx\, dt,
\end{aligned}$$

each term of the last right hand side is of the form $\epsilon(n, j, s)$, which gives

$$\begin{aligned}
&\int_Q (a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(u)\mathcal{X}^s))(\nabla T_k(u_n) - \nabla T_k(u)\mathcal{X}^s)\, dx\, dt \\
&= \int_Q \Big(a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(v_j)\mathcal{X}_j^s)\Big)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)\, dx\, dt \\
&+ (n, j, s).
\end{aligned}$$

Following the same technique used by Porretta [2], we have for all $r < s$:

$$\int_{Q_r} (a(., T_k(u_n), \nabla T_k(u_n)) - a(., T_k(u_n), \nabla T_k(u)))\, (\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt \to 0.$$

Thus, as in the elliptic case (see [7]), there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \text{ a.e. in } Q.$$

We deduce then that,

$$a(., T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(., T_k(u), \nabla T_k(u)) \text{ in } (L_{\overline{M}}(Q))^N \text{for } \sigma(\prod L_M, \prod E_{\overline{M}}).$$

Lemma 3.7 *For all* $k > 0$,

$$\nabla T_k(u_n) \to \nabla T_k(u) \textit{for the modular convergence in } (L_M(Q))^N.$$

Proof:

We have proved that

$$\int_Q \left(a(.,T_k(u_n),\nabla T_k(u_n)) - a(.,T_k(u_n),\nabla T_k(v_j)\mathcal{X}_j^s)\right)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)\,dx\,dt$$

$\leq \epsilon\ (n,\ j,\ \mu,\ s,\ m)$ (see (3.8)).

We can also deduce that

$$\int_Q (a(.,T_k(u_n),\nabla T_k(u_n)) - a(.,T_k(u_n),\nabla T_k(u)\mathcal{X}^s))(\nabla T_k(u_n) - T_k(u)\mathcal{X}^s)\,dx\,dt$$
$$= \int_Q \left(a(.,T_k(u_n),\nabla T_k(u_n)) - a(.,T_k(u_n),\nabla T_k(v_j)\mathcal{X}_j^s)\right)(\nabla T_k(u_n) - \nabla T_k(v_j)\mathcal{X}_j^s)\,dx\,dt$$
$$+\varepsilon(n,j,s).$$

Then

$$\int_Q a(.,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt$$
$$\leq \int_Q a(.,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\mathcal{X}^s dxdt$$
$$+\int_Q a(.,T_k(u_n),\nabla T_k(u)\mathcal{X}^s)(\nabla T_k(u_n) - T_k(u)\mathcal{X}^s)dxdt + \varepsilon(n,j,\mu,s,m).$$
$$\overline{\lim_n}\int_Q a(.,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq \int_Q a(.,T_k(u),\nabla T_k(u))\nabla T_k(u)\mathcal{X}^s dxdt + \lim_n \varepsilon(n,j,\mu,s,m)$$

then,

$$\overline{\lim_n}\int_Q a(.,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \leq \int_Q a(.,T_k(u),\nabla T_k(u))\nabla T_k(u) \leq \underline{\lim}_n \int_Q a(.,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n).$$

Letting $n \to \infty$, we deduce

$$a(.,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \to a(.,T_k(u),\nabla T_k(u))\nabla T_k(u)\mathcal{X}^s \text{ in } L^1(Q).$$

Using the same argument as above, we obtain

$$a(.,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \to a(.,T_k(u),\nabla T_k(u))\nabla T_k(u) \text{ in } L^1(Q),$$

and Vitali's theorem and (1.1) gives

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ for the modular convergence in } (L_M(Q))^N.$$

3.3.1 The convergence of the problems (P_n) and the completion of the proof of Theorem 3.1

The passage to the limit is an easy task by using the last steps, then

$$a(.,u_n,\nabla u_n) \to a(.,u,\nabla u) \text{ weakly in } L_H(Q) \text{ and a.e. in } Q,$$

then,

$$-\int_Q u\frac{\partial\varphi}{\partial t} + \int_Q a(.,u,\nabla u)\nabla\varphi\,dxdt - \int_\Omega \varphi du_0 = \int_Q \varphi d\mu,$$

for all $\phi \in D(\mathbf{R}^{N+1})$ which are zero in a neighborhood of $(0,T) \times \partial\Omega$ and $\{T\} \times \Omega$.

4 Conclusion

In this article, we have proved the existence of solutions of some class of unilateral problems in the Orlicz-Sobolev spaces when the right-hand side is a Radon measure.

Competing interests

The authors declare that they have no competing interests.

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