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Mixed *g*-monotone property and quadruple fixed point theorems in partially ordered metric spaces

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Abstract

In this manuscript, we prove some quadruple coincidence and common fixed point theorems for $F : X^4 \to X$ and $g : X \to X$ satisfying generalized contractions in partially ordered metric spaces. Our results unify, generalize and complement various known results from the current literature. Also, an application to matrix equations is given.

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1 Introduction and preliminaries

Existence of fixed points in partially ordered metric spaces was first investigated by Turinici [1], where he extended the Banach contraction principle in partially ordered sets. In 2004, Ran and Reurings [2] presented some applications of Turinici's theorem to matrix equations. Following these initial articles, some remarkable results were reported see, e.g., [3-13].

Gnana Bhashkar and Lakshmikantham in [14] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. Later, Lakshmikantham and Ćirić [15] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered complete metric spaces. Various results on coupled fixed point have been obtained, since then see, e.g., [6,9,16-33]. Recently, Berinde and Borcut [34] introduced the concept of tripled fixed point in ordered sets.

For simplicity, we denote $\underbrace{X \times X \cdots X \times X}_{k \text{ times}}$ by X^k where $k \in \mathbb{N}$. Let us recall some basic definitions.

Definition 1.1 *(See* [34]*) Let* $(X, \leq)$ *be a partially ordered set and* $F\colon X^3 \to X$. *The mapping F is said to has the mixed monotone property if for any* $x, y, z \in X$

$$\begin{aligned} &x_1,\ x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ &y_1,\ y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ &z_1,\ z_2 \in X, \quad z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned}$$

Definition 1.2 *Let* $F : X^3 \to X$. *An element* (x, y, z) *is called a tripled fixed point of F if*

$$F(x,y,z)=x,\quad F(y,x,y)=y\quad \text{and}\quad F(z,y,x)=z.$$

Also, Berinde and Borcut [34] proved the following theorem:

Theorem 1.1 *Let* $(X,\leq,d)$ *be a partially ordered set and suppose there is a metric d on X such that* (X, d) *is a complete metric space. Let* $F: X^3 \to X$ *having the mixed monotone property. Suppose there exist* $j, r, l \geq 0$ *with* $j + r + l < 1$ *such that*

$$d(F(x,y,z),F(u,v,w)) \leq jd(x,u)+rd(y,v)+ld(z,w), \tag{1}$$

for any $x, y, z \in X$ *for which* $x \leq u$, $v \leq y$ *and* $z \leq w$. *Suppose either F is continuous or X has the following properties:*

1. if a non-decreasing sequence $x_n \to x$, *then* $x_n \leq x$ *for all n,*

2. if a non-increasing sequence $y_n \to y$, *then* $y \leq y_n$ *for all n.*

If there exist $x_0, y_0, z_0 \in X$ *such that* $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ *and* $z_0 \leq F(z_0, y_0, x_0)$, *then there exist* $x, y, z \in X$ *such that*

$$F(x,y,z)=x,\quad F(y,x,y)=y\quad \text{and}\quad F(z,y,x)=z,$$

that is, F has a tripled fixed point.

Recently, Aydi et al. [35] introduced the following concepts.

Definition 1.3 *Let* $(X, \leq)$ *be a partially ordered set. Let* $F: X^3 \to X$ *and* $g: X \to X$. *The mapping F is said to has the mixed g-monotone property if for any* $x, y, z \in X$

$$\begin{aligned}
&x_1,\ x_2 \in X,\ gx_1 \leq gx_2 \Rightarrow F(x_1,y,z) \leq F(x_2,y,z),\\
&y_1,\ y_2 \in X,\ gy_1 \leq gy_2 \Rightarrow F(x,y_1,z) \geq F(x,y_2,z),\\
&z_1,\ z_2 \in X,\ gz_1 \leq gz_2 \Rightarrow F(x,y,z_1) \leq F(x,y,z_2).
\end{aligned}$$

Definition 1.4 *Let* $F: X^3 \to X$ *and* $g: X \to X$. *An element* (x, y, z) *is called a tripled coincidence point of F and g if*

$$F(x,y,z)=gx,\quad F(y,x,y)=gy,\quad \text{and}\quad F(z,y,x)=gz.$$

(gx, gy, gz) *is said a tripled point of coincidence of F and g.*

Definition 1.5 *Let* $F: X^3 \to X$ *and* $g: X \to X$. *An element* (x, y, z) *is called a tripled common fixed point of F and g if*

$$F(x,y,z)=gx=x,\quad F(y,x,y)=gy=y,\quad \text{and}\quad F(z,y,x)=gz=z.$$

Definition 1.6 *Let X be a non-empty set. Then we say that the mappings* $F: X^3 \to X$ *and*

$g: X \to X$ *are commutative if for all* $x, y, z \in X$

$$g(F(x,y,z))=F(gx,gy,gz).$$

The notion of fixed point of order $N \geq 3$ was first introduced by Samet and Vetro [36]. Very recently, Karapinar used the concept of quadruple fixed point and proved some fixed point theorems on the topic [37]. Following this study, quadruple fixed point is developed and some related fixed point theorems are obtained in [38-41].

Definition 1.7 [38]*Let X be a nonempty set and* $F: X^4 \to X$ *be a given mapping. An element* $(x, y, z, w) \in X \times X \times X \times X$ *is called a quadruple fixed point of F if*

$$F(x,y,z,w)=x,\ F(y,z,w,x)=y,\ F(z,w,x,y)=z,\quad \text{and}\quad F(w,x,y,z)=w.$$

Let (X, d) *be a metric space. The mapping* $\bar{d} : X^4 \to X$*, given by*

$$\bar{d}((x, y, z, w), (u, v, h, l)) = d(x, y) + d(y, v) + d(z, h) + d(w, l),$$

defines a metric on X^4*, which will be denoted for convenience by* d.

Definition 1.8 [38]*Let* $(X, \leq)$ *be a partially ordered set and* $F : X^4 \to X$ *be a mapping. We say that F has the mixed monotone property if* $F(x, y, z, w)$ *is monotone non-decreasing in x and z and is monotone non-increasing in y and w; that is, for any* $x, y, z, w \in X$,

$$\begin{aligned}
&x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y, z, w) \leq F(x_2, y, z, w),\\
&y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x, y_2, z, w) \leq F(x, y_1, z, w),\\
&z_1, z_2 \in X, \quad z_1 \leq z_2 \quad \text{implies} \quad F(x, y, z_1, w) \leq F(x, y, z_2, w),
\end{aligned}$$

and

$$w_1, w_2 \in X, \quad w_1 \leq w_2 \quad \text{implies} \quad F(x, y, z, w_2) \leq F(x, y, z, w_1).$$

In this article, we establish some quadruple coincidence and common fixed point theorems for $F : X^4 \to X$ and $g : X \to X$ satisfying nonlinear contractions in partially ordered metric spaces. Also, some interesting corollaries are derived and an application to matrix equations is given.

2 Main results

We start this section with the following definitions.

Definition 2.1 *Let* $(X, \leq)$ *be a partially ordered set. Let* $F : X^4 \to X$ *and* $g : X \to X$. *The mapping F is said to has the mixed g-monotone property if for any* $x, y, z, w \in X$

$$\begin{aligned}
&x_1,\ x_2 \in X, \quad gx_1 \leq gx_2 \Rightarrow F(x_1, y, z, w) \leq F(x_2, y, z, w),\\
&y_1,\ y_2 \in X, \quad gy_1 \leq gy_2 \Rightarrow F(x, y_1, z, w) \geq F(x, y_2, z, w),\\
&z_1,\ z_2 \in X, \quad gz_1 \leq gz_2 \Rightarrow F(x, y, z_1, w) \leq F(x, y, z_2, w) \text{ and}\\
&w_1,\ w_2 \in X, \quad gw_1 \leq gw_2 \Rightarrow F(x, y, z, w_1) \geq F(x, y, z, w_2).
\end{aligned}$$

Definition 2.2 *Let* $F : X^4 \to X$ *and* $g : X \to X$. *An element* (x, y, z, w) *is called a quadruple coincidence point of F and g if*

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz, \quad \text{and} \quad F(w, x, y, z) = gw.$$

(gx, gy, gz, gw) *is said a quadruple point of coincidence of F and g.*

Definition 2.3 *Let* $F : X^4 \to X$ *and* $g : X \to X$. *An element* (x, y, z, w) *is called a quadruple common fixed point of F and g if*

$$\begin{aligned}
&F(x, y, z, w) = gx = x, \qquad F(y, z, w, x) = gy = y,\\
&F(z, w, x, y) = gz = z, \text{ and } F(w, x, y, z) = gw = w.
\end{aligned}$$

Definition 2.4 *Let X be a non-empty set. Then we say that the mappings* $F : X^4 \to X$ *and* $g : X \to \times$ *are commutative if for all* $x, y, z, w \in X$

$$g(F(x, y, z, w)) = F(gx, gy, gz, gw).$$

Let Φ be the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that:

1. $\varphi(t) < t$ for all $t \in (0,+\infty)$.

2. $\lim_{r \to t^+} \phi(r) < t$ for all $t \in (0,+\infty)$.

For simplicity, we define the following.

$$M(x, y, z, w, u, v, h, l) = \min \left\{ \begin{array}{c} d(F(x, y, z, w), gx), d(F(x, y, z, w), gu), \\ d(F(u, v, h, l), gu) \end{array} \right\}. \tag{2}$$

Now, we state the first main result of this article.

Theorem 2.1 *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric d on X such that* (X, d) *is a complete metric space. Suppose* $F : X^4 \to X$ *and* $g : X \to X$ *are such that F is continuous and has the mixed g-monotone property. Assume also that there exist* $\varphi \in \Phi$ *and* $L \geq 0$ *such that*

$$\begin{aligned} d(F(x, y, z, w), F(u, v, , h, l)) \leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gh), d(gw, gl)\}) \\ +LM(x, y, z, w, u, v, h, l) \end{aligned} \tag{3}$$

for any $x, y, z, w, u, v, h, l \in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gh$ *and* $gl \leq gw$. *Suppose* $F(X^4) \subset g(X)$, *g is continuous and commutes with F. If there exist* $x_0, y_0, z_0, w_0 \in X$ *such that*

$$\begin{aligned} &gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\ &gz_0 \leq F(z_0, w_0, x_0, y_0), \quad \textit{and} \quad gw_0 \geq F(w_0, x_0, y_0, z_0), \end{aligned}$$

then there exist $x, y, z, w \in X$ *such that*

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz \;\; \textit{and} \;\; F(w, x, y, z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} &gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\ &gz_0 \leq F(z_0, w_0, x_0, y_0) \text{ and } gw_0 \geq F(w_0, x_0, y_0, z_0). \end{aligned}$$

Since $F(X^4) \subset g(X)$, then we can choose $x_1, y_1, z_1, w_1 \in X$ such that

$$\begin{aligned} &gx_1 = F(x_0, y_0, z_0, w_0), \qquad gy_1 = F(y_0, z_0, w_0, x_0), \\ &gz_1 = F(z_0, w_0, x_0, y_0) \quad \text{and} \quad gw_1 = F(w_0, x_0, y_0, z_0). \end{aligned} \tag{4}$$

Taking into account $F(X^4) \subset g(X)$, by continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ in X such that

$$\begin{aligned} &gx_{n+1} = F(x_n, y_n, z_n, w_n), \qquad gy_{n+1} = F(y_n, z_n, w_n, x_n), \\ &gz_{n+1} = F(z_n, w_n, x_n, y_n), \quad \text{and} \quad gw_{n+1} = F(w_n, x_n, y_n, z_n). \end{aligned} \tag{5}$$

We shall show that

$$gx_n \leq gx_{n+1}, \quad gy_{n+1} \leq gy_n, \quad gz_n \leq gz_{n+1}, \;\; \text{and} \;\; gw_{n+1} \leq gw_n \;\; \text{for} \;\; n = 0, 1, 2, \ldots \tag{6}$$

For this purpose, we use the mathematical induction. Since, $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then by (4), we get

$$gx_0 \leq gx_1, \quad gy_1 \leq gy_0, \quad gz_0 \leq gz_1, \;\; \text{and} \;\; gw_1 \leq gw_0$$

that is, (6) holds for $n = 0$.

We presume that (6) holds for some $n > 0$. As F has the mixed g-monotone property and $gx_n \le gx_{n+1}$, $gy_{n+1} \le gy_n$, $gz_n \le gz_{n+1}$ and $gw_{n+1} \le gw_n$, we obtain

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n, w_n) \le F(x_{n+1}, y_n, z_n, w_n) \\ &\le F(x_{n+1}, y_n, z_{n+1}, w_n) \le F(x_{n+1}, y_{n+1}, z_{n+1}, w_n) \\ &\le F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) = gx_{n+2}, \end{aligned}$$

$$\begin{aligned} gy_{n+2} &= F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}) \le F(y_{n+1}, z_n, x_{n+1}, w_{n+1}) \\ &\le F(y_n, z_n, x_{n+1}, w_{n+1}) \le F(y_n, z_n, x_n, w_{n+1}) \\ &\le F(y_n, z_n, x_n, w_n) = gy_{n+1}, \end{aligned}$$

$$\begin{aligned} gz_{n+1} &= F(z_n, y_n, x_n, w_n) \le F(z_{n+1}, y_n, x_n, w_n) \\ &\le F(z_{n+1}, y_{n+1}, x_n, w_n) \le F(z_{n+1}, y_{n+1}, x_{n+1}, w_n) \\ &\le F(z_{n+1}, y_{n+1}, x_{n+1}, w_{n+1}) = gz_{n+2}, \end{aligned}$$

and

$$\begin{aligned} gw_{n+2} &= F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) \le F(w_{n+1}, x_n, y_{n+1}, z_{n+1}) \\ &\le F(w_n, x_n, y_{n+1}, z_{n+1}) \le F(w_n, x_n, y_n, z_{n+1}) \\ &\le F(w_n, x_n, y_n, z_n) = gw_{n+1}. \end{aligned}$$

Thus, (6) holds for any $n \in \mathbb{N}$. Assume for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \quad gy_n = gy_{n+1}, \quad gz_n = gz_{n+1}, \quad \text{and} \quad gw_n = gw_{n+1}$$

then, by (5), (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. From now on, assume for any $n \in \mathbb{N}$ that at least

$$gx_n \ne gx_{n+1} \quad \text{or} \quad gy_n \ne gy_{n+1} \quad \text{or} \quad gz_n \ne gz_{n+1} \quad \text{or} \quad gw_n \ne gw_{n+1}. \tag{7}$$

By (2) and (5), it is easy that

$$\begin{aligned} M(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}, x_n, y_n, z_n, w_n) &= M(y_n, z_n, w_n, x_n, y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \\ &= M(z_{n-1}, y_{n-1}, x_{n-1}, z_n, y_n, x_n) \\ &= M(w_n, x_n, y_n, z_n, w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}) = 0 \quad \text{for all } n \ge 1. \end{aligned} \tag{8}$$

Due to (3) and (8), we have

$$\begin{aligned} & d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), F(x_n, y_n, z_n, w_n)) \\ & \le \phi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}) \\ & + LM(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}, x_n, y_n, z_n, w_n) \\ & = \phi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}), \end{aligned} \tag{9}$$

$$\begin{aligned} & d(gy_n, gy_{n+1}) = d(F(y_n, z_n, w_n, x_n), y_{n-1}, F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1})) \\ & \le \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}), \\ & + LM(y_n, z_n, w_n, w_n, y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \\ & = \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}), \end{aligned} \tag{10}$$

$$\begin{aligned}
d(gz_n, gz_{n+1}) &= d(F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}), F(z_n, w_n, x_n, y_n)) \\
&\le \phi(\max\{, d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\
&+LM(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}, z_n, w_n, x_n, y_n) \\
&= \phi(\max\{d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\})
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
d(gw_n, gw_{n+1}) &= d(F(w_n, x_n, y_n, z_n), F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1})) \\
&\le \phi(\max\{d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}), \\
&+LM(w_n, x_n, y_n, z_n, w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}) \\
&= \phi(\max\{d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}).
\end{aligned} \tag{12}$$

Having in mind that $\varphi(t) < t$ for all $t > 0$, so from (9)-(12) we obtain that

$$\begin{aligned}
0 &< \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\} \\
&\le \phi(\max\{d(gz_{n-1}, gz_n), d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n)\}) \\
&< \max\{d(gz_{n-1}, gz_n), d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n)\}.
\end{aligned} \tag{13}$$

It follows that

$$\max\left\{\begin{array}{c} d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}, \\ d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1}) \end{array}\right\} < \max\left\{\begin{array}{c} d(gz_{n-1}, gz_n), d(gy_{n-1}, gy_n), \\ d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n) \end{array}\right\}. \tag{14}$$

Thus, $\{\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\}\}$ is a positive decreasing sequence. Hence, there exists $r \ge 0$ such that

$$\lim_{n\to+\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\} = r.$$

Suppose that $r > 0$. Letting $n \to +\infty$ in (13), we obtain that

$$0 < r \le \lim_{n\to+\infty} \phi\left(\max\left\{\begin{array}{c} d(gz_{n-1}, gz_n), d(gy_{n-1}, gy_n), \\ d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n) \end{array}\right\}\right) = \lim_{t\to r^+} \phi(t) < r. \tag{15}$$

It is a contradiction. We deduce that

$$\lim_{n\to+\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\} = 0. \tag{16}$$

We shall show that $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in the metric space (X, d). Assume the contrary, that is, one of the sequence $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ or $\{gw_n\}$ is not a Cauchy, that is,

$$\lim_{n,m\to+\infty} d(gx_m, gx_n) \ne 0 \quad \text{or} \quad \lim_{n,m\to+\infty} d(gy_m, gy_n) \ne 0$$

or

$$\lim_{n,m\to+\infty} d(gz_m, gz_n) \ne 0 \quad \text{or} \quad \lim_{n,m\to+\infty} d(gw_m, gw_n) \ne 0.$$

This means that there exists $\varepsilon > 0$, for which we can find subsequences of integers (m_k) and (n_k) with $n_k > m_k > k$ such that

$$\max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k}), d(gw_{m_k}, gw_{n_k})\} \ge \varepsilon. \tag{17}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (17). Then

$$\max\{d(gx_{m_k}, gx_{n_k-1}), d(gy_{m_k}, gy_{n_k-1}), d(gz_{m_k}, gz_{n_k-1}), d(gw_{m_k}, gw_{n_k-1})\} < \varepsilon. \tag{18}$$

By triangular inequality and (18), we have

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &\le d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &< \varepsilon + d(gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Thus, by (16) we obtain

$$\lim_{k\to+\infty} d(gx_{m_k}, gx_{n_k}) \le \lim_{k\to+\infty} d(gx_{m_k}, gx_{n_k-1}) \le \varepsilon. \tag{19}$$

Similarly, we have

$$\lim_{k\to+\infty} d(gy_{m_k}, gy_{n_k}) \le \lim_{k\to+\infty} d(gy_{m_k}, gy_{n_k-1}) \le \varepsilon, \tag{20}$$

$$\lim_{k\to+\infty} d(gz_{m_k}, gz_{n_k}) \le \lim_{k\to+\infty} d(gz_{m_k}, gz_{n_k-1}) \le \varepsilon, \tag{21}$$

and

$$\lim_{k\to+\infty} d(gw_{m_k}, gw_{n_k}) \le \lim_{k\to+\infty} d(gw_{m_k}, gw_{n_k-1}) \le \varepsilon. \tag{22}$$

Again by (18), we have

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &\le d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &\le d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) \\ &\quad + d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &< d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) + \varepsilon + d(gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Letting $k \to +\infty$ and using (16), we get

$$\lim_{k\to+\infty} d(gx_{m_k}, gx_{n_k}) \le \lim_{k\to+\infty} d(gx_{m_k-1}, gx_{n_k-1}) \le \varepsilon, \tag{23}$$

$$\lim_{k\to+\infty} d(gy_{m_k}, gy_{n_k}) \le \lim_{k\to+\infty} d(gy_{m_k-1}, gy_{n_k-1}) \le \varepsilon, \tag{24}$$

$$\lim_{k\to+\infty} d(gz_{m_k}, gz_{n_k}) \le \lim_{k\to+\infty} d(gz_{m_k-1}, gz_{n_k-1}) \le \varepsilon \tag{25}$$

and

$$\lim_{k\to+\infty} d(gw_{m_k}, gw_{n_k}) \le \lim_{k\to+\infty} d(gw_{m_k-1}, gw_{n_k-1}) \le \varepsilon. \tag{26}$$

Using (17) and (23)-(26), we have

$$\begin{aligned} &\lim_{k\to+\infty} \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k}), d(gw_{m_k}, gw_{n_k})\} \\ &= \lim_{k\to+\infty} \max\{d(gx_{m_k-1}, gx_{n_k-1}), d(gy_{m_k-1}, gy_{n_k-1}), d(gz_{m_k-1}, gz_{n_k-1}), d(gw_{m_k-1}, gw_{n_k-1})\} \\ &= \varepsilon. \end{aligned} \tag{27}$$

By (16), it is easy to see that

$$\begin{aligned}&\lim_{k\to+\infty} M(x_{m_k-1},y_{m_k-1},z_{m_k-1},w_{m_k-1},x_{n_k-1},y_{n_k-1},z_{n_k-1},w_{n_k-1})\\&=\lim_{k\to+\infty} M(y_{n_k-1},z_{n_k-1},w_{n_k-1},x_{n_k-1},y_{m_k-1},z_{m_k-1},w_{m_k-1},x_{m_k-1})\\&=\lim_{k\to+\infty} M(z_{m_k-1},w_{m_k-1},x_{m_k-1},y_{m_k-1},z_{n_k-1},w_{n_k-1},x_{n_k-1},y_{m_k-1})\\&=\lim_{k\to+\infty} M(w_{n_k-1},x_{n_k-1},y_{m_k-1},z_{n_k-1},w_{m_k-1},x_{m_k-1},y_{m_k-1},z_{m_k-1})=0.\end{aligned}\tag{28}$$

Now, using inequality (3), we obtain

$$\begin{aligned}d(gx_{m_k},gx_{n_k})&=d(F(x_{m_k-1},y_{m_k-1},z_{m_k-1},w_{m_k-1}),F(x_{n_k-1},y_{n_k-1},z_{n_k-1},w_{n_k-1}))\\&\le\phi(\max\{d(x_{m_k-1},x_{n_k-1}),d(y_{m_k-1},y_{n_k-1}),d(z_{m_k-1},z_{n_k-1}),d(w_{m_k-1},w_{n_k-1})\})\\&+LM(x_{m_k-1},y_{m_k-1},z_{m_k-1},w_{m_k-1},x_{n_k-1},y_{n_k-1},z_{n_k-1},w_{n_k-1}),\end{aligned}\tag{29}$$

$$\begin{aligned}d(gy_{n_k},gy_{m_k})&=d(F(y_{n_k-1},z_{n_k-1},w_{n_k-1},x_{n_k-1}),F(y_{m_k-1},z_{m_k-1},w_{m_k-1},x_{m_k-1}))\\&\le\phi(\max\{d(y_{m_k-1},y_{n_k-1}),d(z_{m_k-1},z_{n_k-1}),d(w_{m_k-1},w_{n_k-1},d(x_{m_k-1},x_{n_k-1})\})\\&+LM(y_{n_k-1},z_{n_k-1},w_{n_k-1},x_{n_k-1},y_{m_k-1},z_{m_k-1},w_{m_k-1},x_{m_k-1}),\end{aligned}\tag{30}$$

$$\begin{aligned}d(gz_{m_k},gz_{n_k})&=d(F(z_{m_k-1},w_{m_k-1},x_{m_k-1},y_{m_k-1}),F(z_{n_k-1},w_{n_k-1},x_{n_k-1},y_{n_k-1}))\\&\le\phi(\max\{d(z_{m_k-1},z_{n_k-1}),d(w_{m_k-1},w_{n_k-1},d(x_{m_k-1},x_{n_k-1}),d(y_{m_k-1},y_{n_k-1})\})\\&+LM(z_{m_k-1},w_{m_k-1},x_{m_k-1},y_{m_k-1},z_{n_k-1},w_{n_k-1},x_{n_k-1},y_{m_k-1})\end{aligned}\tag{31}$$

and

$$\begin{aligned}d(gw_{n_k},gw_{m_k})&=d(F(w_{n_k-1},x_{n_k-1},y_{n_k-1},z_{n_k-1}),F(w_{m_k-1},x_{m_k-1},y_{m_k-1},z_{m_k-1}))\\&\le\phi(\max\{d(w_{m_k-1},w_{n_k-1},d(x_{m_k-1},x_{n_k-1}),d(y_{m_k-1},y_{n_k-1}),d(z_{m_k-1},z_{n_k-1})\})\\&+LM(w_{n_k-1},x_{n_k-1},y_{n_k-1},z_{n_k-1},w_{m_k-1},x_{m_k-1},y_{m_k-1},z_{m_k-1}).\end{aligned}\tag{32}$$

From (29)-(32), we deduce that

$$\begin{aligned}&\max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k}),d(gz_{m_k},gz_{n_k}),d(gw_{m_k},gw_{n_k})\}\\&\le\phi(\max\{d(x_{m_k-1},x_{n_k-1}),d(y_{m_k-1},y_{n_k-1}),d(z_{m_k-1},z_{n_k-1}),d(gw_{m_k},gw_{n_k})\})\\&+LM(x_{m_k-1},y_{m_k-1},z_{m_k-1},w_{m_k-1},x_{n_k-1},y_{n_k-1},z_{n_k-1},w_{n_k-1})\\&+LM(y_{n_k-1},z_{n_k-1},w_{n_k-1},x_{n_k-1},y_{m_k-1},z_{m_k-1},w_{m_k-1},x_{m_k-1})\\&+LM(z_{m_k-1},w_{m_k-1},x_{m_k-1},y_{m_k-1},z_{n_k-1},w_{n_k-1},x_{n_k-1},y_{m_k-1})\\&+LM(w_{n_k-1},x_{n_k-1},y_{n_k-1},z_{n_k-1},w_{m_k-1},x_{m_k-1},y_{m_k-1},z_{m_k-1}).\end{aligned}\tag{33}$$

Letting $k\to+\infty$ in (33) and having in mind (27) and (28), we get that

$$0<\varepsilon\le\lim_{t\to\varepsilon^+}\phi(t)<\varepsilon,$$

it is a contradiction. Thus, $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in (X, d).

Since (X, d) is complete, there exist $x, y, z, w \in X$ such that

$$\lim_{n\to+\infty} gx_n=x,\quad \lim_{n\to+\infty} gy_n=y,\ \lim_{n\to+\infty} gy_n=y,\quad\text{and}\quad \lim_{n\to+\infty} gw_n=w.\tag{34}$$

From (34) and the continuity of g, we have

$$\lim_{n\to+\infty} g(gx_n)=gx,\quad \lim_{n\to+\infty} g(gy_n)=gy,\quad \lim_{n\to+\infty} g(gz_n)=gz,\quad\text{and}\quad \lim_{n\to+\infty} g(gw_n)=gw.\tag{35}$$

From (5) and the commutativity of F and g, we have

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n, w_n)) = F(gx_n, gy_n, gz_n, gw_n), \tag{36}$$

$$g(gy_{n+1}) = g(F(y_n, z_n, w_n, x_n)) = F(gy_n, gz_n, gw_n, gx_n), \tag{37}$$

$$g(gz_{n+1}) = g(F(z_n, w_n, x_n, y_n)) = F(gz_n, gw_n, gx_n, y_n), \tag{38}$$

and

$$g(gw_{n+1}) = g(F(w_n, x_n, y_n, z_n)) = F(gw_n, gx_n, y_n, gz_n). \tag{39}$$

Now we shall show that $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, and $gw = F(w, x, y, z)$.

By letting $n \to +\infty$ in (36) - (39), by (34), (35) and the continuity of F, we obtain

$$\begin{aligned} gx &= \lim_{n\to+\infty} g(gx_{n+1}) = \lim_{n\to+\infty} F(gx_n, gy_n, gz_n, gw_n) \\ &= F(\lim_{n\to+\infty} gx_n, \lim_{n\to+\infty} gy_n, \lim_{n\to+\infty} gz_n, \lim_{n\to+\infty} gw_n) \\ &= F(x, y, z, w), \end{aligned} \tag{40}$$

$$\begin{aligned} gy &= \lim_{n\to+\infty} g(gy_{n+1}) = \lim_{n\to+\infty} F(gy_n, gz_n, gw_n, gx_n) \\ &= F(\lim_{n\to+\infty} gy_n, \lim_{n\to+\infty} gz_n, \lim_{n\to+\infty} gw_n, \lim_{n\to+\infty} gw_n) \\ &= F(y, z, w, x), \end{aligned} \tag{41}$$

$$\begin{aligned} gz &= \lim_{n\to+\infty} g(gz_{n+1}) = \lim_{n\to+\infty} F(gz_n, gw_n, gx_n, gy_n) \\ &= F(\lim_{n\to+\infty} gz_n, \lim_{n\to+\infty} gw_n, \lim_{n\to+\infty} gx_n, \lim_{n\to+\infty} gy_n) \\ &= F(z, w, x, y), \end{aligned} \tag{42}$$

and

$$\begin{aligned} gw &= \lim_{n\to+\infty} g(gw_{n+1}) = \lim_{n\to+\infty} F(gw_n, gx_n, gy_n, gz_n) \\ &= F(\lim_{n\to+\infty} gw_n, \lim_{n\to+\infty} gx_n, \lim_{n\to+\infty} gy_n, \lim_{n\to+\infty} gz_n) \\ &= F(w, x, y, z). \end{aligned} \tag{43}$$

We have proved that F and g have a quadruple coincidence point. This completes the proof of Theorem 2.1.

In the following theorem, we omit the continuity hypothesis of F. We need the following definition.

Definition 2.5 *Let* $(X, \leq)$ *be a partially ordered metric set and* d *be a metric on* X. *We say that* $(X, d, \leq)$ *is regular if the following conditions hold*:

(i) if non-decreasing sequence $a_n \to a$, *then* $a_n \leq a$ *for all* n,

(ii) if non-increasing sequence $b_n \to b$, *then* $b \leq b_n$ *for all* n.

Theorem 2.2 *Let* $(X, \leq)$ *be a partially ordered set and* d *be a metric on* X *such that* $(X, d, \leq)$ *is regular. Suppose* $F : X^4 \to X$ *and* $g : X \to X$ *are such that* F *has the mixed* g*-monotone property. Assume that there exist* $\varphi \in \Phi$ *and* $L \geq 0$ *such that*

$$d(F(x,y,z,w),F(u,v,,h,l)) \leq \phi(\max\{d(gx,gu),d(gy,gv),d(gz,gh),d(gw,gl)\}) \\ +LM(x,y,z,w,u,v,h,l)$$

for any $x, y, z, w, u, v, h, l \in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gh$, *and* $gl \leq gw$. *Also, suppose* $F(X^4) \subset g(X)$ *and* $(g(X), d)$ *is a complete metric space. If there exist* $x_0, y_0, z_0, w_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$ *and* $gw_0 \geq F(w_0, x_0, y_0, z_0)$, *then there exist* $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz \text{ and } F(w,x,y,z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. Proceeding exactly as in Theorem 2.1, we have that $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in the complete metric space $(g(X), d)$. Then, there exist $x, y, z, w \in X$ such that

$$gx_n \to gx, \quad gy_n \to gy, \quad gz_n \to gz, \quad \text{and} \quad gw_n \to gw. \tag{44}$$

Since $\{gx_n\}$, $\{gz_n\}$ are non-decreasing and $\{gy_n\}$, $\{gw_n\}$ are non-increasing, then since $(X, d, \leq)$ is regular we have

$$gx_n \leq gx, \; gy_n \geq gy, \; gz_n \leq gz, \; gw_n \geq gw$$

for all n. If $gx_n = gx$, $gy_n = gy$, $gz_n = gz$, and $gw_n = gw$ for some $n \geq 0$, then $gx = gx_n \leq gx_{n+1} \leq gx = gx_n$, $gy \leq gy_{n+1} \leq gy_n = gy$, $gz = gz_n \leq gz_{n+1} \leq gz = gz_n$, and $gw \leq gw_{n+1} \leq gw_n = gw$, which implies that

$$gx_n = gx_{n+1} = F(x_n, y_n, z_n, w_n), \quad gy_n = gy_{n+1} = F(y_n, z_n, w_n, x_n),$$

and

$$gz_n = gz_{n+1} = F(z_n, w_n, x_n, y_n), \quad gw_n = gw_{n+1} = F(w_n, w_n, y_n, z_n),$$

that is, (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n, gw_n) \neq (gx, gy, gz, gw)$ for all $n \geq 0$. By (3), consider now

$$\begin{aligned} d(gx, F(x,y,z,w)) &\leq d(gx, gx_{n+1}) + d(gx_{n+1}, F(x,y,z,w)) \\ &= d(gx, gx_{n+1}) + d(F(x_n,y_n,z_n,w_n), F(x,y,z,w)) \\ &\leq d(gx, gx_{n+1}) + \phi\left(\max\left\{\begin{array}{l} d(gx_n, gx), d(gy_n, gy), \\ d(gz_n, gz), d(gw_n, gw) \end{array}\right\}\right) + LM(x_n,y_n,z_n,w_n,x,y,z,w) \\ &< d(gx, gx_{n+1}) + \max\{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz), d(gw_n, gw)\} + LM(x_n,y_n,z_n,w_n,x,y,z,w). \end{aligned} \tag{45}$$

Taking $n \to \infty$ and using (44), the quantity $M(x_n, y_n, z_n, w_n, x, y, z, w)$ tends to 0 and so the right-hand side of (45) tends to 0, hence we get that $d(gx, F(x, y, z, w)) = 0$. Thus, $gx = F(x, y, z, w)$. Analogously, one finds

$$F(x,y,z,w) = gy, \qquad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw.$$

Thus, we proved that F and g have a quartet coincidence point. This completes the proof of Theorem 2.2.

Corollary 2.1 *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric d on X such that* (X, d) *is a complete metric space. Suppose* $F : X^4 \to X$ *and* $g : X \to X$ *are such that F is continuous and has the mixed g-monotone property. Assume also that there exist* $\varphi \in \Phi$ *a non-decreasing function and* $L \geq 0$ *such that*

$$d(F(x,y,z,w),F(u,v,h,l)) \leq \phi\left(\frac{d(gx,gu)+d(gy,gv)+d(gz,gh)+d(gw,gl}{4}\right) + LM(x,y,z,w,u,v,h,l),$$

for any $x, y, z, w, u, v, h, l, \in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gw$, *and* $gl \leq gw$. *Suppose* $F(X^4) \subset g(X)$, g *is continuous and commutes with* F .

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, *and* $gw_0 \geq F(w_0, x_0, y_0, z_0)$, *then there exist* $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz, \text{ and } F(w,x,y,z) = gw.$$

Proof. It suffices to remark that

$$\frac{d(gx,gu)+d(gy,gv)+d(gz,ph),d(gw,gl)}{4} \leq \max\left\{\begin{matrix} d(gx,gu), d(gu,gv), \\ d(gz,gh), d(gw,gl) \end{matrix}\right\}.$$

Then, we apply Theorem 2.1, since φ is assumed to be non-decreasing.

Similarly, as an easy consequence of Theorem 2.2 we have the following corollary.

Corollary 2.2 *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* $(X, d, \leq)$ *is regular. Suppose* $F: X^4 \to X$ *and* $g: X \to X$ *are such that* F *has the mixed g-monotone property. Assume also that there exist* $\varphi \in \Phi$ *a non-decreasing function and* $L \geq 0$ *such that*

$$d(F(x,y,z,w),F(u,v,h,l)) \leq \phi\left(\frac{d(gx,gu)+d(gy,gv)+d(gz,gh)+d(gw,gl)}{4}\right) + LM(x,y,z,w,u,v,h,l),$$

for any $x, y, z, w, u, v, h, l \in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gw$, *and* $gl \leq gw$. *Also, suppose* $F(X^4) \subset g(X)$ *and* $(g(X), d)$ *is a complete metric space.*

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, *and* $gw_0 \geq F(w_0, x_0, y_0, z_0)$, *then there exist* $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw.$$

Corollary 2.3 *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose* $F: X^4 \to X$ *and* $g: X \to X$ *are such that* F *is continuous and has the mixed g-monotone property. Assume that there exist* $k \in [0, 1)$ *and* $L \geq 0$ *such that*

$$d(F(x,y,z,w),F(u,v,,h,l)) \leq k \max\left\{\begin{matrix} d(gx,gu), d(gy,gv), \\ d(gz,gh), d(gw,gl) \end{matrix}\right\} + LM(x,y,z,w,u,v,h,l),$$

for any $x, y, z, w, u, v, h, l \in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gw$, *and* $gl \leq gw$. *Suppose* $F(X^4) \subset g(X)$, g *is continuous and commutes with* F.

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, *and* $gw_0 \geq F(w_0, x_0, y_0, z_0)$, *then there exist* $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad f(y,z,w,x) = gy, \quad f(z,w,x,y) = gz, \text{ and } F(w,x,y,z) = gw.$$

Proof. It suffices to take $\varphi(t) = kt$ in Theorem 2.1.

Corollary 2.4 *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric d on X such that* $(X, d, \leq)$ *is regular. Suppose* $F : X^4 \to X$ *and* $g : X \to X$ *are such that F has the mixed g-monotone property. Assume that there exist* $k \in [0, 1)$ *and* $L \geq 0$ *such that*

$$d(F(x,y,z,w),F(u,v,,h,l)) \leq k \max\left\{\begin{matrix} d(gx,gu), d(gy,gv), \\ d(gz,gh), d(gw,gl) \end{matrix}\right\} + LM(x,y,z,w,u,v,h,l),$$

for any x, y, z, w, u, v, h, $l \in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gw$, *and* $gl \leq gw$. *Suppose* $F(X^4) \subset g(X)$ *and* $(g(X), d)$ *is a complete metric space.*

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, *and* $gw_0 \geq F(w_0, x_0, y_0, z_0)$, *then there exist x, y, z,* $w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,x) = gy, \quad F(z,w,x,y) = gz, \quad \text{and} \quad F(w,x,y,z) = gw.$$

Proof. It suffices to take $\varphi(t) = kt$ in Theorem 2.2.

Corollary 2.5 *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric d on X such that* (X, d) *is a complete metric space. Suppose* $F : X^4 \to X$ *and* $g : X \to X$ *are such that F is continuous and has the mixed g-monotone property. Assume that there exist* $k \in [0, 1)$ *and* $L \geq 0$ *such that*

$$d(F(x,y,z,w),F(u,v,h,l)) \leq \frac{k}{4}\left\{\begin{matrix} d(gx,gu) + d(gy,gv) + \\ d(gz,gh) + d(gw,gl) \end{matrix}\right\} + LM(x,y,z,w,u,v,h,l),$$

for any x, y, z, w, $\in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gw$, *and* $gl \leq gw$. *Also, suppose* $F(X^4) \subset g(X)$ *and* $(g(X)$, *g is continuous and commutes with F.*

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, *and* $gw_0 \geq F(w_0, x_0, y_0, z_0)$, *then there exist x, y, z,* $w \in X$ *such that*

$$F(x, y, z, w) = gx, \; F(y, z, w, x) = gy, \; F(z, w, x, y) = gz, \text{ and } F(w, x, y, z) = gw.$$

Proof. It suffices to take $\varphi(t) = kt$ in Corollary 2.1.

Corollary 2.6 *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric d on X such that* $(X, d, \leq)$ *is regular. Suppose* $F : X^4 \to X$ *and* $g : X \to X$ *are such that F has the mixed g-monotone property. Assume that there exist* $k \in [0, 1)$ *and* $L \geq 0$ *such that*

$$d(F(x,y,z,w),F(u,v,h,l)) \leq \frac{k}{4}\left\{\begin{matrix} d(gx,gu) + d(gy,gv) + \\ d(gz,gh) + d(gw,gl) \end{matrix}\right\} + LM(x,y,z,w,u,v,h,l),$$

for any x, y, z, w, $\in X$ *for which* $gx \leq gu$, $gv \leq gy$, $gz \leq gw$, *and* $gl \leq gw$. *Suppose* $F(X^4) \subset g(X)$ *and* $(g(X), d)$ *is a complete metric space.*

If there exist $x_0, y_0, z_0, w_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, *and* $gw_0 \geq F(w_0, x_0, y_0, z_0)$, *then there exist x, y, z,* $w \in X$ *such that*

$$F(x,y,z,w) = gx, \quad F(y,z,w,r) = gy, \quad F(z,w,x,y) = gz, \quad \text{and } F(w,x,y,z = gw).$$

Proof. It suffices to take $\varphi(t) = kt$ in Corollary 2.2.

Remark 1 • *Corollary 2.4 of Karapinar* [39]*is a particular case of Corollary 2.5 by taking* $L = 0$ *and* $g = I_X$ *the identity on X.*

- *Corollary 2.4 of Karapinar* [39]*is a particular case of Corollary 2.6 by taking* $L = 0$ *and* $g = I_X$.
- *Theorem 2.6 of Berinde and Karapinar* [40]*is a particular case of Corollary 2.1 by taking* $L = 0$.
- *Theorem 2.6 of Berinde and Karapinar* [40]*is a particular case of Corollary 2.1 by taking* $L = 0$.

Now, we shall prove the existence and uniqueness of quadruple common fixed point. For a product X^4 of a partial ordered set $(X, \leq)$, we define a partial ordering in the following way: For all $(x, y, z, w), (u, v, r, h) \in X^4$

$$(x,y,z,w) \leq (u,v,r,h) \Leftrightarrow x \leq u, \quad y \geq v, \quad z \leq r \text{ and } w \geq l \tag{46}$$

We say that (x, y, z, w) and (u, v, r, l) are comparable if

$$(x,y,z,w) \leq (u,v,r,l) \quad \text{or} \quad (u,v,r,l) \leq (x,y,z,w).$$

Also, we say that (x, y, z, w) is equal to (u, v, r, l) if and only if $x = u$, $y = v$, $z = r$ and $w = l$.

Theorem 2.3 *In addition to hypotheses of Theorem 2.1, suppose that for all* $(x, y, z, w), (u, v, r, l) \in X^4$, *there exists*$(a, b, c, d) \in X^4$ *such that*

$$(F(a,b,c,d),\ F(b,c,d,a),\ F(c,d,a,b),\ F(d,a,b,c))$$

is comparable to

$$(F(x,y,z,w), F(y,z,w,x), F(z,w,x,y), F(w,x,y,z)) \text{ and }$$
$$(F(u,v,r,l), F(v,r,l,u), F(r,l,u,v), F(l,u,v,r)).$$

Then, F and g have a unique quadruple common fixed point (x, y, z, w) *such that*

$$\begin{aligned} &x = gx = F(x,y,z,w), &&y = gy = F(y,z,w,x),\\ &z = gz = F(z,w,x,y), \quad \text{and} &&w = gw = F(w,x,y,z). \end{aligned}$$

Proof. The set of quadruple coincidence points of F and g is not empty due to Theorem 2.1. Assume, now, (x, y, z, w) and (u, v, r, l) are two quadruple coincidence points of F and g, that is,

$$\begin{aligned} &F(x,y,z,w) = gx, \quad F(u,v,r,l) = gu,\\ &F(y,z,w,x) = gy, \quad F(v,r,l,u) = gv,\\ &F(z,w,x,y) = gz, \quad F(r,l,u,v) = gr,\\ &F(w,x,y,z) = gw, \quad F(l,u,v,r) = gl. \end{aligned} \tag{47}$$

We shall show that (gx, gy, gz, gw) and (gu, gv, gr, gl) are equal. By assumption, there exists $(a, b, c, d) \in X^4$ such that $(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c))$

is comparable to ($F(x, y, z, w)$, $F(y, z, w, x)$, $F(z, w, x, y)$, $F(w, x, y, z)$) and ($F(u, v, r, l)$, $F(v, r, l, u)$, $F(r, l, u, v)$, $F(l, u, v, r)$).

Define sequences $\{ga_n\}$, $\{gb_n\}$, $\{gc_n\}$, and $\{gd_n\}$ such that

$$a_0 = a,\ b_0 = b,\ c_0 = \mathrm{c},\ d_0 = \mathrm{d} \text{ and for any } n \geq 1$$

$$\begin{aligned} ga_n &= F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}),\\ gb_n &= F(b_{n-1}, c_{n-1}, d_{n-1}, a_{n-1}),\\ gc_n &= F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}),\\ gd_n &= F(d_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}), \end{aligned} \tag{48}$$

for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $w_0 = w$ and $u_0 = u$, $v_0 = v$, $r_0 = r$, $l_0 = l$ and on the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, $\{gw_n\}$ and $\{gu_n\}$, $\{gv_n\}$, $\{gr_n\}$, $\{gl_n\}$. Then, it is easy that

$$\begin{aligned} gx_n &= F(x, y, z, w), \quad & gu_n &= F(u, v, r, l),\\ gy_n &= F(y, z, w, x), \quad & gv_n &= F(v, r, l, u),\\ gz_n &= F(z, w, x, y), \quad & gr_n &= F(r, l, u, v),\\ gw_n &= F(w, x, y, z), \quad & gl_n &= F(l, u, v, r) \end{aligned} \tag{49}$$

for all $n \geq 1$. Since

$$\begin{aligned} (F(x,y,z,w), F(y,z,w,x), F(z,w,x,y), F(w,x,y,z)) &= (gx_1, gy_1, gz_1, gw_1)\\ &= (gx, gy, gz, gw) \end{aligned}$$

is comparable to

$$(F(a, b, c, d),\ F(b, c, d, a),\ F(c, d, a, b),\ F(d, a, b, c)) = (ga_1, gb_1, gc_1, gd_1),$$

then it is easy to show $(gx, gy, gz, gw) \geq (ga_1, gb_1, gc_1, gd_1)$. Recursively, we get that

$$(ga_n, gb_n, gc_n, gd_n) \leq (gx, gy, gz, gw) \text{ for all } n. \tag{50}$$

From (2) and (47), it is obvious that

$$\begin{aligned} &M(a_n, b_n, c_n, d_n, x, y, z, w) = M(y, z, w, x, b_n, c_n, d_n, a_n))\\ &= M(c_n, d_n, a_n, b_n, z, w, x, y) = M(w, x, y, z, d_n, a_n, b_n, c_n) = 0. \end{aligned} \tag{51}$$

By (50), (51), and (3), we have

$$\begin{aligned} d(ga_{n+1}, gx) &= d(F(a_n, b_n, c_n, d_n), F(x, y, z, w))\\ &\leq \phi(\max\{d(gx, ga_n), d(gy, gb_n), d(gz, gc_n), d(gw, gd_n)\}),\\ &\quad + LM(a_n, b_n, c_n, d_n, x, y, z, w)\\ &= \phi(\max\{d(gx, ga_n), d(gy, gb_n), d(gz, gc_n), d(gw, gd_n)\}), \end{aligned} \tag{52}$$

$$\begin{aligned} d(gy, gb_{n+1}) &= d(F(y, z, w, x), F(b_n, c_n, d_n, a_n))\\ &\leq \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\})\\ &\quad + LM(y, z, w, x, b_n, c_n, d_n, a_n)\\ &= \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}), \end{aligned} \tag{53}$$

$$\begin{aligned} d(gc_{n+1}, gz) &= d(F(c_n, d_n, a_n, b_n), F(z, w, x, y)) \\ &\le \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}) \\ &\quad + LM(c_n, d_n, a_n, b_n, z, w, x, y) \\ &= \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}) \end{aligned} \tag{54}$$

and

$$\begin{aligned} d(gw, gd_{n+1}) &= d(F(w, x, y, z), F(d_n, a_n, b_n, c_n)) \\ &\le \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}) \\ &\quad + LM(w, x, y, z, d_n, a_n, b_n, c_n) \\ &= \phi(\max\{d(gd_n, gw), d(ga_n, gx), d(gb_n, gy), d(gc_n, gz)\}). \end{aligned} \tag{55}$$

From (52)-(55), it follows that

$$\max\left\{\begin{matrix} d(gz, gc_{n+1}), d(gy, gb_{n+1}), \\ d(gx, ga_{n+1}), d(gw, gd_{n+1}) \end{matrix}\right\} \le \phi\left(\max\left\{\begin{matrix} d(gz, gc_n), d(gy, gb_n), \\ d(gx, ga_n), d(gw, gd_n) \end{matrix}\right\}\right). \tag{56}$$

Therefore, for each $n \ge 1$,

$$\max\left\{\begin{matrix} d(gz, gc_n), d(gy, gb_n), \\ d(gx, ga_n), d(gw, gd_n) \end{matrix}\right\} \le \phi^n\left(\max\left\{\begin{matrix} d(gz, gc_0), d(gy, gb_0), \\ d(gx, ga_0), d(gw, gd_0) \end{matrix}\right\}\right). \tag{57}$$

It is known that $\varphi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ imply $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$. Thus, from (57)

$$\lim_{n\to\infty} \max\{d(gz, gc_n), d(gy, gb_n), d(gx, ga_n), d(gw, gd_n)\} = 0.$$

This yields that

$$\lim_{n\to\infty} d(gx, ga_n) = 0, \ \lim_{n\to\infty} d(gy, gb_n) = 0, \lim_{n\to\infty} d(gz, gc_n) = 0 \text{ and } \lim_{n\to\infty} d(gw, gd_n) = 0. \tag{58}$$

Analogously, we may show that

$$\lim_{n\to\infty} d(gu, ga_n) = 0, \ \lim_{n\to\infty} d(gv, gb_n) = 0, \lim_{n\to\infty} d(gr, gc_n) = 0 \text{ and } \lim_{n\to\infty} d(gl, gd_n) = 0. \tag{59}$$

Combining (58) and (59) yields that (gx, gy, gz, gw) and (gu, gv, gr, gl) are equal.

Since $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x. y)$, and $gz = F(z, w, x, y)$, by commutativity of F and g we have

$$\begin{aligned} gx' &= g(gx) = g(F(x, y, z, w)) = F(gx, gy, gz, gw), \\ gy' &= g(gy) = g(F(y, z, w, x)) = F(gy, gz, gw, gx), \\ gz' &= g(gz) = g(F(z, w, x, y)) = F(gz, gw, gx, gy) \end{aligned}$$

and

$$gw' = g(gw) = g(F(w, x, y, z)) = F(gw, gx, gy, gz)$$

where $gx = x'$, $gy = y'$, $gz = z'$, and $gw = w'$. Thus, (x', y', z', w') is a quadruple coincidence point of F and g. Consequently, (gx', gy', gz', gz') and (gx, gy, gz, gw) are equal. We deduce

$$gx' = gx = x', \quad gy' = gy = y' \text{ and } gz' = gz = z', \quad gw' = gw = w'.$$

Therefore, (x', y', z', w') is a quadruple common fixed of F and g. Its uniqueness follows easily from (3).

Example 2.1 *Let $X = \mathbb{R}$ be endowed with the usual ordering and the usual metric, which is complete.*

Let $g: X \to X$ *and* $F: X^4 \to X$ *be defined by*

$$g(x) = \frac{3}{4}x, \quad F(x, y, z, w) = \frac{x - y + z - w}{8}, \text{ for all } x, y, z, w \in X$$

Take $\varphi : [0, \infty) \to [0, \infty)$ *be given by* $\phi(t) = \frac{2}{3}t$ *for* all $t \in [0, \infty)$.

We will check that the contraction (3) is satisfied for all $x, y, z, w, u, v, h, l \in X$ *satisfying* $gx \leq gu$, $gv \leq gy$, $gz \leq gh$, *and* $gl \leq gw$. *In this case, we have*

$$\begin{aligned} d(F(x, y, z, w), F(u, v, h, l)) &= \frac{u - x}{8} + \frac{y - v}{8} + \frac{h - z}{8} + \frac{w - l}{8} \\ &\leq \frac{1}{2}[\max\{(u - x), (y - v), (h - z), (w - l)\}] \\ &= \frac{2}{3}\max\{d(gx, gu), d(gy, gv), d(gz, gh), d(gw, gl)\} \\ &\leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gh), d(gw, gl)\}) \\ &\quad + LM(x, y, z, w, u, v, h, l), \end{aligned}$$

for arbitrary $L \geq 0$.

It is obvious that the other hypotheses of Theorem 2.3 are satisfied. We deduce that (0, 0, 0, 0) *is the unique quadruple common fixed point of F and g.*

3 Application to matrix equations

In this section, we study the existence and uniqueness of solutions (X, Y, Z, T) to the system of matrix equations

$$\begin{cases} X = Q + A_1^* X A_1 - B_1^* Y B_1 + A_2^* Z A_2 - B_2^* T B_2 \\ Y = Q + A_1^* Y A_1 - B_1^* Z B_1 + A_2^* T A_2 - B_2^* X B_2 \\ Z = Q + A_1^* Z A_1 - B_1^* T B_1 + A_2^* X A_2 - B_2^* Y B_2 \\ T = Q + A_1^* T A_1 - B_1^* X B_1 + A_2^* Y A_2 - B_2^* Z B_2, \end{cases} \tag{60}$$

where $A_1, A_2, B_1, B_2 \in \mathcal{M}(n)$: the set of all $n \times n$ matrices, $Q \in \mathcal{P}(n)$: the set of all $n \times n$ positive definite matrices, and $\mathcal{H}(n)$ is the set of all $n \times n$ Hermitian matrices.

We endow $\mathcal{H}(n)$ with the partial order $\preccurlyeq$ given by

$$M, N \in \mathcal{H}(n), \ \ M \preccurlyeq N \Leftrightarrow N - M \in \mathcal{P}(n).$$

For a fixed $P \in \mathcal{P}(n)$, we consider

$$||H||_{1,P} = tr(P^{\frac{1}{2}} H P^{\frac{1}{2}}).$$

for all $H \in \mathcal{H}(n)$, where tr is the trace operator. The space $\mathcal{H}(n)$ equipped with the metric induced by $||.||_{1,P}$ is a complete metric space for any positive definite matrix P (see [42]).

The following lemma will be useful for our application.

Lemma 3.1 *Let* $A \succcurlyeq 0$ *and* $B \succcurlyeq 0$ *be* $n \times n$ *matrices. Then, we have*

$$0 \le tr(AB) = tr(BA) \le ||A|| tr(B),$$

where $||.||$ *is the spectral norm.*

Theorem 3.1 *Suppose that there exists* $P \in \mathcal{P}(n)$ *such that*

$$k = 4\max\{||P^{-\frac{1}{2}}A_1^*PA_1P^{-\frac{1}{2}}||, ||P^{-\frac{1}{2}}A_2^*PA_2P^{-\frac{1}{2}}||, ||P^{-\frac{1}{2}}B_1^*PB_1P^{-\frac{1}{2}}||, ||P^{-\frac{1}{2}}B_2^*PB_2P^{-\frac{1}{2}}||\} < 1. \qquad (61)$$

Suppose also that

$$0 \preccurlyeq \sum_{i=1}^{2} A_i^*QA_i \text{ and } Q \preccurlyeq \sum_{i=1}^{2} B_i^*QB_i. \qquad (62)$$

Then, the system (60) has one and only one solution $(X_1, X_2, X_3, X_4) \in (\mathcal{H}(n))^4$.

Proof. Consider the mappings $F : (\mathcal{H}(n))^4 \to \mathcal{H}(n)$ and $g : \mathcal{H}(n) \to \mathcal{H}(n)$ defined by

$$F(X_1, X_2, X_3, X_4) = Q + A_1^*X_1A_1 - B_1^*X_2B_1 + A_2^*X_3A_2 - B_2^*X_4B_2 \text{ and } gX = X,$$

for all $X, X_i \in \mathcal{H}(n)$ i =1, . . . , 4.

For all $X_i, Y_i \in \mathcal{H}(n)$ i = 1. . . , 4 with $gX_1 \preccurlyeq gY_1$, $gY_2 \preccurlyeq gX_2$, $gX_3 \preccurlyeq gY_3$ and $gY_4 \preccurlyeq gX_4$, by using Lemma 3.1, we have

$$\begin{aligned}
&||F(Y_1, Y_2, Y_3, Y_4) - F(X_1, X_2, X_3, X_4)||_{1,P} \\
&= ||A_1^*(Y_1 - X_1)A_1 - B_1^*(Y_2 - X_2)B_1 + A_2^*(Y_3 - X_3)A_2 - B_2^*(Y_4 - X_4)B_2||_{1,P} \\
&= tr\left[P^{\frac{1}{2}}\left(A_1^*(Y_1 - X_1)A_1 - B_1^*(Y_2 - X_2)B_1 + A_2^*(Y_3 - X_3)A_2 - B_2^*(Y_4 - X_4)B_2\right)P^{\frac{1}{2}}\right] \\
&= tr[A_1PA_1^*(Y_1 - X_1)] + tr[B_1PB_1^*(X_2 - Y_2)] + tr[A_2PA_2^*(Y_3 - X_3)] + tr[B_2PB_2^*(X_4 - Y_4)] \\
&= tr[A_1PA_1^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(Y_1 - X_1)P^{\frac{1}{2}}P^{-\frac{1}{2}}] + tr[B_1PB_1^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(X_2 - Y_2)P^{\frac{1}{2}}P^{-\frac{1}{2}}] \\
&+ tr[A_2PA_2^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(Y_3 - X_3)P^{\frac{1}{2}}P^{-\frac{1}{2}}] + tr[B_2PB_2^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(X_4 - Y_4)P^{\frac{1}{2}}P^{-\frac{1}{2}}] \\
&\le ||P^{-\frac{1}{2}}A_1PA_1^*P^{-\frac{1}{2}}||tr(P^{\frac{1}{2}}(Y_1 - X_1)P^{\frac{1}{2}}) + ||P^{-\frac{1}{2}}B_1PB_1^*P^{-\frac{1}{2}}||tr(P^{\frac{1}{2}}(X_2 - Y_2)P^{\frac{1}{2}}) \\
&+ ||P^{-\frac{1}{2}}A_2PA_2^*P^{-\frac{1}{2}}||tr(P^{\frac{1}{2}}(Y_3 - X_3)P^{\frac{1}{2}}) + ||P^{-\frac{1}{2}}B_2PB_2^*P^{-\frac{1}{2}}||tr(P^{\frac{1}{2}}(X_4 - Y_4)P^{\frac{1}{2}}) \\
&= ||P^{-\frac{1}{2}}A_1PA_1^*P^{-\frac{1}{2}}||\,||Y_1 - X_1||_{1,P} + ||P^{-\frac{1}{2}}B_1PB_1^*P^{-\frac{1}{2}}||\,||X_2 - Y_2||_{1,P} \\
&+ ||P^{-\frac{1}{2}}A_2PA_2^*P^{-\frac{1}{2}}||\,||Y_3 - X_3||_{1,P} + ||P^{-\frac{1}{2}}B_2PB_2^*P^{-\frac{1}{2}}||\,||X_4 - Y_4||_{1,P} \\
&\le \frac{k}{4}\left(||gY_1 - gX_1||_{1,P} + ||gX_2 - gY_2||_{1,P} + ||gY_3 - gX_3||_{1,P} + ||gX_4 - gY_4||_{1,P}\right).
\end{aligned}$$

Thus, we proved that the contractive condition given in Corollary 2.5 is satisfied for all $L \ge 0$. Moreover, from (62), we have letting $gQ \preccurlyeq F$ (Q, 0, Q, 0) and $g0 \succcurlyeq F$ (0, Q, 0, Q). Applying Corollary 2.5, F and g have a coupled coincidence point (and so a quadrupled fixed point since g is the identity on $\mathcal{H}(n)$). Then, there exist $X_1, X_2, X_3, X_4 \in \mathcal{H}(n)$ such that

$$F(X_1, X_2, X_3, X_4) = X_1, \quad F(X_2, X_3, X_4, X_1) = X_2,$$
$$F(X_3, X_4, X_1, X_2) = X_3 \quad \text{and} \quad F(X_4, X_1, X_2, X_4) = X_4.$$

On the other hand, for all $X, Y \in \mathcal{H}(n)$ there is a greatest lower bound and a least upper bound, hence it is obvious that the hypotheses of Theorem 2.3 hold, so the uniqueness of that quadrupled fixed point of F, which is also the unique solution of the system (60).

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Authors' contributions
All authors have contributed in obtaining the new results presented in this article. All authors read and approve the final manuscript.

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