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On critical extremal length for the existence of holomorphic mappings of once-holed tori

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Abstract

Let $\mathfrak{T}_a[Y_0]$ be the set of marked once-holed tori which allows a holomorphic mapping into a given Riemann surface Y_0 with marked handle. We compare it with the subset $\mathfrak{T}_\infty[Y_0]$ of marked once-holed tori X such that there is a holomorphic mapping $f : X \rightarrow Y_0$ for which the cardinal numbers of $f^{-1}(p)$, $p \in Y_0$, are bounded. We show that while $\mathfrak{T}_\infty[Y_0]$ is a proper subset of $\mathfrak{T}_a[Y_0]$ apart from a few exceptions, their critical extremal lengths are identical.

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1 Introduction

Let R be a Riemann surface of positive genus. By a *mark of handle* of R we mean an ordered pair $\chi = \{a, b\}$ of simple loops a and b on R whose geometric intersection number $a \times b$ is equal to one. The pair $Y = (R, \chi)$ is called a *Riemann surface with marked handle*.

Let $Y' = (R', \chi')$ be another Riemann surface with marked handle, where $\chi' = \{a', b'\}$. If $f : R \rightarrow R'$ is holomorphic and maps a and b onto loops freely homotopic to a' and b' on R' , respectively, then we say that f is a holomorphic mapping of Y into Y' and use the notation $f : Y \rightarrow Y'$. If, in addition, $f : R \rightarrow R'$ is conformal, that is, if $f : R \rightarrow R'$ is holomorphic and injective, then $f : Y \rightarrow Y'$ is called conformal.

A noncompact Riemann surface of genus one with exactly one boundary component is called a *once-holed torus*. A *marked once-holed torus* means a once-holed torus with marked handle. Let \mathfrak{T} denote the set of marked once-holed tori, where two marked once-holed tori are identified with each other if there is a conformal mapping of one onto the other. It is a three-dimensional real analytic manifold with boundary (see [1, §7]).

For later use, we introduce some notations. Let \mathbb{H} stand for the upper half-plane: $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. For $\tau \in \mathbb{H}$, let G_τ denote the additive group generated by 1 and τ . Then $T_\tau := \mathbb{C}/G_\tau$ is a torus, that is, a compact Riemann surface of genus one. The two oriented segments $[0, 1]$ and $[0, \tau]$ are projected onto simple loops a_τ and b_τ forming a mark χ_τ of handle of T_τ . Set $X_\tau = (T_\tau, \chi_\tau)$. For $l \in [0, 1)$, we define $T_\tau^{(l)} = T_\tau \setminus \pi_\tau([0, l])$, where $\pi_\tau : \mathbb{C} \rightarrow T_\tau$ is the natural projection. Then $T_\tau^{(l)}$ is a once-holed torus. We choose a mark $\chi_\tau^{(l)}$ of handle of $T_\tau^{(l)}$ so that the inclusion mapping of $T_\tau^{(l)}$ into T_τ is a conformal mapping of $X_\tau^{(l)} := (T_\tau^{(l)}, \chi_\tau^{(l)})$ into X_τ . The correspondence $(\tau, l) \mapsto X_\tau^{(l)}$ defines a bijection of $\mathbb{H} \times [0, 1)$

onto \mathfrak{T} (see, for example, [2]). In other words, every marked once-holed torus is realized as a horizontal slit domain of a torus with marked handle, or a marked torus, uniquely up to conformal automorphisms of the marked torus.

Let Y_0 be a Riemann surface with marked handle. We are interested in the set $\mathfrak{T}_a[Y_0]$ of marked once-holed tori X for which there is a holomorphic mapping of X into Y_0 . It possesses an interesting quantitative property. In [3] (see also [4]) we have established that there is a nonnegative number $\lambda_a[Y_0]$ such that

- (i) if $\text{Im } \tau \geq 1/\lambda_a[Y_0]$, then $X_\tau^{(l)} \notin \mathfrak{T}_a[Y_0]$ for any l , while
- (ii) if $\text{Im } \tau < 1/\lambda_a[Y_0]$, then $X_\tau^{(l)} \in \mathfrak{T}_a[Y_0]$ for some l ,

where $1/0 = +\infty$. If Y_0 is a marked torus, then $\mathfrak{T}_a[Y_0] = \mathfrak{T}$ and hence $\lambda_a[Y_0] = 0$. Otherwise, $\lambda_a[Y_0] > 0$ by [5, Theorem 1 and Proposition 1].

In this article we compare $\mathfrak{T}_a[Y_0]$ with the set $\mathfrak{T}_\infty[Y_0]$ of marked once-holed tori X such that there is a holomorphic mapping $f : X \rightarrow Y_0$ for which the supremum $d(f)$ of the cardinal numbers of $f^{-1}(p)$, $p \in R_0$, is finite. As is shown in [3], it possesses a property similar to that of $\mathfrak{T}_a[Y_0]$: There is a nonnegative number $\lambda_\infty[Y_0]$ such that

- (i) if $\text{Im } \tau \geq 1/\lambda_\infty[Y_0]$, then $X_\tau^{(l)} \notin \mathfrak{T}_\infty[Y_0]$ for any l , while
- (ii) if $\text{Im } \tau < 1/\lambda_\infty[Y_0]$, then $X_\tau^{(l)} \in \mathfrak{T}_\infty[Y_0]$ for some l .

Since

$$\mathfrak{T}_\infty[Y_0] \subset \mathfrak{T}_a[Y_0],$$

we have

$$\lambda_\infty[Y_0] \geq \lambda_a[Y_0]. \tag{1}$$

We first establish the following theorem.

Theorem 1 *If Y_0 is not a marked torus or a marked once-holed torus, then $\mathfrak{T}_\infty[Y_0]$ is a proper subset of $\mathfrak{T}_a[Y_0]$.*

Nevertheless, the sign of equality actually holds in (1).

Theorem 2 *For any marked Riemann surface Y_0 , the equality $\lambda_\infty[Y_0] = \lambda_a[Y_0]$ holds.*

The proofs of Theorems 1 and 2 will be given in the next section.

2 Proofs

We begin with the proof of Theorem 1. Let $Y_0 = (R_0, \chi_0)$, where $\chi_0 = \{a_0, b_0\}$, be a Riemann surface with marked handle which is not a marked torus or a marked once-holed torus. We consider the loops a_0 and b_0 as elements of the fundamental group $\pi_1(R_0)$ of R_0 . Let \tilde{R}_0 be the covering Riemann surface of R_0 corresponding to the subgroup $\langle a_0, b_0 \rangle$ of $\pi_1(R_0)$ generated by a_0 and b_0 . Since R_0 is not a torus, \tilde{R}_0 is a once-holed torus. We choose a mark $\tilde{\chi}_0 = \{\tilde{a}_0, \tilde{b}_0\}$ of handle of \tilde{R}_0 so that the natural projection $\pi_0 : \tilde{R}_0 \rightarrow R_0$ is a holomorphic mapping of the marked once-holed torus $\tilde{Y}_0 := (\tilde{R}_0, \tilde{\chi}_0)$ onto Y_0 . Then \tilde{Y}_0 is an element of $\mathfrak{T}_a[Y_0]$.

Let f be an arbitrary holomorphic mapping of \tilde{Y}_0 into Y_0 . Since it maps \tilde{a}_0 and \tilde{b}_0 onto loops freely homotopic to a_0 and b_0 , respectively, it is lifted to a holomorphic mapping \tilde{f} of

\tilde{Y}_0 into itself satisfying $\pi_0 \circ \tilde{f} = f$. By Huber [6, Satz II] (see also Marden-Richards-Rodin [7, Theorem 5]), we infer that \tilde{f} is a conformal automorphism of \tilde{Y}_0 . Since R_0 is not a torus or a once-holed torus, we conclude that $d(f) = d(\pi_0) = \infty$ and hence $\tilde{Y}_0 \notin \mathfrak{T}_\infty[Y_0]$. This completes the proof of Theorem 1.

For the proof of Theorem 2, we make a remark. Let $X = (T, \chi)$, where $\chi = \{a, b\}$, be a marked once-holed torus. Then the extremal length $\lambda(X)$ of the free homotopy class of a is called the *basic extremal length* of X . Note that $\lambda(X_\tau^{(l)}) = 1/\text{Im } \tau$ (see [8, Proposition 1]).

Now, take an arbitrary $\tau \in \mathbb{H}$ with $\text{Im } \tau < 1/\lambda_a[Y_0]$. Then, for some $l \in [0, 1)$, there is a holomorphic mapping f of $X_\tau^{(l)}$ into Y_0 . Recall that $T_\tau^{(l)}$ is the horizontal slit domain $T_\tau \setminus \pi_\tau([0, l])$ of the torus T_τ . Choose a canonical exhaustion $\{S_n\}$ of $T_\tau^{(l)}$ so that each S_n is a once-holed torus including the loops in $\chi_\tau^{(l)}$. Since the inclusion mapping $S_n \rightarrow T_\tau^{(l)}$ is a conformal mapping of the marked once-holed torus $W_n := (S_n, \chi_\tau^{(l)})$ into $X_\tau^{(l)}$, the restriction f_n of f to S_n is a holomorphic mapping of W_n into Y_0 . As S_n is relatively compact in $T_\tau^{(l)}$, we know that $d(f_n) < \infty$. Consequently, W_n belongs to $\mathfrak{T}_\infty[Y_0]$.

To estimate the basic extremal length of W_n , take an arbitrary positive number ε less than $\text{Im } \tau/2$. Let H_ε be the horizontal strip $\{z \in \mathbb{C} \mid \varepsilon < \text{Im } z < \text{Im } \tau - \varepsilon\}$. Since $\{S_n\}$ is increasing with $\bigcup_n S_n = T_\tau^{(l)}$, for all sufficiently large n , the subdomain S_n includes the ring domain $\pi_\tau(H_\varepsilon)$. It follows that

$$\text{Im } \tau - 2\varepsilon < \frac{1}{\lambda(W_n)} \leq \frac{1}{\lambda_\infty[Y_0]},$$

which implies that

$$\text{Im } \tau \leq \frac{1}{\lambda_\infty[Y_0]}.$$

As τ was an arbitrary point of \mathbb{H} satisfying $\text{Im } \tau < 1/\lambda_a[Y_0]$, we deduce that

$$\frac{1}{\lambda_a[Y_0]} \leq \frac{1}{\lambda_\infty[Y_0]},$$

or

$$\lambda_\infty[Y_0] \leq \lambda_a[Y_0].$$

Theorem 2 has been proved.

3 Topological relations between $\mathfrak{T}_a[Y_0]$ and $\mathfrak{T}_\infty[Y_0]$

The arguments in the proof of Theorem 2 easily lead us to the following theorem.

Theorem 3 *The closure of $\mathfrak{T}_\infty[Y_0]$ is identical with $\mathfrak{T}_a[Y_0]$.*

Proof We begin with recalling a global coordinate system on the space \mathfrak{T} of marked once-holed tori. Let $X = (T, \chi)$ be a marked once-holed torus, where $\chi = \{a, b\}$. Observe that $\dot{\chi} := \{b, a^{-1}\}$ is a mark of handle of T . Also, if c is a simple loop homotopic to ab^{-1} , then $\ddot{\chi} := \{c, a\}$ is another mark of handle of T . Set $\dot{X} = (T, \dot{\chi})$ and $\ddot{X} = (T, \ddot{\chi})$. Then the basic extremal lengths of X, \dot{X} and \ddot{X} define a global coordinate system on \mathfrak{T} . In fact, we introduce a real

analytic structure into \mathfrak{T} so that the mapping $\Lambda : X \mapsto (\lambda(X), \lambda(\dot{X}), \lambda(\ddot{X}))$ is a real analytic diffeomorphism of \mathfrak{T} into \mathbb{R}^3 (see [1]).

Now, let $X = (T, \chi)$ be an arbitrary element of $\mathfrak{T}_a[Y_0]$. Take a canonical exhaustion $\{S_n\}$ of T for which each S_n is a once-holed torus including the loops in χ , and set $W_n = (S_n, \chi)$. Since $X = X_\tau^{(l)}$ for some $\tau \in \mathbb{H}$ and $l \in [0, 1)$, the proof of Theorem 2 shows that the basic extremal length $\lambda(W_n)$ tends to $\lambda(X)$ as $n \rightarrow \infty$. By changing marks of handles, we infer that $\{\lambda(\dot{W}_n)\}$ and $\{\lambda(\ddot{W}_n)\}$ converge to $\lambda(\dot{X})$ and $\lambda(\ddot{X})$, respectively, and hence that $\Lambda(W_n) \rightarrow \Lambda(X)$ as $n \rightarrow \infty$. Since each W_n belongs to $\mathfrak{T}_\infty[Y_0]$, the marked once-holed torus X belongs to the closure $\overline{\mathfrak{T}_\infty[Y_0]}$ of $\mathfrak{T}_\infty[Y_0]$. We thus obtain $\mathfrak{T}_a[Y_0] \subset \overline{\mathfrak{T}_\infty[Y_0]}$. Because $\mathfrak{T}_a[Y_0]$ is closed and includes $\mathfrak{T}_\infty[Y_0]$, we conclude that $\overline{\mathfrak{T}_\infty[Y_0]} = \mathfrak{T}_a[Y_0]$. \square

Since $\mathfrak{T}_\infty[Y_0]$ and $\mathfrak{T}_a[Y_0]$ are (noncompact) domains with Lipschitz boundary by [3], we see that Theorem 3 is an improvement of Theorem 2. Also, we obtain the following corollary.

Corollary 1 *The interiors of $\mathfrak{T}_\infty[Y_0]$ and $\mathfrak{T}_a[Y_0]$ coincide with each other.*

If Y_0 is a marked torus, then $\mathfrak{T}_a[Y_0]$ is identical with \mathfrak{T} (see [1]). Hence so is $\mathfrak{T}_\infty[Y_0]$ by Corollary 1.

Competing interests

The author declares that he has no competing interests.

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