

Quantum algebraic representation of localization and motion of a Dirac electron

Marc-Thierry Jaekel [†] and Serge Reynaud [‡]

[†] *Laboratoire de Physique Théorique* ¹, ENS, 24 rue Lhomond, F75231 Paris France

[‡] *Laboratoire Kastler Brossel* ², UPMC, case 74, Campus Jussieu, F75252 Paris France

Frontier Tests of QED and Physics of the Vacuum, Trieste, October 5-11, 2000

Abstract. Quantum algebraic observables representing localization in space-time of a Dirac electron are defined. Inertial motion of the electron is represented in the quantum algebra with electron mass acting as the generator of motion. Since transformations to uniformly accelerated frames are naturally included in this conformally invariant description, the quantum algebra is also able to deal with uniformly accelerated motion.

INTRODUCTION

Modern discussions of time and space revive a basic distinction already stated by Newton [1]. Theoretical physics deals with two different definitions of time and space. On one hand, the equations of motion are written in terms of coordinate parameters representing ideal mathematical points on a classical map of space-time. This was true for classical Newtonian physics, but this is still true for quantum field theory or general relativity. On the other hand, time and space measurements necessarily reflect the properties of physical observables.

Einstein emphasized this point in his first paper on relativity [2] :

If we wish to describe the *motion* of a material point, we give the values of its co-ordinates as functions of time. Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by “time”...

The “time” of an event is that which is given simultaneously with the event by a stationary clock located at the place of the event...

Einstein then explains that time indications delivered by remote clocks have to be compared through synchronization procedures consisting in particular in the

¹) Laboratoire du CNRS, de l’Ecole Normale Supérieure et de l’Université Paris Sud

²) Laboratoire de l’Ecole Normale Supérieure, de l’Université Pierre et Marie Curie et du CNRS

transfer of light pulses between the two clocks. The position of an event in space-time can be deduced from the exchange of several light pulses. As is well known, these thought experiments about localization in space-time entail that space and time are relativistic observables mixed under frame transformations. And they are not only thought experiments since they are nowadays implemented as practical applications such as the Global Positioning System [3].

At the same time, space and time belong to the quantum domain with their metrological definition now rooted in atomic physics [4]. The time delivered by an atomic clock is the phase of a quantum oscillator while electromagnetic signals used to synchronize remote clocks are quantum fields. This raises the question of the compatibility of quantum and relativistic descriptions of space and time, as explicitly stated by Schrödinger [5] :

In Lorentz transformations, time and space coordinates enter in a completely symmetrical manner. But in quantum mechanics, ... time is a quite different thing than the co-ordinates.

Time is not treated as an observable... It is a parameter the value of which is supposed to be exactly known : it is in fact the old good time of Newton and quantum mechanics does not worry about the existence of the old good clock which it would need in order to know the value of this parameter t .

But it seems to me doubtless that we will have to give up this too classical notion of time, and not only because of relativity...

The knowledge of the variable t is obtained in the same manner as that of any other variable, by observing a physical system, namely a clock. t is therefore an observable and must be treated as an observable ; time must in general have “statistics” and not a “value”.

In other words, space and time have to be treated as quantum observables with their proper quantum fluctuations. And, moreover, this treatment has to be compatible with the effects of relativistic frame transformations. It is well known in ordinary quantum mechanics that space positions may be represented as quantum operators conjugated to momentum operators. This is still the case in standard quantum field theory [6,7]. But it is also commonly thought that standard quantum formalism does not allow for time being treated as an operator conjugated to energy [8,9] which forbids to relate the energy-time Heisenberg inequality to a quantum commutation relation. This also entails that Lorentz transformations cannot be represented properly with positions in space described by quantum observables and position in time described by a classical coordinate parameter.

Lorentz invariance may be restored by abandoning the observable character of space variables as well [10]. This clears up the unacceptable difference between space and time with however the drawback of no longer having explicit representations of space-time observables. Physical fluctuations are then described by quantum fields. This situation makes the quantum implementation of relativistic

symmetries quite unsatisfactory [11] and plagues the attempts to construct a quantum theory including gravity [12,13]. Coming back to the quotation of Einstein, this also challenges the very representation of “motion” in quantum theory.

LOCALIZATION OBSERVABLES

We develop here an alternative approach to localization and motion in space-time where quantum and relativistic requirements are treated simultaneously and consistently [14,15]. The main idea is to define localization observables generalizing Einstein’s definition of the position of the center-of-mass in special theory of relativity [16]. The latter definition relies on the invariance of equations of motion under Lorentz frame transformations. Positions in space are then expressed in terms of the symmetry generators, *i.e.* the components P_μ of energy-momentum vector and the components $J_{\mu\nu}$ of angular momentum tensor. The more general definition advocated here uses invariance of Maxwell equations not only under Lorentz frame transformations but also under the larger group of conformal transformations. This definition will allow one to give a quantum description of motion of an electron, as it was demanded by Schrödinger.

Conformal symmetry of electromagnetism has been discussed by Bateman and Cunningham [17,18], very early after the advent of relativity theory. It has been studied in a number of papers [19] and it still holds for free electromagnetic fields in quantum field theory [20]. In particular conformal symmetry includes a dilatation generator D which, together with the Poincaré generators, allows one to construct 4 quantum observables X_μ representing positions in space-time. We recall below the main building blocks of this construction described in more details in [14,15].

The whole framework is based upon the conformal algebra, that is the set of commutators which characterize conformal symmetry. Conformal algebra firstly contains a sub-algebra with Poincaré and dilatation generators

$$\begin{aligned}
(P_\mu, P_\nu) &= 0 & (J_{\mu\nu}, P_\rho) &= \eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu \\
(J_{\mu\nu}, J_{\rho\sigma}) &= \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} \\
(D, P_\mu) &= P_\mu & (D, J_{\mu\nu}) &= 0
\end{aligned} \tag{1}$$

P_μ and $J_{\mu\nu}$ are the translation and rotation generators, that is also components of energy-momentum vector and angular momentum tensor. D is the dilatation generator producing a change of space-time and energy-momentum units which preserves the velocity of light c and the Planck constant \hbar . Quantum algebraic commutators (1) represent the relativistic shifts of observables under translations and rotations with $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ standing for Minkowski tensor as well as the commutation relations of observables. We use the following notations for the commutator, the symmetrized product and the symmetrized division of observables

$$(A, B) \equiv \frac{AB - BA}{i\hbar} \quad A \cdot B \equiv \frac{AB + BA}{2} \quad \frac{A}{B} \equiv A \cdot \frac{1}{B} \tag{2}$$

Localization observables are constructed from the algebra (1) as the solutions X_μ of the equations

$$J_{\mu\nu} = P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu} \quad D = P^\mu \cdot X_\mu \quad (3)$$

Angular momenta $J_{\mu\nu}$ are sums of orbital and spin contributions. As the spin tensor $S_{\mu\nu}$ is transverse with respect to momentum, the first equation fixes the transverse components of position observables with respect to momentum. This explains why D has to be involved to fix the longitudinal components of position observables. Position observables are obtained as the following expressions which generalize Einstein's definition of the center-of-mass position as soon as the square $P^2 \equiv P_\mu P^\mu$ of the momentum vector differs from 0

$$X_\mu = \frac{D \cdot P_\mu - J_{\mu\nu} \cdot P^\nu}{P^2} \quad (4)$$

Elementary algebraic calculus then leads to their transformation properties

$$\begin{aligned} (P_\mu, X_\nu) &= -\eta_{\mu\nu} \\ (J_{\mu\nu}, X_\rho) &= \eta_{\nu\rho} X_\mu - \eta_{\mu\rho} X_\nu \\ (D, X_\mu) &= -X_\mu \end{aligned} \quad (5)$$

These equations respectively mean that position observables are canonically conjugated to momenta, that they are transformed as components of a Lorentz vector under Lorentz transformations and that they have a conformal weight opposite to that of momenta. These results meet the expectations based on classical relativity but they bear on quantum observables.

However different position components are found to have a non null commutator which is directly connected to spin, implying that dispersions obey a Heisenberg inequality

$$(X_\mu, X_\nu) = \frac{S_{\mu\nu}}{P^2} \quad (\Delta X)^2 \gtrsim \frac{\hbar^2}{P^2} \quad (6)$$

This important output of the quantum algebraic formalism clearly indicates that the conceptions of space-time inherited from classical relativity have to be revised for quantum objects. Positions in space-time cannot be associated with sizeless classical points but they have rather to be thought of as fuzzy spots with a size ΔX given by Compton relation.

It is possible to give a geometrical interpretation of these results by coming back to the Einstein construction of a position in space-time as the intersection of 2 light rays propagating in different directions. For such a field state, the squared mass P^2 associated with the field state differs from 0. It may then be proved that positions (4) correspond to the coincidence of the two light rays. Precisely, two real light rays never intersect exactly and each ray has a transverse dimension due to diffraction

but the positions correspond to the middle point on the segment which joins the two rays while being perpendicular to both rays [15]. The fuzziness in (6) due to the spin associated with the 2-light-rays state is directly connected to the length of the segment involved in the geometrical construction.

Let us recall at this point that conformal symmetry includes four additional generators C_μ which represent transformations to uniformly accelerated frames. The algebra (1) is thus complemented by the commutators

$$\begin{aligned} (D, C_\mu) &= -C_\mu & (C_\mu, C_\nu) &= 0 \\ (J_{\mu\nu}, C_\rho) &= \eta_{\nu\rho}C_\mu - \eta_{\mu\rho}C_\nu \\ (P_\mu, C_\nu) &= -2\eta_{\mu\nu}D - 2J_{\mu\nu} \end{aligned} \tag{7}$$

Generators C_μ are commuting components of a Lorentz vector with a conformal weight opposite to that of momenta. Their commutators (P_μ, C_ν) with translations describe the redshifts of momenta under transformations to accelerated frames and thus constitute quantum versions of the Einstein redshift law [21]. This entails that the quantum algebraic construction allows one to discuss relativistic effects associated with accelerated frames from invariance properties represented by conformal algebra [22]. We will show below that the same algebraic framework allows one to represent inertial as well as uniformly accelerated motion.

CONFORMAL DIRAC ELECTRON

Up to now, localization observables have been constructed on Einstein procedures with light rays so that conformal symmetry has been used in the case of free electromagnetic fields. The localization observables thus constitute a theoretical representation equivalent to standard QED theory although the interpretation may be quite different from the standard one. In particular conformal symmetry has been used for the field state associated with 2-light-rays localization although the mass does not vanish for this state. Also, conformal generators are space-time integrals of the quantum stress tensor and localization observables are highly non linear and non local expressions of the fields. Since they are not defined in vacuum or one photon states, these hermitian observables are not self-adjoint. This is just the point where the Pauli theorem forbidding the definition of a quantum time operator has been circumvented [14,15].

We want now to show that quantum algebra also allows one to deal with Dirac electrons [23]. To this aim, we first introduce a second set x_μ of position observables and a new spin tensor $s_{\mu\nu}$. Poincaré and dilatation generators keep the same form (3) when X_μ and $S_{\mu\nu}$ are replaced by x_μ and $s_{\mu\nu}$ but the new observables obey canonical commutation relations. In particular the components x_μ commute with other components of positions x_ν or spin $s_{\nu\rho}$. The canonical positions may be written

$$x_\mu = X_\mu - i\gamma \frac{W_\mu}{P^2} \quad W^\mu \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \quad \gamma^2 = 1 \tag{8}$$

where W^μ is the Pauli-Lubanski spin vector and γ is a sign representing the orientation of the spin tensor $s_{\mu\nu}$. γ is invariant under translations, rotations and dilatation and plays the same role here as γ_5 in Dirac electron theory [24].

Canonical positions are transformed under translations, rotations and dilatation as expected from classical relativity : they obey equations (5) with X_μ replaced by x_μ . This entails that the requirements enounced by Schrödinger have now been met. Positions may be represented as quantum relativistic observables which are conjugate with respect to momentum observables while properly representing Lorentz symmetry. A position in time has been defined besides positions in space and it is conjugated to energy as positions in space are conjugated to spatial momenta. The 4 positions in space-time are mixed under Lorentz transformations according to the classical laws. It has however to be emphasized that the canonical variables x_μ and $s_{\mu\nu}$ are not hermitian, as shown by (8). This is a further important output of the quantum algebraic approach to the localization problem. One may define either hermitian observables with non canonical commutators or canonical variables which are not hermitian.

In order to build up a quantum algebraic theory of electrons, we use the general properties already deduced from conformal symmetry and add two further assumptions. First we assume that the spin number is $\frac{1}{2}$

$$\frac{W^2}{P^2} = -\hbar^2 s(s+1) \quad s = \frac{1}{2} \quad (9)$$

and invariant under transformations to accelerated frames. As it is already invariant under translations, rotations and dilatation, s is thus invariant under the whole conformal algebra. Then we introduce the sign ε of mass which commutes with translations, rotations and dilatation. γ commutes with $\varepsilon^2 = 1$ while ε commutes with $\gamma^2 = 1$ and these conditions are fulfilled as soon as γ and ε either commute or anticommute. We assume here that they anticommute

$$M = \varepsilon\sqrt{P^2} \quad \varepsilon^2 = 1 \quad \gamma \cdot \varepsilon = 0 \quad (10)$$

Hence the spin orientation γ changes the mass sign ε into its opposite while the mass sign ε changes the spin orientation γ into its opposite.

From these assumptions, we deduce that there exist Clifford symbols in the quantum algebra which allow one to write a Dirac equation

$$\gamma_\mu = \frac{P_\mu}{\hbar M} - 2\gamma \frac{W_\mu}{\hbar M} \quad \gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} \quad M = P^\mu \gamma_\mu = \gamma_\mu P^\mu \quad (11)$$

The symbols γ_μ commute with canonical positions and momenta and their commutators reproduce the canonical spin tensor $s_{\mu\nu}$. These relations constitute a quantum algebraic extension of Dirac electron theory [24]. At this point, it is worth emphasizing that the Clifford relations and the Dirac equation have been derived from conformal symmetry and the 2 further assumptions (9,10).

The quantum algebraic formalism leads to a striking difference with the standard Dirac theory. The mass M is now a quantum operator which, like its sign ε , anticommutes with γ . Though being a Lorentz scalar invariant under translations and rotations, M is not invariant under dilatation since it has the same conformal weight as momenta. These properties are certainly incompatible with the classical treatment of the mass commonly associated with Dirac electron. Notice however that electron mass is, at least partly, generated by electromagnetic self-energy and that it should therefore present intrinsic quantum fluctuations. Since electron-positron pairs may decay into 2-photon pairs, it seems hard to forbid treating the mass of the $e^- - e^+$ pair similarly to that of the 2-photon pair. But, as already noticed, the second one is commonly considered to obey conformal invariance.

These arguments plead for mass having its proper conformal dimension, so that the conformal symmetry holds for free electron states as well as for photon states. They also correspond to the motivations of Weyl aiming at a conformal description of space-time [25] or Dirac attempting to describe the electron field in a conformal space [26]. In any case, modern descriptions of the electron are no longer identical to the original Dirac theory. Electron mass is now considered to be generated through an interaction with Higgs fields [27] and Higgs models obeying conformal invariance are available [28]. Here, we do not specify a particular field model but we consider that the treatment of electron mass is compatible with conformal symmetry and, thus, given by the previously written equations.

INERTIAL MOTION

As just discussed, mass M is a quantum observable. We show now that commutators with this observable play an important role since they generate inertial motion. To this aim, we introduce a prime symbol representing an algebraic derivative and obeying Leibniz rule

$$F' = (F, M) \quad (FG)' = F'G + FG' \quad (12)$$

The same property is obtained for the commutator with any observable but the present definition implies that the Poincaré generators are constants of motion $P'_\mu = J'_{\mu\nu} = 0$. These features resemble the common Hamiltonian formalism with however the mass as the motion generator and, hence, a full compatibility with Lorentz invariance. The derivative defined by (12) may be thought of as associated with a quantum “evolution time” $\frac{D}{M}$ since the latter observable has a unit derivative

$$D' = M \quad M' = 0 \quad \left(\frac{D}{M}\right)' = 1 \quad (13)$$

This evolution time is distinct from the date X_0 of an event, that is its position in time. Meanwhile, both times have to be distinguished from their classical analogs.

The motion of hermitian positions corresponds to the simple expectations of classical mechanics. The hermitian velocity, that is the derivative of the hermitian

position, is the standard mechanical velocity given by the ratio of momentum to mass while the hermitian acceleration vanishes

$$X'_\mu = \frac{P_\mu}{M} \quad X''_\mu = 0 \quad (14)$$

Hermitian spin is also conserved. The orientation γ oscillates at twice the rest mass frequency

$$\gamma' = \frac{2}{i\hbar}\gamma M = -\frac{2}{i\hbar}M\gamma \quad \gamma'' = -\frac{4M^2}{\hbar^2}\gamma \quad (15)$$

It follows that the canonical velocities are identical to the Clifford generators and that they undergo a “Zitterbewegung”

$$x'_\mu = \gamma_\mu \quad x''_\mu = \gamma'_\mu = -i\gamma'' \frac{W_\mu}{P^2} \quad (16)$$

These relations are analogous to well known results of standard Dirac theory with, once again, an algebraic representation of the motion generated by mass observable.

ACCELERATED MOTION

As already explained, the quantum algebraic formalism relies upon conformal symmetry and, therefore, has the ability of dealing with uniformly accelerated frames as well as inertial frames. It can therefore describe uniformly accelerated motion as well as inertial motion. In order to prove this statement, we write the shift of the mass observable from M in a frame to \overline{M} in another frame as a conjugation by an element of the conformal group

$$\overline{M} = \exp\left(-\frac{a^\rho C_\rho}{2i\hbar}\right) M \exp\left(\frac{a^\rho C_\rho}{2i\hbar}\right) = M - \frac{a^\rho}{2}(C_\rho, M) + \dots \quad (17)$$

The parameters a^ρ are classical accelerations along the 4 space-time directions. Here, we restrict our attention to the linear approximation with respect to these parameters. It is important to notice that conjugations preserve the structure of quantum algebraic relations. For example, position and momentum observables are transformed under conjugation but the canonical commutators between them are preserved since they are classical numbers. Canonical commutators have the same form in accelerated and inertial frames and are written in terms of the same Minkowski metric $\eta_{\mu\nu}$. This result had to be expected in a quantum algebraic approach but it clearly stands in contradistinction with covariant techniques of classical relativity.

Relation (17) gives the redshift of mass under the frame transformation

$$(C_\rho, M) = 2M \cdot X_\rho \quad (18)$$

The redshift has exactly the form expected from Einstein classical law since it is proportional to M and to the gravitational potential $a^\rho X_\rho$ arising from the transformation according to Einstein equivalence principle. The shift may also be read as a conformal metric factor which depends on position observables as the classical metric factor depends on classical coordinates [23]. To fix ideas, we may consider the coordinate frame denoted with bars to be an inertial frame and the coordinate frame without bars to be the accelerated frame. We may then define “inertial motion” as the algebraic derivative $F' = (F, \overline{M})$ associated with the “inertial mass” \overline{M} . This choice leads to conservation of Poincaré generators \overline{P}_μ and $\overline{J}_{\mu\nu}$ defined in the inertial frame. The laws of inertial motion have also the same form as previously in the inertial frame. Now we can write the motion of observables P_μ or $J_{\mu\nu}$ as they are defined in the accelerated frame. Proceeding in this manner we obtain for the hermitian positions X_μ

$$X''_\mu = a_\mu + \dots \tag{19}$$

This is a quantum algebraic expression of the law of free fall in the gravity force arising in the accelerated frame from the Einstein equivalence principle. The dots mean that this expression is linearized in the gravity acceleration. A more general expression with non linear terms is deduced from conformal symmetry in [29].

CONCLUSIONS

We have shown that positions in space-time may be represented as quantum relativistic observables conjugated with respect to momentum observables while properly obeying Lorentz symmetry. The whole framework constitutes a “quantum algebraic relativity” where relativistic effects under frame transformations and quantum commutation relations are described by a unique algebraic calculus. Mass is a quantum observable and this has important consequences. First, it suffers shifts under transformations to accelerated frames and these shifts reproduce the effect of the gravitational potential arising as a consequence of Einstein equivalence principle. Then, mass is the generator of inertial motion and the choice of inertial frames among all possible definitions allows one to represent uniformly accelerated motion as well, still in full consistency with Einstein equivalence principle.

The quantum algebraic description of localization in space-time relies on conformal symmetry and it can be written so that it explicitly displays invariance under the $SO(4, 2)$ algebra [29]. This allows one to write the laws of inertial motion as well as the Newton equation of motion in a gravity field under an invariant form consistent with this algebra. The whole description may thus be thought of as a “quantum geometry” [30] relying on conformal symmetry.

REFERENCES

1. Newton I., *Philosophiae Naturalis Principia Mathematica* (1687); the definitions of time and space are reproduced in the introduction of [14].
2. Einstein A., *Annalen der Physik* **17** 891 (1905); the quotation is taken from the english translation in Einstein A., Lorentz H.A., Weyl H. and Minkowski H. *The Principle of Relativity* (Methuen, 1923; reprinted by Dover, 1952).
3. Special Issue on Time and Frequency, *Proceedings of IEEE* **79** 891-1079 (1991); in particular, Lewandowski W. and Thomas C. p.991 and Leschiutta S. p.1001.
4. Quinn T.J., in [3] p. 894.
5. E. Schrödinger, *Sitz. preuss. Akad. Wissen.* 418 (1930); 63, 144, 238 (1931); *Ann. Inst. Henri Poincaré* **19** 269 (1932); the quotation is translated by the authors from the latter french text.
6. Pryce M.H.L., *Proc. Roy. Soc.* **A195** 62 (1948).
7. Newton T.D. and Wigner E., *Rev. Mod. Phys.* **21** 400 (1949).
8. Pauli W., *Die allgemeinen Prinzipien der Wellenmechanik* (Springer Verlag, 1958); *General Principles of Quantum Mechanics* (Springer Verlag, 1980).
9. Wightman A.S., *Rev. Mod. Phys.* **34** 845 (1962).
10. Heisenberg W., *The Physical Principles of the Quantum Theory* (Dover, 1930).
11. de Witt B. in *Relativity, Groups and Topology* eds C. de Witt and B. de Witt (Gordon Breach, 1963).
12. Rovelli C. *Phys. Rev.* **D42** 2638 (1990); *ibidem* **D43** 442 (1991); *Class. Quantum Grav.* **8** 297, 317 (1991).
13. Isham C. J. in *Integrable Systems, Quantum Groups and Quantum Field Theories* (Kluwer, 1997) p.157.
14. Jaekel M.T. and Reynaud S. in *Frontier tests of QED and Physics of Vacuum* eds E. Zavattini, D. Bakalov and C. Rizzo (Heron Press, 1998) p. 426.
15. Jaekel M.T. and Reynaud S., *Eur. Phys. J.* **D5** 9 (1999).
16. Einstein A., *Annalen der Physik* **20** 627 (1906).
17. Bateman H. *Proc. London Math. Soc.* **8** 223 (1909).
18. Cunningham E. *Proc. London Math. Soc.* **8** 77 (1909).
19. Fulton T., Rohrlich F. and Witten L., *Rev. Mod. Phys.* **34** 442 (1962) and references therein.
20. Binengar B., Fronsdal C. and Heidenreich W., *J. Math. Phys.* **24** 2828 (1983).
21. Jaekel M.T. and Reynaud S., *EuroPhys. Lett.* **38** 1 (1997).
22. Jaekel M.T. and Reynaud S., *Found. of Phys.* **28** 439 (1998).
23. Jaekel M.T. and Reynaud S., *Phys. Lett.* **A256** 95 (1999).
24. Dirac P.A.M., *Proc. Roy. Soc. London* **117 A** 610; **118 A** 351 (1928).
25. Weyl H., *Space Time Matter* (Methuen) § 35 (1922).
26. Dirac P.A.M., *Annals of Mathematics* **37** 429 (1936).
27. Itzykson C. and Zuber J.-B., *Quantum Field Theory* (McGraw Hill, 1985).
28. Pawlowski M. preprint xxx hep-th/9808021 (1998).
29. Jaekel M.T. and Reynaud S., *Annalen der Physik* **9** 589 (2000).
30. Connes A., *Noncommutative geometry* (Academic Press, 1994); *Commun. Math. Phys.* **182** 155 (1996).