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Common fixed points of ordered g-contractions in partially ordered metric spaces

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Abstract

The concept of ordered g-contraction is introduced, and some fixed and common fixed point theorems for g-nondecreasing ordered g-contraction mapping in partially ordered metric spaces are proved. We also show the uniqueness of the common fixed point in the case of an ordered g-contraction mapping. The theorems presented are generalizations of very recent fixed point theorems due to Golubović *et al.* (Fixed Point Theory Appl. 2012:20, 2012).

MSC: 47H10; 47N10

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1 Introduction

The Banach fixed point theorem for contraction mappings has been extended in many directions (*cf.* [1–48]). Very recently, Golubović *et al.* [49] presented some new results for ordered quasicontractions and ordered g-quasicontractions in partially ordered metric spaces.

Recall that if $(X, \preceq)$ is a partially ordered set and $f : X \to X$ is such that, for $x, y \in X$, $x \preceq y$ implies $fx \preceq fy$, then a mapping f is said to be nondecreasing. The main result of Golubović *et al.* [49] is the following common fixed point theorem.

Theorem 1.1 (See [49], Theorem 1) *Let $(X, d, \preceq)$ be a partially ordered metric space and let $f, g : X \to X$ be two self-maps on X satisfying the following conditions*:

(i) $fX \subset gX$;
(ii) gX *is complete*;
(iii) f *is g-nondecreasing*;
(iv) f *is an ordered g-quasicontraction*;
(v) *there exists* $x_0 \in X$ *such that* $gx_0 \preceq fx_0$;
(vi) *if* $\{gx_n\}$ *is a nondecreasing sequence that converges to some* $gz \in gX$, *then* $gx_n \preceq gz$ *for each* $n \in \mathbb{N}$ *and* $gz \preceq g(gz)$.

Then f and g have a coincidence point, i.e., there exists $z \in X$ such that $fz = gz$. If, in addition,

(vii) f *and* g *are weakly compatible* [50, 51], *i.e.*, $fx = gx$ *implies* $fgx = gfx$, *for each* $x \in X$, *then they have a common fixed point*.

An open problem is to find sufficient conditions for the uniqueness of the common fixed point in the case of an ordered g-quasicontraction in Theorem 1.1.

In Section 2 of this article, we introduce ordered g-contractions in partially ordered metric spaces and prove the respective (common) fixed point results, which generalizes the results of Theorem 1.1.

In Section 3 of this article, a theorem on the uniqueness of a common fixed point is obtained when for all $x, u \in X$, there exists $a \in X$ such that fa is comparable to fx and fu, in addition to the hypotheses in Theorem 2.1 of Section 2. Our result is an answer to finding sufficient conditions for the uniqueness of the common fixed point in the case of ordered g-contractions in Theorem 1.1. Finally, two examples show that the comparability is a sufficient condition for the uniqueness of common fixed point in the case of ordered g-contractions, so our results are extensions of known ones.

2 Common fixed points of ordered g-contractions

We start this section with the following definitions. Consider a partially ordered set $(X, \preceq)$ and two mappings $f : X \to X$ and $g : X \to X$ such that $f(X) \subset g(X)$.

Definition 2.1 (See [1]) We shall say that the mapping f is g-nondecreasing (resp., g-nonincreasing) if

$$gx \preceq gy \quad \Rightarrow \quad fx \preceq fy \tag{1}$$

(resp., $gx \preceq gy \Rightarrow fx \succeq fy$) holds for each $x, y \in X$.

Definition 2.2 (See [49]) We shall say that the mapping f is an ordered g-quasicontraction if there exists $\alpha \in (0, 1)$ such that for each $x, y \in X$ satisfying $gy \preceq gx$, the inequality

$$d(fx, fy) \le \alpha \cdot M(x, y)$$

holds, where

$$M(x, y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}.$$

Definition 2.3 We shall say that the mapping f is an ordered g-contraction if there is a continuous and nondecreasing function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(0) = 0$ and if there exists $\alpha \in (0, 1)$, the inequality

$$\begin{aligned} \psi\bigl(d(fx, fy)\bigr) \le \max\bigl\{&\psi\bigl(\alpha d(gx, gy)\bigr), \psi\bigl(\alpha d(gx, fx)\bigr), \psi\bigl(\alpha d(gy, fy)\bigr), \\ &\psi\bigl(\alpha d(gx, fy)\bigr), \psi\bigl(\alpha d(gy, fx)\bigr)\bigr\} \end{aligned} \tag{2}$$

holds for all $x, y \in X$ for which $gy \preceq gx$.

It is obviously that if $\psi = I$, then ordered g-contraction reduces to ordered g-quasicontraction.

For arbitrary $x_0 \in X$ one can construct a so-called Jungck sequence $\{y_n\}$ in the following way: denote $y_0 = fx_0 \in f(X) \subset g(X)$; there exists $x_1 \in X$ such that $gx_1 = y_0 = fx_0$; now $y_1 =$

$fx_1 \in f(X) \subset g(X)$ and there exists $x_2 \in X$ such that $gx_2 = y_1 = fx_1$ and the procedure can be continued.

Theorem 2.1 *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $f, g : X \to X$ be two self-maps on X satisfying the following conditions:*
(i) $f(X) \subset g(X)$;
(ii) *$g(X)$ is closed*;
(iii) *f is a g-nondecreasing mapping*;
(iv) *f is an ordered g-contraction*;
(v) *there exists an $x_0 \in X$ with $gx_0 \preceq fx_0$*;
(vi) *$\{g(x_n)\} \subset X$ is a nondecreasing sequence with $g(x_n) \to gz$ in $g(X)$, then $gx_n \preceq gz$, $gz \preceq g(gz)$, $\forall n$ hold.*

Then f and g have a coincidence point. Further, if f and g are weakly compatible, then f and g have a common fixed point.

Proof Let $x_0 \in X$ be such that $gx_0 \preceq fx_0$. Since $f(X) \subset g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Again from $f(X) \subset g(X)$, we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process we can choose a sequence $\{y_n\}$ in X such that

$$gx_{n+1} = fx_n = y_n, \quad \forall n \geq 0. \tag{3}$$

Since $gx_0 \preceq fx_0$ and $gx_1 = fx_0$, we have $gx_0 \preceq gx_1$. Then by (1),

$$fx_0 \preceq fx_1. \tag{4}$$

Thus, by (3), $gx_1 \preceq gx_2$. Again by (1),

$$fx_1 \preceq fx_2, \tag{5}$$

that is, $gx_2 \preceq gx_3$. Continuing this process, we obtain

$$fx_0 \preceq fx_1 \preceq fx_2 \preceq fx_3 \preceq \cdots \preceq fx_n \preceq fx_{n+1}. \tag{6}$$

Now, we will claim that $\{y_n\}$ is a Cauchy sequence. In what follows, we will suppose that $d(fx_n, fx_{n+1}) > 0$ for all n, since if $fx_n = fx_{n+1}$ for some n, by (3),

$$fx_{n+1} = gx_{n+1}, \tag{7}$$

that is, f and g have a coincidence at $x = x_{n+1}$, and so we have finished the proof. Thus we assume that $d(fx_n, fx_{n+1}) > 0$ for all n. We will show that

$$d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n), \quad \forall n \geq 1. \tag{8}$$

From (3) and (6), it follows that $gx_n \preceq gx_{n+1}$ for all $n > 0$. Then apply the contractivity condition (2) with $x = x_n$ and $y = x_{n+1}$,

$$\begin{aligned}\psi\big(d(fx_n,fx_{n+1})\big) \le \max\big\{&\psi\big(\alpha d(gx_n,gx_{n+1})\big), \psi\big(\alpha d(gx_n,fx_n)\big), \psi\big(\alpha d(gx_{n+1},fx_{n+1})\big),\\ &\psi\big(\alpha d(gx_n,fx_{n+1})\big), \psi\big(\alpha d(gx_{n+1},fx_n)\big)\big\}.\end{aligned} \tag{9}$$

Thus by (3),

$$\begin{aligned}\psi\big(d(fx_n,fx_{n+1})\big) &\le \max\big\{\psi\big(\alpha d(fx_{n-1},fx_n)\big), \psi\big(\alpha d(fx_{n-1},fx_n)\big), \psi\big(\alpha d(fx_n,fx_{n+1})\big),\\ &\qquad \psi\big(\alpha d(fx_{n-1},fx_{n+1})\big), \psi\big(\alpha d(fx_n,fx_n)\big)\big\}\\ &= \max\big\{\psi\big(\alpha d(fx_{n-1},fx_n)\big), \psi\big(\alpha d(fx_n,fx_{n+1})\big),\\ &\qquad \psi\big(\alpha d(fx_{n-1},fx_{n+1})\big)\big\}.\end{aligned} \tag{10}$$

We divide the proof of (8) into three cases in the following:

(I) If $\max\{\psi(\alpha d(fx_{n-1},fx_n)), \psi(\alpha d(fx_n,fx_{n+1})), \psi(\alpha d(fx_{n-1},fx_{n+1}))\} = \psi(\alpha d(fx_{n-1},fx_n))$, from (10), then

$$\psi\big(d(fx_n,fx_{n+1})\big) \le \psi\big(\alpha d(fx_{n-1},fx_n)\big). \tag{11}$$

Since ψ is nondecreasing, $d(fx_n,fx_{n+1}) \le \alpha d(fx_{n-1},fx_n)$. By virtue of $\alpha \in (0,1)$, it follows that $d(fx_n,fx_{n+1}) \le d(fx_{n-1},fx_n)$. Thus (8) holds.

(II) If $\max\{\psi(\alpha d(fx_{n-1},fx_n)), \psi(\alpha d(fx_n,fx_{n+1})), \psi(\alpha d(fx_{n-1},fx_{n+1}))\} = \psi(\alpha d(fx_n,fx_{n+1}))$, from (10), then

$$\psi\big(d(fx_n,fx_{n+1})\big) \le \psi\big(\alpha d(fx_n,fx_{n+1})\big). \tag{12}$$

Since ψ is nondecreasing, $d(fx_n,fx_{n+1}) \le \alpha d(fx_n,fx_{n+1})$. By virtue of $\alpha \in (0,1)$, $d(fx_n,fx_{n+1}) = 0$, and it is a contraction with the assumption that $d(fx_n,fx_{n+1}) > 0$ for all n!

(III) If $\max\{\psi(\alpha d(fx_{n-1},fx_{n+1})), \psi(\alpha d(fx_n,fx_{n+1})), \psi(\alpha d(fx_{n-1},fx_{n+1}))\} = \psi(\alpha d(fx_{n-1},fx_{n+1}))$, from (10) and the triangle inequality, we have

$$\begin{aligned}\psi\big(d(fx_n,fx_{n+1})\big) &\le \psi\big(\alpha d(fx_{n-1},fx_{n+1})\big)\\ &\le \psi\big(\alpha d(fx_{n-1},fx_n) + \alpha d(fx_n,fx_{n+1})\big).\end{aligned} \tag{13}$$

Since ψ is nondecreasing,

$$d(fx_n,fx_{n+1}) \le \alpha d(fx_{n-1},fx_n) + \alpha d(fx_n,fx_{n+1}).$$

Then it follows that

$$d(fx_n,fx_{n+1}) \le \frac{\alpha}{1-\alpha} d(fx_{n-1},fx_n). \tag{14}$$

Thus (8) trivially holds when $\alpha \in (0, \frac{1}{2})$. Hence

$$d(fx_n,fx_{n+1}) \le d(fx_{n-1},fx_n), \quad \forall n \ge 1.$$

Taking into account the previous considerations, we proved that (8) holds. From (8), it follows that the sequence $d(fx_n,fx_{n+1})$ of real non-negative numbers is monotone nonincreasing. Therefore, there exists some $\sigma \geq 0$ such that

$$\lim_{n\to\infty} d(fx_n,fx_{n+1}) = \sigma. \tag{15}$$

Next we will prove that $\sigma = 0$. We suppose that $\sigma > 0$. By the triangle inequality,

$$d(fx_{n-1},fx_{n+1}) \leq d(fx_{n-1},fx_n) + d(fx_n,fx_{n+1}). \tag{16}$$

Hence, by (8),

$$d(fx_{n-1},fx_{n+1}) \leq 2d(fx_{n-1},fx_n). \tag{17}$$

Taking the upper limit as $n \to \infty$, we get

$$\limsup_{n\to\infty} \frac{1}{2} d(fx_{n-1},fx_{n+1}) \leq \lim_{n\to\infty} d(fx_{n-1},fx_n). \tag{18}$$

Set

$$\limsup_{n\to\infty} \frac{1}{2} d(fx_{n-1},fx_{n+1}) = \rho. \tag{19}$$

Then it follows that $0 \leq \rho \leq \sigma$. Now, taking the upper limit on the both sides of (10) and $\psi(t)$ being continuous, we get

$$\begin{aligned}\lim_{n\to\infty} \psi\bigl(d(fx_n,fx_{n+1})\bigr) \leq \max\Bigl\{&\psi\Bigl(\lim_{n\to\infty} \alpha d(fx_{n-1},fx_n)\Bigr), \psi\Bigl(\lim_{n\to\infty} \alpha d(fx_n,fx_{n+1})\Bigr),\\ &\psi\Bigl(\lim_{n\to\infty} \alpha d(fx_{n-1},fx_{n+1})\Bigr)\Bigr\}.\end{aligned} \tag{20}$$

From (15) and (19),

$$\psi(\sigma) \leq \max\bigl\{\psi(\alpha\sigma), \psi(2\alpha\rho)\bigr\}. \tag{21}$$

If $\max\{\psi(\alpha\sigma), \psi(2\alpha\rho)\} = \psi(2\alpha\rho)$, from (21), it yields $\psi(\sigma) \leq \psi(2\alpha\rho)$. Since ψ is nondecreasing, then $\sigma \leq 2\alpha\rho$. When $\alpha \in (0, \frac{1}{2})$, then $\sigma \leq 2\alpha\rho < \rho$, it is a contradiction! When $\alpha = \frac{1}{2}$, then $\sigma \leq \rho$, it is a contradiction! When $\alpha \in (\frac{1}{2}, 1)$, then $\sigma \leq 2\alpha\rho \leq 2\alpha\sigma$, it yields $(1 - 2\alpha)\sigma \leq 0$. Since $1 - 2\alpha < 0$ and $\sigma > 0$, it is also a contradiction!

If $\max\{\psi(\alpha\sigma), \psi(2\alpha\rho)\} = \psi(\alpha\sigma)$, then from (21), it yields $\psi(\sigma) \leq \psi(\alpha\sigma)$. Since ψ is nondecreasing, then $\sigma \leq \alpha\sigma$. By virtue of $\alpha \in (0,1)$, it yields σ. It is a contradiction with the assumption that $\sigma > 0$!

Taking into account the previous consideration, $\sigma = 0$. Therefore, we proved that

$$\lim_{n\to\infty} d(fx_n,fx_{n+1}) = 0. \tag{22}$$

Now, we prove that $\{fx_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{fx_n\}$ is not a Cauchy sequence. Then there exist an $\epsilon > 0$ and two sequences of integers $\{n(k)\}$, $\{m(k)\}$,

$m(k) > n(k) \geq k$ with

$$t_k = d(fx_{n(k)}, fx_{m(k)}) \geq \epsilon \quad \text{for } k = 1, 2, 3, \ldots. \tag{23}$$

We may also assume

$$d(fx_{n(k)}, fx_{m(k)-1}) < \epsilon \tag{24}$$

by choosing $m(k)$ to be the smallest number which satisfies $m(k) > n(k)$, and (23) holds. From (23), (24), and by the triangle inequality,

$$\epsilon \leq t_k \leq d(fx_{n(k)}, fx_{m(k)-1}) + d(fx_{m(k)-1}, fx_{m(k)}) < \epsilon + d(fx_{m(k)-1}, fx_{m(k)}). \tag{25}$$

Hence, by (22),

$$\lim_{n\to\infty} t_k = \epsilon. \tag{26}$$

Since from (3) and (6), we have $gx_{n(k)+1} = fx_{n(k)} \preceq fx_{m(k)} = gx_{m(k)+1}$, from (2) and (3) with $x = x_{m(k)+1}$ and $y = x_{n(k)+1}$, we get

$$\begin{aligned}
\psi\left(d(fx_{n(k)+1}, fx_{m(k)+1})\right) &\leq \max\bigl\{\psi\left(\alpha d(gx_{m(k)+1}, gx_{n(k)+1})\right), \psi\left(\alpha d(gx_{m(k)+1}, fx_{m(k)+1})\right), \\
&\quad \psi\left(\alpha d(gx_{n(k)+1}, fx_{n(k)+1})\right), \psi\left(\alpha d(gx_{m(k)+1}, fx_{n(k)+1})\right), \\
&\quad \psi\left(\alpha d(gx_{n(k)+1}, fx_{m(k)+1})\right)\bigr\} \\
&= \max\bigl\{\psi\left(\alpha d(fx_{m(k)}, fx_{n(k)})\right), \psi\left(\alpha d(fx_{m(k)}, fx_{m(k)+1})\right), \\
&\quad \psi\left(\alpha d(fx_{n(k)}, fx_{n(k)+1})\right), \psi\left(\alpha d(fx_{m(k)}, fx_{n(k)+1})\right), \\
&\quad \psi\left(\alpha d(fx_{n(k)}, fx_{m(k)+1})\right)\bigr\}.
\end{aligned} \tag{27}$$

Denote $\sigma_n = d(fx_n, fx_{n+1})$. Then we have

$$\begin{aligned}
\psi\left(d(fx_{n(k)+1}, fx_{m(k)+1})\right) &\leq \max\bigl\{\psi(\alpha t_k), \psi(\alpha\sigma_{m(k)}), \psi(\alpha\sigma_{n(k)}), \psi\left(\alpha d(fx_{m(k)}, fx_{n(k)+1})\right), \\
&\quad \psi\left(\alpha d(fx_{n(k)}, fx_{m(k)+1})\right)\bigr\} \\
&= \psi\left(\max\bigl\{\alpha t_k, \alpha\sigma_{m(k)}, \alpha\sigma_{n(k)}, \alpha d(fx_{m(k)}, fx_{n(k)+1}), \right. \\
&\quad \left. \alpha d(fx_{n(k)}, fx_{m(k)+1})\bigr\}\right) \\
&= \psi\left(\alpha \max\bigl\{t_k, \sigma_{m(k)}, \sigma_{n(k)}, d(fx_{m(k)}, fx_{n(k)+1}), \right. \\
&\quad \left. d(fx_{n(k)}, fx_{m(k)+1})\bigr\}\right),
\end{aligned} \tag{28}$$

where the first equality holds, since ψ is nondecreasing, and $\psi(\max(s_1, s_2, \ldots, s_n)) = \max(\psi(s_1), \psi(s_2), \ldots, \psi(s_n))$ for all $s_1, s_2, \ldots, s_n \in [0, +\infty)$. Again, since ψ is nondecreasing, by (28), it follows that

$$d(fx_{n(k)+1}, fx_{m(k)+1}) \leq \alpha \max\bigl\{t_k, \sigma_{m(k)}, \sigma_{n(k)}, d(fx_{m(k)}, fx_{n(k)+1}), d(fx_{n(k)}, fx_{m(k)+1})\bigr\}.$$

Therefore, since

$$\begin{aligned} t_k &\le d(fx_{n(k)},fx_{n(k)+1}) + d(fx_{n(k)+1},fx_{m(k)+1}) + d(fx_{m(k)+1},fx_{m(k)}) \\ &= \sigma_{n(k)} + \sigma_{m(k)} + d(fx_{n(k)+1},fx_{m(k)+1}), \end{aligned} \tag{29}$$

we have

$$\begin{aligned} \epsilon &\le t_k \\ &\le \sigma_{n(k)} + \sigma_{m(k)} + \alpha \max\bigl\{t_k, \sigma_{m(k)}, \sigma_{n(k)}, d(fx_{m(k)},fx_{n(k)+1}), \\ &\quad d(fx_{n(k)},fx_{m(k)+1})\bigr\}. \end{aligned} \tag{30}$$

By the triangle inequality, (23), and (24),

$$\begin{aligned} \epsilon &\le t_k \\ &\le d(fx_{n(k)},fx_{m(k)+1}) + d(fx_{m(k)+1},fx_{m(k)}) \\ &= d(fx_{n(k)},fx_{m(k)+1}) + \sigma_{m(k)} \\ &\le d(fx_{n(k)},fx_{m(k)-1}) + d(fx_{m(k)-1},fx_{m(k)}) + d(fx_{m(k)},fx_{m(k)+1}) + \sigma_{m(k)} \\ &\le \epsilon + \sigma_{m(k)-1} + 2\sigma_{m(k)}. \end{aligned} \tag{31}$$

From the equality of (31) and the last inequality of (31), it yields

$$\begin{aligned} \epsilon - \sigma_{m(k)} &\le d(fx_{n(k)},fx_{m(k)+1}) \\ &\le \epsilon + \sigma_{m(k)-1} + \sigma_{m(k)}. \end{aligned} \tag{32}$$

Similarly, we obtain

$$\begin{aligned} \epsilon &\le t_k \\ &\le d(fx_{n(k)},fx_{n(k)+1}) + d(fx_{n(k)+1},fx_{m(k)}) \\ &= \sigma_{n(k)} + d(fx_{n(k)+1},fx_{m(k)}). \end{aligned}$$

And it follows that

$$\begin{aligned} d(fx_{n(k)+1},fx_{m(k)}) &\le d(fx_{n(k)+1},fx_{n(k)}) + d(fx_{n(k)},fx_{m(k)-1}) + d(fx_{m(k)-1},fx_{m(k)}) \\ &= \sigma_{n(k)} + d(fx_{n(k)},fx_{m(k)-1}) + \sigma_{m(k)-1} \\ &\le \epsilon + \sigma_{n(k)} + \sigma_{m(k)-1}. \end{aligned} \tag{33}$$

Adding the two inequalities above,

$$\begin{aligned} \epsilon - \sigma_{n(k)} &\le d(fx_{n(k)+1},fx_{m(k)}) \\ &\le \epsilon + \sigma_{n(k)} + \sigma_{m(k)-1}. \end{aligned} \tag{34}$$

From (32) and (34), we have

$$\begin{aligned}\epsilon - \frac{\sigma_{n(k)} + \sigma_{m(k)}}{2} &\leq \frac{d(fx_{n(k)}, fx_{m(k)+1}) + d(fx_{n(k)+1}, fx_{m(k)})}{2} \\ &\leq \epsilon + \sigma_{m(k)-1} + \frac{\sigma_{n(k)} + \sigma_{m(k)}}{2}. \end{aligned} \tag{35}$$

Thus from (22) and (35), we get

$$\lim_{n\to\infty} \frac{d(fx_{n(k)}, fx_{m(k)+1}) + d(fx_{n(k)+1}, fx_{m(k)})}{2} = \epsilon. \tag{36}$$

Letting $n \to \infty$ in (30), then by (22), (26), and (36), we get, as ψ is continuous,

$$\epsilon \leq \alpha \max\{\epsilon, 0, 0, \epsilon, \epsilon\} = \alpha\epsilon < \epsilon, \tag{37}$$

it is a contradiction! Thus our assumption (23) is wrong. Therefore, $\{fx_n\}$ is a Cauchy sequence. Since by (3), we have $\{fx_n = gx_{n+1} \subseteq g(X)\}$ and $g(X)$ is closed, there exists $z \in X$ such that

$$\lim_{n\to\infty} gx_n = gz. \tag{38}$$

Now we show that z is a coincidence point of f and g. Since from condition (vi) and (39), we have $gx_n \preceq gz$ for all n, then by the triangle inequality and (2), we have

$$\begin{aligned}\psi\bigl(d(fx_n, fz)\bigr) \leq \max\bigl\{&\psi\bigl(\alpha d(gx_n, gz)\bigr), \psi\bigl(\alpha d(gx_n, fx_n)\bigr), \psi\bigl(\alpha d(gz, fz)\bigr), \\ &\psi\bigl(\alpha d(gx_n, fz)\bigr), \psi\bigl(\alpha d(gz, fx_n)\bigr)\bigr\}.\end{aligned} \tag{39}$$

So letting $n \to \infty$, and ψ being continuous, we have

$$\begin{aligned}\psi\bigl(d(fz, gz)\bigr) &\leq \max\bigl\{0, 0, \psi\bigl(\alpha d(fz, gz)\bigr), \psi\bigl(\alpha d(fz, gz)\bigr), 0\bigr\} \\ &= \psi\bigl(\alpha d(fz, gz)\bigr).\end{aligned}$$

Since ψ is nondecreasing, then $d(fz, gz) \leq \alpha d(fz, gz)$. Since $\alpha \in (0, 1)$, it follows that $d(fz, gz) = 0$. Hence $fz = gz$. Thus we proved that f and g have a coincidence point.

Suppose now that f and g commute at z. Set $w = fz = gz$. Since f and g are weakly compatible,

$$fw = f(gz) = g(fz) = gw. \tag{40}$$

Since from condition (vi), we have $gz \preceq g(gz) = gw$ and as $fz = gz$ and $fw = gw$, from (2), we have

$$\begin{aligned}\psi\bigl(d(fz, fw)\bigr) &\leq \max\bigl\{\psi\bigl(\alpha d(gz, gw)\bigr), \psi\bigl(\alpha d(gz, fz)\bigr), \psi\bigl(\alpha d(gw, fw)\bigr), \\ &\qquad \psi\bigl(\alpha d(gz, fw)\bigr), \psi\bigl(\alpha d(gw, fz)\bigr)\bigr\} \\ &= \psi\bigl(\alpha d(gz, gw)\bigr).\end{aligned} \tag{41}$$

Since ψ is nondecreasing, $d(fz,fw) \leq \alpha d(gz,gw)$, *i.e.*, $d(fz,fw) \leq \alpha d(fz,fw)$. Again from $\alpha \in (0,1)$, $d(fz,fw) = 0$, that is, $d(w,fw) = 0$. Therefore,

$$fw = gw = w. \tag{42}$$

Thus, we have proved that f and g have a common fixed point. The proof is completed. □

If we replace some conditions in Theorem 2.1, then we can obtain the following conclusions. Note that the way followed in Theorem 2.2 is different from that in the proof of Theorem 2.1. In fact, we can use the way in Theorem 2.2 to prove the conclusions in Theorem 2.1. Similarly, we can also use the way in Theorem 2.1 to prove Theorem 2.2. Here, our aim is to show two different methods of proof. Comparing Theorem 2.1 with Theorem 2.2, we can find that the conclusions cover Theorem 2.2; in other words, the condition of Theorem 2.2 is more extensive than that in Theorem 2.1. Now, let us treat the following theorem.

Theorem 2.2 *Let the conditions of Theorem* 2.1 *be satisfied, except that* (iii), (v) *and* (vi) *are, respectively, replaced by*

(iii′) f *is a* g*-nonincreasing mapping*;
(v′) *there exists* $x_0 \in X$ *such that* fx_0 *and* gx_0 *are comparable*;
(vi′) *if* $\{gx_n\}$ *is a sequence in* $g(X)$ *which has comparable adjacent terms and that converges to some* $gz \in gX$, *then there exists a subsequence* gx_{n_k} *of* $\{gx_n\}$ *having all the terms comparable with* gz *and* gz *is comparable with* ggz. *Then all the conclusions of Theorem* 2.1 *hold*.

Proof Regardless of whether $fx_0 \preceq gx_0$ or $gx_0 \preceq fx_0$ (condition (v′)), Lemma 1 of [49] implies that two arbitrary adjacent terms of the Jungck sequence $\{y_n\}$ are comparable. This is again sufficient to imply that $\{y_n\}$ is a Cauchy sequence. In the following, we assume the other case to prove the conclusions of Theorem 2.2.

Let $x_0 \in X$ be such that $fx_0 \preceq gx_0$, where it is different from $gx_0 \preceq fx_0$ in Theorem 2.1. Since $f(X) \subset g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Again from $f(X) \subset g(X)$, we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process, we can choose a sequence $\{y_n\}$ in X such that

$$gx_{n+1} = fx_n = y_n, \quad \forall n \geq 0. \tag{43}$$

Since $fx_0 \preceq gx_0$ and $gx_1 = fx_0$, we have $gx_1 \preceq gx_0$. Then by condition (iii′), f is a g-nonincreasing mapping,

$$fx_0 \preceq fx_1. \tag{44}$$

Thus, by (43), it follows that $gx_1 \preceq gx_2$. Again by condition (iii′),

$$fx_2 \preceq fx_1, \tag{45}$$

that is, $gx_3 \preceq gx_2$. Continuing this process, we obtain the result that two arbitrary adjacent terms of the Jungck sequence $\{y_n\}$ are comparable.

Let $O(y_k, n) = \{y_k, y_{k+1}, \dots, y_{k+n}\}$. We will show that $\{y_n\}$ is a Cauchy sequence. To prove our claim, we follow the arguments of Das and Naik [12] again. Fix $k \geq 0$ and $n \in \{1, 2, \dots\}$. If $\operatorname{diam}[O(y_k; n)] = 0$, then $\{y_n\}$ is also a Cauchy sequence. Thus our claims holds. Now we suppose that $\operatorname{diam}[O(y_k; n)] > 0$. Now for i, j with $1 \leq i < j$, by (2), we have

$$\begin{aligned}
\psi\bigl(d(y_i, y_j)\bigr) &= \psi\bigl(d(fx_i, fx_j)\bigr) \\
&\leq \max\bigl\{\psi\bigl(\alpha d(gx_i, gx_j)\bigr), \psi\bigl(\alpha d(gx_i, fx_i)\bigr), \psi\bigl(\alpha d(gx_j, fx_j)\bigr), \\
&\quad \psi\bigl(\alpha d(gx_i, fx_j)\bigr), \psi\bigl(\alpha d(gx_j, fx_i)\bigr)\bigr\} \\
&= \max\bigl\{\psi\bigl(\alpha d(y_{i-1}, y_{j-1})\bigr), \psi\bigl(\alpha d(y_{i-1}, y_i)\bigr), \psi\bigl(\alpha d(y_{j-1}, y_j)\bigr), \\
&\quad \psi\bigl(\alpha d(y_{i-1}, y_j)\bigr), \psi\bigl(\alpha d(y_{j-1}, y_i)\bigr)\bigr\} \\
&= \psi\bigl(\max\bigl\{\alpha d(y_{i-1}, y_{j-1}), \alpha d(y_{i-1}, y_i), \alpha d(y_{j-1}, y_j), \\
&\quad \alpha d(y_{i-1}, y_j), \alpha d(y_{j-1}, y_i)\bigr\}\bigr) \\
&= \psi\bigl(\alpha \max\bigl\{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{j-1}, y_j), \\
&\quad d(y_{i-1}, y_j), d(y_{j-1}, y_i)\bigr\}\bigr) \\
&\leq \psi\bigl(\alpha \operatorname{diam}\bigl[O(y_{i-1}; j-i+1)\bigr]\bigr),
\end{aligned}$$

where the third equality holds, since ψ is nondecreasing, and $\psi(\max(s_1, s_2, \dots, s_n)) = \max(\psi(s_1), \psi(s_2), \dots, \psi(s_n))$ for all $s_1, s_2, \dots, s_n \in [0, +\infty)$. Since ψ is nondecreasing,

$$d(y_i, y_j) \leq \alpha \operatorname{diam}\bigl[O(y_{i-1}; j-i+1)\bigr]. \tag{46}$$

Now for some i, j with $k \leq i < j \leq k+n$, $\operatorname{diam}[O(y_k; n)] = d(y_i, y_j)$. If $i > k$, by (2) and (46), then we have

$$\begin{aligned}
\operatorname{diam}\bigl[O(y_k; n)\bigr] &\leq \alpha \operatorname{diam}\bigl[O(y_{i-1}; j-i+1)\bigr] \\
&\leq \alpha \operatorname{diam}\bigl[O(y_k; n)\bigr],
\end{aligned} \tag{47}$$

where the inequality (47) holds as $\operatorname{diam}[O(y_{i-1}; j-i+1)] \leq \operatorname{diam}[O(y_k; n)]$. Then from (47) and $\alpha \in (0, 1)$, we have $\operatorname{diam}[O(y_k; n)] = 0$. It is a contradiction with the assumption that $\operatorname{diam}[O(y_k; n)] > 0$! Thus,

$$\operatorname{diam}\bigl[O(y_k; n)\bigr] = d(y_k, y_j) \quad \text{for } j \text{ with } k < j \leq k+n. \tag{48}$$

Also, by (46) and (48), we have

$$\begin{aligned}
\operatorname{diam}\bigl[O(y_k; n)\bigr] &= d(y_k, y_j) \\
&\leq \alpha \operatorname{diam}\bigl[O(y_{k-1}; j-k+1)\bigr] \\
&\leq \alpha \operatorname{diam}\bigl[O(y_{k-1}; n+1)\bigr].
\end{aligned} \tag{49}$$

Using the triangle inequality, by (46), (48), and (49), we obtain

$$\begin{aligned}
\operatorname{diam}\bigl[O(y_l;m)\bigr] &= d(y_l,y_j) \\
&\le d(y_l,y_{l+1}) + d(y_{l+1},y_j) \\
&\le d(y_l,y_{l+1}) + \alpha \operatorname{diam}\bigl[O(y_{l+1};m-1)\bigr] \\
&\le d(y_l,y_{l+1}) + \alpha \operatorname{diam}\bigl[O(y_l;m)\bigr],
\end{aligned} \tag{50}$$

and so

$$\operatorname{diam}\bigl[O(y_l;m)\bigr] \le \frac{1}{1-\alpha} d(y_l,y_{l+1}). \tag{51}$$

As a result, we have

$$\begin{aligned}
\operatorname{diam}\bigl[O(y_k;n)\bigr] &\le \alpha \operatorname{diam}\bigl[O(y_{k-1};n+1)\bigr] \\
&\le \alpha \cdot \alpha \operatorname{diam}\bigl[O(y_{k-2};n+2)\bigr] \\
&\quad \cdots \\
&\le \alpha^k \operatorname{diam}\bigl[O(y_0;n+k)\bigr] \\
&\le \frac{\alpha^k}{1-\alpha} d(y_0,y_1),
\end{aligned} \tag{52}$$

where the first inequality holds by the expression (49) and the last inequality holds by (51). Now let $\epsilon > 0$; there exists an integer n_0 such that

$$\alpha^k d(y_0,y_1) < (1-\alpha)\epsilon \quad \text{for all } k > n_0. \tag{53}$$

For $m > n > n_0$, we have

$$\begin{aligned}
d(y_m,y_n) &\le \operatorname{diam}\bigl[O(y_{n_0};m-n_0)\bigr] \\
&\le \frac{\alpha^{n_0}}{1-\alpha} d(y_0,y_1) \\
&< \epsilon.
\end{aligned} \tag{54}$$

Therefore, $\{y_n\}$ is a Cauchy sequence. Since $g(X)$ is closed, it converges to some $gz \in g(X)$.

By condition (vi′), there exists a subsequence $y_{n_k} = fx_{n_k} = gx_{n_k+1}$, $k \in \mathbb{N}$, having all the terms comparable with gz. Hence, we can apply the contractivity condition to obtain

$$\begin{aligned}
\psi\bigl(d(fz,fx_{n_k})\bigr) \le \max\bigl\{&\psi\bigl(\alpha d(gz,gx_{n_k})\bigr), \psi\bigl(\alpha d(gz,fz)\bigr), \psi\bigl(\alpha d(gx_{n_k},fx_{n_k})\bigr), \\
&\psi\bigl(\alpha d(gz,fx_{n_k})\bigr), \psi\bigl(\alpha d(gx_{n_k},fz)\bigr)\bigr\}.
\end{aligned} \tag{55}$$

So letting $n \to \infty$, and as ψ is continuous, we have

$$\begin{aligned}
\psi\bigl(d(fz,gz)\bigr) &\le \max\bigl\{0, \psi\bigl(\alpha d(fz,gz)\bigr), 0, 0, \psi\bigl(\alpha d(fz,gz)\bigr)\bigr\} \\
&= \psi\bigl(\alpha d(fz,gz)\bigr).
\end{aligned}$$

Since ψ is nondecreasing, $d(fz,gz) \le \alpha d(fz,gz)$. Since $\alpha \in (0,1)$, it follows that $d(fz,gz) = 0$. Hence $fz = gz$. Thus we proved that f and g have a coincidence point.

Suppose now that f and g commute at z. Set $w = fz = gz$. Since f and g are weakly compatible,

$$fw = f(gz) = g(fz) = gw. \tag{56}$$

Since from condition (vi′), we have $gz \preceq g(gz) = gw$ and as $fz = gz$ and $fw = gw$, from (2), we have

$$\begin{aligned}\psi\bigl(d(fz,fw)\bigr) &\le \max\bigl\{\psi\bigl(\alpha d(gz,gw)\bigr),\psi\bigl(\alpha d(gz,fz)\bigr),\psi\bigl(\alpha d(gw,fw)\bigr),\\ &\qquad \psi\bigl(\alpha d(gz,fw)\bigr),\psi\bigl(\alpha d(gw,fz)\bigr)\bigr\}\\ &= \psi\bigl(\alpha d(fz,fw)\bigr).\end{aligned} \tag{57}$$

Since ψ is nondecreasing, $d(fz,fw) \le \alpha d(gz,gw)$, *i.e.*, $d(fz,fw) \le \alpha d(fz,fw)$. Again from $\alpha \in (0,1)$, we have $d(fz,fw) = 0$, that is, $d(w,fw) = 0$. Therefore,

$$fw = gw = w. \tag{58}$$

Thus, we have proved that f and g have a common fixed point. The proof is completed. □

Corollary 2.1 (a) *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $f : X \to X$ be a nondecreasing self-mapping such that for some $\alpha \in (0,1)$*

$$d(fx,fy) \le \alpha \max\bigl\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\bigr\}$$

for all $x,y \in X$ for which $x \succeq y$. Suppose also that either

(i) *$\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to u$ in X, then $x_n \preceq u$, $\forall n$ holds, or*
(ii) *f is continuous.*

If there exists an $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point.

(b) *The same holds if f is nonincreasing*; *there exists x_0 comparable with fx_0 and* (i) *is replaced by*

(i′) *if a sequence $\{x_n\}$ converging to some $u \in X$ has every two adjacent terms comparable, then there exists a subsequence $\{x_{n_k}\}$ having each term comparable with x.*

Proof (a) If (i) holds, then take $\psi = I$ and $g = I$ (I = the identity mapping) in Theorem 2.1.

If (ii) holds, then from (3) with $g = I$, we get

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\Bigl(\lim_{n\to\infty} x_n\Bigr) = fz. \tag{59}$$

(b) Let u be the limit of the Picard sequence $\{f^n x_0\}$ and let $f^{n_k}x_0$ be a subsequence having all the terms comparable with u. Then we can apply the contractivity condition in the (a) term to obtain

$$\begin{aligned} d(fu,u) &\le d\big(u,f^{n_k+1}x_0\big) + d\big(fu,f^{n_k+1}x_0\big) \\ &\le d\big(u,f^{n_k+1}x_0\big) + \alpha \max\big\{d\big(u,f^{n_k}x_0\big), d(u,fu), \\ &\quad d\big(f^{n_k}x_0,f^{n_k+1}x_0\big), d\big(u,f^{n_k+1}x_0\big), d\big(f^{n_k}x_0,fu\big)\big\}. \end{aligned}$$

Letting $k \to \infty$, we have

$$\begin{aligned} d(fu,u) &\le \alpha \max\big\{0, d(u,fu), 0, 0, d(u,fu)\big\} \\ &= \alpha d(u,fu). \end{aligned}$$

It follows that $d(fu,u) = 0$. Therefore, $fu = u$.

Note also that instead of the completeness of X, its f-orbitally completeness is sufficient to obtain the conclusion of the corollary. The proof is completed. □

3 Uniqueness of common fixed point of *f* and *g*

The following theorem gives the sufficient condition for the uniqueness of the common fixed point of f and g in the case of ordered g-contractions in partially ordered metric spaces.

Theorem 3.1 *In addition to the hypotheses of Theorem* 2.1, *suppose that for all* $x, u \in X$, *there exists* $a \in X$ *such that*

$$fa \text{ is comparable to } fx \text{ and } fu. \tag{60}$$

Then f and g have a unique common fixed point x such that

$$x = fx = gx. \tag{61}$$

Proof Since a set of common fixed points of f and g is not empty due to Theorem 2.1, assume now that x and u are two common fixed points of f and g, *i.e.*,

$$fx = gx = x, \qquad fu = gu = u. \tag{62}$$

We claim that $gx = gu$.

By the assumption, there exists $a \in X$ such that fa is comparable to fx and fu. Define a sequence $\{ga_n\}$ such that $a_0 = a$ and

$$ga_n = fa_{n-1} \quad \text{for all } n. \tag{63}$$

Further, set $x_0 = x$ and $u_0 = u$ and in the same way, define $\{gx_n\}$ and $\{gu_n\}$ such that

$$gx_n = fx_{n-1}, \qquad gu_n = fu_{n-1} \quad \text{for all } n. \tag{64}$$

Since $fx(= gx_1 = gx)$ is comparable to $fa(= fa_0 = ga_1)$, without loss of generality, we assume that $fa \succeq fx$, *i.e.*, $ga_1 \succeq gx$; then it is easy to show

$$ga_1 \succeq gx. \tag{65}$$

Since f is g-nondecreasing, we obtain $fa_1 \succeq fx$. Since $ga_2 \succeq ga_1$, it follows that $ga_2 \succeq fx$, *i.e.*, $ga_2 \succeq gx$. Recursively, we get

$$ga_n \succeq gx \quad \text{for all } n. \tag{66}$$

By (66), we have

$$\begin{aligned}\psi\bigl(d(ga_{n+1},gx)\bigr) &= \psi\bigl(d(fa_n,fx)\bigr)\\ &\le \max\bigl\{\psi\bigl(\alpha d(ga_n,gx)\bigr),\psi\bigl(\alpha d(ga_n,fa_n)\bigr),\psi\bigl(\alpha d(gx,fx)\bigr),\\ &\quad \psi\bigl(\alpha d(ga_n,fx)\bigr),\psi\bigl(\alpha d(gx,fa_n)\bigr)\bigr\}.\end{aligned} \tag{67}$$

By the proof of Theorem 2.1, we find that $\{ga_n\}$ is a convergent sequence, and there exists $g\bar{a}$ such that $ga_n \to g\bar{a}$. Letting $n \to \infty$ in (67), we obtain

$$\begin{aligned}\lim_{n\to\infty}\psi\bigl(d(ga_{n+1},gx)\bigr) &= \psi\bigl(d(g\bar{a},gx)\bigr)\\ &\le \max\bigl\{\psi\bigl(\alpha d(g\bar{a},gx)\bigr),0,0,\psi\bigl(\alpha d(g\bar{a},fx)\bigr),\psi\bigl(\alpha d(gx,g\bar{a})\bigr)\bigr\}\\ &= \psi\bigl(\alpha d(g\bar{a},gx)\bigr).\end{aligned}$$

Therefore, it yields

$$d(g\bar{a},gx) = 0.$$

Hence

$$g\bar{a} = gx. \tag{68}$$

Similarly, we can also show that

$$ga_n \succeq gu \quad \text{for all } n.$$

Apply the contractivity condition, we obtain

$$\begin{aligned}\lim_{n\to\infty}\psi\bigl(d(ga_{n+1},gu)\bigr) &= \psi\bigl(d(g\bar{a},gu)\bigr)\\ &\le \max\bigl\{\psi\bigl(\alpha d(g\bar{a},gu)\bigr),0,0,\psi\bigl(\alpha d(g\bar{a},fx)\bigr),\psi\bigl(\alpha d(gx,g\bar{a})\bigr)\bigr\}\\ &= \psi\bigl(\alpha d(g\bar{a},gu)\bigr).\end{aligned}$$

Therefore, it yields

$$d(g\bar{a},gu) = 0.$$

Hence

$$g\bar{a} = gu. \tag{69}$$

Combining (68) with (69), we obtain $gx = gu$. It follows that

$$x = fx = gx = gu = fu = u. \tag{70}$$

The proof is completed. □

Remark 3.1 Theorem 3.1 can be considered as an answer to Theorem 3 in [49]. We find the sufficient conditions for the uniqueness of the common fixed point in the case of an ordered g-quasicontraction. In this paper, condition (vi) in Theorem 2.1 is weaker than that ordered g-quasicontraction in [49]. When $\psi = I$ (I = the identity mapping), our condition (vi) reduces to an ordered g-quasicontraction in [49].

Example 3.1 Let $X = \{(0,-1),(1,2)\}$, let $(a,b) \preceq (c,d)$ if and only if $a \leq c$ and $b \geq d$, and let d be the Euclidean metric. We define the functions as follows:

$$f(x,y) = \left(x^2, y^3 - 2y^2 + 2\right), \qquad g(x,y) = \left(x^3 + x^2 - x, y^2 - 2\right) \quad \text{for all } (x,y) \in X.$$

Let $\phi, \psi : [0,\infty) \to [0,\infty)$ be given by

$$\psi(t) = \frac{2}{5}t \quad \text{for all } t \in [0,\infty).$$

The only comparable pairs of points in X are (x,x) for $x \in X$ and then the contractivity condition (2) reduces to $d(fx,fx) = 0$, and condition (iv) of Theorem 2.1 is trivially fulfilled. The other conditions of Theorem 2.1 are also satisfied. It is obvious that for $(0,-1)$ and $(1,2) \in X$, $f(0,-1) = (0,-1)$ is not comparable to $f(1,2) = (1,2)$, *i.e.*, comparability in Theorem 3.1 is not satisfied. In fact, f and g have two common fixed points $(0,-1)$ and $(1,2)$, since

$$f(0,-1) = g(0,-1) = (0,-1), \qquad f(1,2) = g(1,2) = (1,2).$$

Example 3.2 Let $X = (-\infty,+\infty)$ with the usual metric $d(x,y) = |x-y|$, for all $x, y \in X$. Let $f : X \to X$ and $g : X \to X$ be given by

$$f(x) = \frac{x}{16}, \qquad g(x) = \frac{3}{4}x \quad \text{for all } x \in X.$$

Let $\phi : [0,\infty) \to [0,\infty)$ be given by

$$\psi(t) = \frac{1}{2}t \quad \text{for all } t \in [0,\infty).$$

It is easy to check that all the conditions of Theorem 2.1 are satisfied. We have

$$\begin{aligned} \psi\left(d(fx,fy)\right) &= \frac{1}{2} \cdot \frac{1}{16}|x-y| = \frac{1}{32}|x-y| \\ &\leq \frac{\alpha}{2} \cdot \frac{3}{4}|x-y| \end{aligned}$$

$$\begin{aligned}
&= \max\left\{\frac{\alpha}{2}\cdot\frac{3}{4}|x-y|, \frac{\alpha}{2}\cdot\left|\frac{3}{4}x-\frac{x}{16}\right|, \frac{\alpha}{2}\cdot\left|\frac{3}{4}y-\frac{y}{16}\right|,\right.\\
&\quad \left.\frac{\alpha}{2}\cdot\left|\frac{3}{4}x-\frac{y}{16}\right|, \frac{\alpha}{2}\cdot\left|\frac{3}{4}y-\frac{x}{16}\right|\right\}\\
&= \max\{\psi(\alpha d(gx,gy)), \psi(\alpha d(gx,fx)), \psi(\alpha d(gy,fy)),\\
&\quad \psi(\alpha d(gx,fy)), \psi(\alpha d(gy,fx))\},
\end{aligned}$$

and this holds when $\alpha \geq \frac{1}{12}$ and $gx \geq gy$, *i.e*, $\frac{3}{4}x \geq \frac{3}{4}y$, *i.e.*, $x \geq y$. This means that the contractivity condition (2) holds when $\alpha \in [\frac{1}{12}, 1)$.

In addition, $\forall x, u \in X$, there exists $a \in X$ such that $fa = \frac{a}{16}$ is comparable to $fx = \frac{x}{16}$ and $fu = \frac{u}{16}$. So, all the conditions of Theorem 3.1 are satisfied.

By applying Theorem 3.1, we conclude that f and g has a unique common fixed point. In fact, f and g has only one common fixed point. It is $x = 0$.

Example 3.3 Let $X = [0, \frac{1}{2}]$ be the closed interval with usual metric and let $f, g : X \to X$ and $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be mappings defined as follows:

$$\begin{aligned}
&f(x) = x^2 - x^4 \quad \text{for all } x \in X,\\
&g(x) = x^2 \quad \text{for all } x \in X,\\
&\psi(t) = t^2 \quad \text{for } 0 \leq t \leq \frac{1}{2},\\
&\psi(t) = \frac{1}{2}t \quad \text{for } t > \frac{1}{2}.
\end{aligned}$$

Let x, y in X be arbitrary. We say that $y \preceq x$ if $y \leq x$. For any $x, y \in X$ such that $y \preceq x$, we have

$$d(fx,fy) = \left|x^2 - x^4 - (y^2 - y^4)\right|.$$

Since $f_{\max} = f(\frac{\sqrt{2}}{2}) = \frac{1}{4}$ and $f_{\min} = f(0) = 0$, f is nondecreasing at $[0, \frac{1}{2}]$, then $x^2 - x^4 - (y^2 - y^4) \in [0, \frac{1}{2}]$. By the definition of ψ, we have

$$\psi(d(fx,fy)) = \left(\left|x^2 - x^4 - (y^2 - y^4)\right|\right)^2 = \left[x^2 - x^4 - (y^2 - y^4)\right]^2$$

and

$$\begin{aligned}
&\max\{\psi(\alpha d(gx,gy)), \psi(\alpha d(gx,fx)), \psi(\alpha d(gy,fy)), \psi(\alpha d(gx,fy)), \psi(\alpha d(gy,fx))\}\\
&\quad = \max\{[\alpha(x^2 - y^2)]^2, [\alpha(x^2 - (x^2 - x^4))]^2, [\alpha(y^2 - (y^2 - y^4))]^2,\\
&\qquad [\alpha(x^2 - (y^2 - y^4))]^2, [\alpha(y^2 - (x^2 - x^4))]^2\}\\
&\quad = [\alpha(x^2 - (y^2 - y^4))]^2.
\end{aligned}$$

Since $x^2 \geq x^2 - y^2(1 - y^2)$ for all $x \in [0, \frac{1}{2}]$, it follows that

$$-x^4 \leq -\left(x^2 - y^2(1 - y^2)\right)^2.$$

Thus we have

$$\begin{aligned}\psi\big(d(fx,fy)\big) &= \big[x^2 - x^4 - \big(y^2 - y^4\big)\big]^2 \\ &\le \big[x^2 - \big(x^2 - y^2\big(1 - y^2\big)\big)^2 - \big(y^2 - y^4\big)\big]^2 \\ &= \big[x^2 - \big(y^2 - y^4\big)\big]^2 - 2\cdot\big(x^2 - \big(y^2 - y^4\big)\big)\cdot\big(x^2 - y^2\big(1 - y^2\big)\big)^2 \\ &\quad + \big(x^2 - y^2\big(1 - y^2\big)\big)^4 \\ &= \big[x^2 - \big(y^2 - y^4\big)\big]^2 - 2\big[x^2 - \big(y^2 - y^4\big)\big]^3 + \big[x^2 - \big(y^2 - y^4\big)\big]^4 \\ &\le \big[\alpha\big(x^2 - \big(y^2 - y^4\big)\big)\big]^2,\end{aligned}$$

where the last inequality holds whenever $\alpha \in (0,1)$. Therefore, f and g satisfy (2). Also it is easy to see that the mappings $\psi(t)$ possess all properties in Definition 2.3, as well as hypotheses (v), (vi), and (vii) in Theorem 2.1. Thus we can apply Theorem 2.1 and Theorem 3.1.

On the other hand, for $y = 0$ and each $x > 0$ the contractive condition in Theorem 1 and Theorem 2 of Golubović, Kadelburg and Radenović [49]:

$$d(fx,fy) \le \lambda \cdot M(x,y), \tag{71}$$

where $0 < \lambda < 1$ and

$$M(x,y) = \max\big\{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\big\}$$

is not satisfied. Indeed,

$$\begin{aligned}M(x;0) &= \max\big\{d\big(g(x),g(0)\big), d\big(g(x),f(x)\big), d\big(g(0),f(0)\big), d\big(g(x),f(0)\big), d\big(g(0),f(x)\big)\big\} \\ &= \max\big\{x^2, x^4, 0, x^2, x^2 - x^4\big\} = x^2.\end{aligned}$$

Thus, for any fixed λ; $0 < \lambda < 1$, we have, for $y = 0$ and each $x \in X$ with $0 < x < \sqrt{1-\lambda}$,

$$\begin{aligned}d\big(f(x),f(0)\big) &= x^2 - x^4 = \big(1 - x^2\big)x^2 > \lambda\cdot x^2 \\ &= \lambda\cdot d\big(g(x),g(0)\big) = \lambda\cdot M(x,0).\end{aligned}$$

Thus, f does not satisfy the contractive condition in Definition 2.2. Therefore, the theorems of Jungck and Hussain [52], Al-Thagafi and Shahzad [53] and Das and Naik [54] also cannot be applied.

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author read and approved the final manuscript.

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