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On generalized absolute Cesàro summability factors

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Abstract

In this paper, a known theorem dealing with $|C,\alpha,\gamma;\delta|_k$ summability factors has been generalized for $|C,\alpha,\beta,\gamma;\delta|_k$ summability factors. Some results have also been obtained.

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Keywords: Hölder's inequality; quasi-monotone sequence; summability factors

1 Introduction

A sequence (b_n) of positive numbers is said to be quasi-monotone if $n\Delta b_n \geq -\rho b_n$ for some $\rho > 0$ and is said to be δ-quasi-monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the nth Cesàro means of order (α,β), with $\alpha+\beta > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [2]),

$$u_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v, \tag{1}$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{2}$$

where

$$A_n^{\alpha+\beta} = O\left(n^{\alpha+\beta}\right), \quad \alpha+\beta > -1, \qquad A_0^{\alpha+\beta} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for } n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta|_k$, $k \geq 1$ and $\alpha+\beta > -1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} \left|u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta}\right|^k < \infty. \tag{4}$$

Since $t_n^{\alpha,\beta} = n(u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta})$ (see [3]), condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|t_n^{\alpha,\beta}\right|^k < \infty. \tag{5}$$

The series $\sum a_n$ is said to be summable $|C,\alpha,\beta,\gamma;\delta|_k$, $k \geq 1$, $\alpha + \beta > -1$, $\delta \geq 0$ and γ is a real number, if (see [4])

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)} \left|u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta}\right|^k = \sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k} \left|t_n^{\alpha,\beta}\right|^k < \infty. \tag{6}$$

If we take $\beta = 0$, then $|C,\alpha,\beta,\gamma;\delta|_k$ summability reduces to $|C,\alpha,\gamma;\delta|_k$ summability (see [5]).

2 Known result

In [6], we have proved the following theorem dealing with $|C,\alpha,\gamma;\delta|_k$ summability factors of infinite series.

Theorem A *Let $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, and γ be a real number such that $-\gamma(\delta k + k - 1) + (\alpha + 1)k > 1$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $|\Delta\lambda_n| \leq |B_n|$, $\lambda_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} n\delta_n \log n < \infty$ and $\sum_{n=1}^{\infty} nB_n \log n$ is convergent. If the sequence (w_n^α) defined by (see [7])*

$$w_n^\alpha = \left|t_n^\alpha\right|, \quad \alpha = 1, \tag{7}$$

$$w_n^\alpha = \max_{1\leq v\leq n} \left|t_v^\alpha\right|, \quad 0 < \alpha < 1, \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\gamma(\delta k+k-1)-k} \left(w_n^\alpha\right)^k = O(\log m) \quad \text{as } m \to \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ is summable $|C,\alpha,\gamma;\delta|_k$.

3 The main result

The aim of this paper is to generalize Theorem A for $|C,\alpha,\beta,\gamma;\delta|_k$ summability. We shall prove the following theorem.

Theorem *Let $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, and γ be a real number such that $(\alpha + \beta + 1 - \gamma(\delta + 1))k > 1$, and let there be sequences (B_n) and (λ_n) such that the conditions of Theorem* A *are satisfied. If the sequence $(w_n^{\alpha,\beta})$ defined by*

$$w_n^{\alpha,\beta} = \left|t_n^{\alpha,\beta}\right|, \quad \alpha = 1, \beta > -1, \tag{10}$$

$$w_n^{\alpha,\beta} = \max_{1\leq v\leq n} \left|t_v^{\alpha,\beta}\right|, \quad 0 < \alpha < 1, \beta > -1, \tag{11}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\gamma(\delta k+k-1)-k} \left(w_n^{\alpha,\beta}\right)^k = O(\log m) \quad \text{as } m \to \infty, \tag{12}$$

then the series $\sum a_n\lambda_n$ is summable $|C,\alpha,\beta,\gamma;\delta|_k$. It should be noted that if we take $\beta = 0$, then we get Theorem A.

We need the following lemmas for the proof of our theorem.

Lemma 1 ([8]) *Under the conditions on* (B_n), *as taken in the statement of the theorem, we have the following*:

$$nB_n \log n = O(1), \tag{13}$$

$$\sum_{n=1}^{\infty} n \log n |\Delta B_n| < \infty. \tag{14}$$

Lemma 2 ([9]) *If* $0 < \alpha \leq 1$, $\beta > -1$, *and* $1 \leq v \leq n$, *then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p\right| \leq \max_{1\leq m\leq v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p\right|. \tag{15}$$

4 Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C,α,β) mean of the sequence $(na_n\lambda_n)$. Then by (2), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Firstly applying Abel's transformation and then using Lemma 2, we have that

$$\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \\
\left|T_n^{\alpha,\beta}\right| &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p\right| + \frac{|\lambda_n|}{A_n^{\alpha+\beta}} \left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v\right| \\
&\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} w_v^{\alpha,\beta} |\Delta\lambda_v| + |\lambda_n| w_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \quad \text{say}
\end{aligned}$$

since

$$\left|T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}\right|^k \leq 2^k\left(\left|T_{n,1}^{\alpha,\beta}\right|^k + \left|T_{n,2}^{\alpha,\beta}\right|^k\right). \tag{16}$$

In order to complete the proof of the theorem, by (6), it is sufficient to show that for $r = 1, 2$,

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k} \left|T_{n,r}^{\alpha,\beta}\right|^k < \infty.$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
&\sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} \left|T_{n,1}^{\alpha,\beta}\right|^k \\
&\quad\leq \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} \left|\frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} w_v^{\alpha,\beta} \Delta\lambda_v\right|^k
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}}\left\{\sum_{v=1}^{n-1}v^{\alpha k}v^{\beta k}|\Delta\lambda_v|\left(w_v^{\alpha,\beta}\right)^k\right\}\left\{\sum_{v=1}^{n-1}|\Delta\lambda_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}}\left\{\sum_{v=1}^{n-1}v^{\alpha k}v^{\beta k}|B_v|\left(w_v^{\alpha,\beta}\right)^k\right\}\left\{\sum_{v=1}^{n-1}|B_v|\right\}^{k-1}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}|B_v|\left(w_v^{\alpha,\beta}\right)^k\sum_{n=v+1}^{m+1}\frac{1}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}|B_v|\left(w_v^{\alpha,\beta}\right)^k\int_v^{\infty}\frac{dx}{x^{(\alpha+\beta+1-\gamma(\delta+1))k}}\\
&= O(1)\sum_{v=1}^{m}|B_v|v^{\gamma(\delta k+k-1)-k+1}\left(w_v^{\alpha,\beta}\right)^k\\
&= O(1)\sum_{v=1}^{m}v|B_v|v^{\gamma(\delta k+k-1)-k}\left(w_v^{\alpha,\beta}\right)^k\\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\left(v|B_v|\right)\right|\sum_{p=1}^{v}p^{\gamma(\delta k+k-1)-k}\left(w_p^{\alpha,\beta}\right)^k + O(1)m|B_m|\sum_{v=1}^{m}v^{\gamma(\delta k+k-1)-k}\left(w_v^{\alpha,\beta}\right)^k\\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\left(v|B_v|\right)\right|\log v + O(1)m|B_m|\log m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta B_v|\log v + O(1)\sum_{v=1}^{m-1}|B_{v+1}|\log v + O(1)m|B_m|\log m\\
&= O(1)\quad \text{as } m\to\infty,
\end{aligned}
$$

in view of the hypotheses of the theorem and Lemma 1. Similarly, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}n^{\gamma(\delta k+k-1)-k}\left|T_{n,2}^{\alpha,\beta}\right|^k &= O(1)\sum_{n=1}^{m}|\lambda_n|n^{\gamma(\delta k+k-1)-k}\left(w_n^{\alpha,\beta}\right)^k\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|\sum_{v=1}^{n}v^{\gamma(\delta k+k-1)-k}\left(w_v^{\alpha,\beta}\right)^k\\
&\quad + O(1)|\lambda_m|\sum_{v=1}^{m}v^{\gamma(\delta k+k-1)-k}\left(w_v^{\alpha,\beta}\right)^k\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|\log n + O(1)|\lambda_m|\log m\\
&= O(1)\sum_{n=1}^{m-1}|B_n|\log n + O(1)|\lambda_m|\log m\\
&= O(1)\quad \text{as } m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 1. Therefore, by (6), we get that for $r = 1, 2$,

$$
\sum_{n=1}^{\infty}n^{\gamma(\delta k+k-1)-k}\left|T_{n,r}^{\alpha,\beta}\right|^k < \infty.
$$

This completes the proof of the theorem. If we take $\delta = 0$ and $\gamma = 1$, then we get a result for $|C,\alpha,\beta|_k$ summability factors. Also, if we take $\beta = 0$, $\delta = 0$, and $\alpha = 1$, then we get a result for $|C,1|_k$ summability.

Competing interests

The author declares that they have no competing interests.

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