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Generalized open sets in grill N-topology

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Abstract

The aim of this paper is to give a systematic development of grill N-topological spaces and discuss a few properties of local function. We build a topology for the corresponding grill by using the local function. Furthermore, we investigate the properties of weak forms of open sets in the grill N-topological spaces and discuss the relationships between them.

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1. Introduction

The grill concept proved to be an important and useful tool like nets and filters, for studying some topological concepts such as proximity spaces, closure spaces, the theory of compactifications and other similar extension problems. The idea of grill on a topological space was first introduced by Choquet [4]. Later Chattopadhyay and Thorn [3] proved that grills are always unions of ultra filters. Further Roy and Mukherjee [13] defined and studied the typical topology associated with grill on a given topological space. Recently, Hatir and Jafari [6] and Al-Omari and Noiri [1] investigated new classes of generalized open sets and the relevant generalizations of continuity in grill topological spaces. Many more researchers [5, 7, 9, 10, 11, 12] defined and established the

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properties of generalized open sets in classical topology. We note that Corson and Michael [5] used the term locally dense for pre open sets. Lellis Thivagar et al. [8] introduced the concept of N-topological space that is a set equipped with $\tau_1, \tau_2, ..., \tau_N$, and also established its open sets. In this paper, we extend the notion of grill topological spaces into the grill N-topological spaces and we obtain a kind of topology by an operator which satisfies Kuratowski's closure axioms for the corresponding grill. We also investigate the properties of some generalized open sets in grill N-topological spaces.

2. Preliminaries

In this section we recall some known results of N-topological spaces and grill topological spaces which are used in the following sections. By a space X, we mean a grill N-topological space $(X, N\tau, G)$ with N-topology $N\tau$ and grill G on X on which no separation axioms are assumed unless explicitly stated.

Definition 2.1 ([8])**.** Let X be a non empty set, τ_1, τ_2, ... , τ_N be N-arbitrary topologies defined on X and let the collection $N\tau$ be defined by

$$N\tau = \{S \subseteq X : S = (\bigcup_{i=1}^{N} A_i) \cup (\bigcap_{i=1}^{N} B_i),\ A_i, B_i \in \tau_i\},$$

satisfying the following axioms:

(i) $X, \varnothing \in N\tau$
(ii) $\bigcup_{i=1}^{\infty} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{\infty} \in N\tau$
(iii) $\bigcap_{i=1}^{n} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{n} \in N\tau$.

Then the pair $(X, N\tau)$ is called a N-topological space on X and the elements of the collection $N\tau$ are known as $N\tau$-open sets on X. A subset A of X is said to be $N\tau$-closed on X if the complement of A is $N\tau$-open on X. The set of all $N\tau$-open sets on X and the set of all $N\tau$-closed sets on X are respectively denoted by $N\tau$O(X) and $N\tau$C(X).

Definition 2.2 ([8])**.** Let $(X, N\tau)$ be a N-topological space and S be a subset of X. Then

(i) the $N\tau$-interior of S, denoted by $N\tau$-$int(S)$, and is defined by

$$N\tau\text{-}int(S) = \cup\{G : G \subseteq S \text{ and G is } N\tau\text{-open}\}.$$

(ii) the $N\tau$-closure of S, denoted by $N\tau$-$cl(S)$, and is defined by

$$N\tau\text{-}cl(S) = \cap\{F : S \subseteq F \text{ and F is } N\tau\text{-closed}\}.$$

Theorem 2.3 ([8])**.** *Let $(X, N\tau)$ be a N-topological space on X and $A \subseteq X$. Then $x \in N\tau$-$cl(A)$ if and only if $O \cap A \neq \varnothing$, for every $N\tau$-open set O containing x.*

Definition 2.4 ([4])**.** A non empty collection G of non empty subsets of a topological space (X, τ) is called a grill on X if

(i) $A \in G$ and $A \subset B \Rightarrow B \in G$ and
(ii) $A, B \subset X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$.

A topological space (X, τ) together with a grill G on X is called a grill topological space and is denoted by (X, τ, G). For any point x of a topological space (X, τ), $\tau(x)$ means the collection of all open sets containing x.

Definition 2.5 ([13])**.** Let (X, τ, G) be a grill topological space and for every $A \subseteq X$, the operator $\Phi_G(A, \tau) = \{x \in X : A \cap U \in G, \forall U \in \tau(x)\}$ is called the local function associated with the grill G and the topology τ.

Definition 2.6 ([13])**.** Corresponding to a grill G on a topological space (X, τ), then the operator τ_G-$cl : P(X) \to P(X)$ defined by τ_G-$cl(A) = A \cup \Phi(A)$ $\forall A \subseteq X$, satisfies Kuratowski's closure axioms and also there exists a unique topology $\tau_G = \{U \subseteq X : \tau_G\text{-}cl(U^c) = U^c\}$ which is finer than τ.

3. Closure Operator in Grill N-Topological Spaces

In this section we introduce grill N-topological spaces and investigate the properties of the local function $\Phi_G(A, N\tau)$. Further we derive a topology by the closure operator τ_G-cl and we discuss some of its properties.

Definition 3.1. A non empty collection G of non empty subsets of a N-topological space $(X, N\tau)$ is called a grill on X if

(i) $A \in G$ and $A \subset B \Rightarrow B \in G$ and
(ii) $A, B \subset X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$.

Then a N-topological space $(X, N\tau)$ together with a grill G is called a grill N-topological space and is denoted by $(X, N\tau, G)$. Particularly, if $N = 1$, then $(X, 1\tau = \tau, G)$ is called the grill topological space, if $N = 2$, then $(X, 2\tau, G)$ is called the grill bitopological space, if $N = 3$, then $(X, 3\tau, G)$ is called the grill tritopological space defined on X and so on.

Remark 3.2.

(i) The grill $G = P(X) - \{\varnothing\}$ is the maximal grill in any N-topological space $(X, N\tau)$.
(ii) The grill $G = \{X\}$ is the minimal grill in any N-topological space $(X, N\tau)$.

Definition 3.3. Let $(X, N\tau, G)$ be a grill N-topological space and for each $A \subseteq X$, the operator $\Phi_G(A, N\tau) = \{x \in X : A \cap U \in G, \forall U \in N\tau(x)\}$, is called the local function associated with the grill G and the N-topology $N\tau$. It is denoted as $\Phi_G(A)$. For any point x of a N-topological space $(X, N\tau)$, $N\tau(x)$ means the collection of all $N\tau$-open sets containing x.

Theorem 3.4. *Let $(X, N\tau)$ be a N-topological space. Then the following are true:*

(i) *If G is any grill on X, then Φ_G is an increasing function in the sense that $A \subseteq B$ implies $\Phi_G(A, N\tau) \subseteq \Phi_G(B, N\tau)$.*
(ii) *If G_1 and G_2 are two grills on X with $G_1 \subseteq G_2$, then $\Phi_{G_1}(A, N\tau) \subseteq \Phi_{G_2}(A, N\tau)$, for all $A \subseteq X$.*
(iii) *For any grill G on X and if $A \notin G$, then $\Phi_G(A, N\tau) = \varnothing$.*

Proof. It trivially follows from the Definition 3.3. ☐

Theorem 3.5. *Let $(X, N\tau, G)$ be a grill N-topological space. Then for all $A, B \subseteq X$,*

(i) $\Phi_G(A \cup B) \supseteq \Phi_G(A) \cup \Phi_G(B)$.
(ii) $\Phi_G(\Phi_G(A)) \subseteq \Phi_G(A) = N\tau\text{-}cl(\Phi_G(A)) \subseteq N\tau\text{-}cl(A)$.

Proof. We prove the part (ii) only and part(i) is trivial.
(ii). If $x \notin N\tau\text{-}cl(A)$, then there exists $U \in N\tau(x)$ such that $U \cap A = \varnothing \notin G$ implies $x \notin \Phi_G(A)$. Thus $\Phi_G(A) \subseteq N\tau\text{-}cl(A)$. Now we shall show that $N\tau\text{-}cl(\Phi_G(A)) \subseteq \Phi_G(A)$. Suppose that $x \in N\tau\text{-}cl(\Phi_G(A))$, then there exists a $U \in N\tau(x)$ such that $U \cap \Phi_G(A) \neq \varnothing$. Let $y \in U \cap \Phi_G(A)$. Then $U \cap A \in G$ and so $x \in \Phi_G(A)$. Thus $N\tau\text{-}cl(\Phi_G(A)) = \Phi_G(A)$. Hence $\Phi_G(\Phi_G(A)) \subseteq N\tau\text{-}cl(\Phi_G(A)) = \Phi_G(A) \subseteq N\tau\text{-}cl(A)$. □

Remark 3.6. Equality does not always hold in (i) of Theorem 3.5. Let $N = 2$ and $X = \{a, b, c, d\}$, and consider $\tau_1 O(X) = \{\varnothing, X, \{a\}\}$, $\tau_2 O(X) = \{\varnothing, X, \{a, b\}\}$. Then $2\tau O(X) = \{\varnothing, X, \{a\}, \{a, b\}\}$ is a bitopology and consider the grill $G = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Thus $(X, 2\tau, G)$ is a grill bitopological space on X. If $A = \{a\}$ and $B = \{b, c\}$, then $\Phi_G(A) \cup \Phi_G(B) = \varnothing \subset \{b, c, d\} = \Phi_G(A \cup B)$.

Definition 3.7. Corresponding to a grill G on a N-topological space $(X, N\tau)$, the operator $N\tau_G\text{-}cl : P(X) \to P(X)$ defined by $N\tau_G\text{-}cl(A) = A \cup \Phi_G(A)$ $\forall A \subseteq X$, satisfies Kuratowski's closure axioms and also there exists a unique topology $N\tau_G = \{U \subseteq X : N\tau_G\text{-}cl(U^c) = U^c\}$ which is finer than $N\tau$.

Example 3.8. Let $N = 3$ and $X = \{a, b, c\}$ and consider $\tau_1 O(X) = \{\varnothing, X, \{a\}\}$, $\tau_2 O(X) = \{\varnothing, X, \{b\}\}$ and $\tau_3 O(X) = \{\varnothing, X, \{a, b\}\}$. Then $3\tau O(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ is a tritopology and consider the grill $G = \{\{a\}, \{a, b\}, \{a, c\}, X\}$. Thus $(X, 3\tau, G)$ is a grill tritopological space on X and $3\tau_G = \{U \subseteq X : 3\tau_G\text{-}cl(U^c) = U^c\} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ which is finer than $3\tau O(X)$.

Theorem 3.9.

(i) *If G_1 and G_2 are two grills on a N-topological space $(X, N\tau)$ with $G_1 \subseteq G_2$, then $N\tau_{G_2} \subseteq N\tau_{G_1}$.*
(ii) *If G is a grill on a N-topological space $(X, N\tau)$ and $B \notin G$, then B is $N\tau_G$-closed set in $(X, N\tau_G)$.*
(iii) *For any subset A of a N-topological space $(X, N\tau)$ and any grill G on X, $\Phi_G(A)$ is $N\tau_G$-closed set in $(X, N\tau_G)$.*
(iv) *If A is a $N\tau_G$-closed, then $\Phi_G(A) \subseteq A$.*

Proof.

(i) $U \in N\tau_{G_2} \Rightarrow N\tau_{G_2}\text{-}cl(U^c) = U^c \Rightarrow \Phi_{G_2}(U^c) \subseteq U^c \Rightarrow \Phi_{G_1}(U^c) \subseteq U^c \Rightarrow N\tau_{G_1}\text{-}cl(U^c) = U^c \Rightarrow U \in N\tau_{G_1}$.
(ii) If $B \notin G$, then $\Phi_G(B) = \varnothing$ and $N\tau_G\text{-}cl(B) = B$.
(iii) We have, $N\tau_G\text{-}cl(\Phi_G(A)) = \Phi_G(A) \cup \Phi_G(\Phi_G(A)) = \Phi_G(A) \Rightarrow \Phi_G(A)$ is $N\tau_G$-closed.
(iv) Assume that $x \notin A = N\tau_G\text{-}cl(A) \Rightarrow x \notin \Phi_G(A)$. Thus $\Phi_G(A) \subseteq A$. □

Theorem 3.10. *Let $(X, N\tau, G)$ be a grill N-topological space. Then the collection $\beta(G, N\tau) = \{V - A : V \in N\tau$ and $A \notin G\}$ is an open basis for $N\tau_G$.*

Proof. Let $(X, N\tau, G)$ be a grill N-topological space and $U \in N\tau_G$ and $x \in U \Rightarrow X - U$ is $N\tau_G$-closed $\Rightarrow \Phi_G(X - U) \subseteq X - U \Rightarrow U \subseteq X - \Phi_G(X - U)$. Therefore, $x \in U$ which implies that $x \notin \Phi_G(X - U)$. Then there exists a $V \in N\tau(x)$ such that $V \cap (X - U) \notin G$. Let us take $A = V \cap (X - U) \notin G$ and we have $x \in V - A \subseteq U$ where V is $N\tau$-open set and $A \notin G$. Thus U is the union of sets in $\beta(G, N\tau)$. Clearly, $\beta(G, N\tau)$ is closed under finite intersections, that is if $V_1 - A, V_2 - B \in \beta(G, N\tau)$, then $V_1, V_2 \in N\tau$ and $A, B \notin G$ and also $V_1 \cap V_2 \in N\tau$ and $A \cup B \notin G$. Now, $(V_1 - A) \cap (V_2 - B) = (V_1 \cap V_2) - (A \cup B) \in \beta(G, N\tau)$, and hence $\beta(G, N\tau) = \{V - A : V \in N\tau$ and $A \notin G\}$ is an open base for $N\tau_G$. □

Theorem 3.11. *In a grill N-topological space $(X, N\tau, G)$, $N\tau \subseteq \beta(G, N\tau) \subseteq N\tau_G$ and in particular if $G = P(X) - \{\varnothing\}$, then $N\tau = \beta(G, N\tau) = N\tau_G$.*

Proof. Let $V \in N\tau$. Then $V = V - \varnothing \in \beta(G, N\tau)$. Hence $N\tau \subseteq \beta(G, N\tau)$. Now, let $A \in \beta(G, N\tau)$, then there exists $V \in N\tau$ and $H \notin G$ such that $A = V - H$. Then, $N\tau_G\text{-}cl(A^c) = N\tau_G\text{-}cl((V - H)^c) = (V - H)^c \cup \Phi_G((V - H)^c) = (V^c \cup H) \cup (\Phi_G(V^c) \cup \Phi_G(H))$. But, $H \notin G$, then, by Theorem 3.4(iii), $\Phi_G(V^c) \cup \Phi_G(H) = \Phi_G(V^c)$. Since V^c is $N\tau$-closed and by Theorem 3.9(iv), $\Phi_G(V^c) \subseteq V^c$. Thus, $N\tau_G\text{-}cl(A^c) \subseteq A^c$ and hence $A \in N\tau_G$. In particular, if $G = P(X) - \{\varnothing\}$, then $N\tau_G = N\tau$. Now $V \in \beta(G, N\tau) \Rightarrow V = U - A$ with $U \in N\tau$ and $A \notin G$, we have $A = \varnothing$, so that $V = U \in N\tau$ and so $N\tau = \beta(G, N\tau) = N\tau_G$. □

Corollary 3.12. *Let $(X, N\tau, G)$ be a grill N-topological space. If $U \in N\tau$, then $U \cap \Phi_G(A) = U \cap \Phi_G(U \cap A)$, for any $A \subseteq X$.*

Proof. Clearly, $U \cap \Phi_G(A) \supseteq U \cap \Phi_G(U \cap A)$. On the other hand, let $x \in U \cap \Phi_G(A)$ and $V \in N\tau(x)$. Then $U \cap V \in N\tau(x)$ and $x \in \Phi_G(A) \Rightarrow (U \cap V) \cap A \in G$, that is, $(U \cap A) \cap V \in G \Rightarrow x \in \Phi_G(U \cap A) \Rightarrow x \in U \cap \Phi_G(U \cap A)$. Thus $U \cap \Phi_G(A) = U \cap \Phi_G(U \cap A)$. □

Corollary 3.13. *Let $(X, N\tau, G)$ be a grill N-topological space. If $N\tau - \{\varnothing\} \subseteq G$, then $U \subseteq \Phi_G(U)$ for all $U \in N\tau$.*

Proof. If $U = \varnothing$, then $\Phi_G(U) = \varnothing = U$ and if $N\tau - \{\varnothing\} \subseteq G$, then $\Phi_G(X) = X$. By Corollary 3.12, we have for any $U \in N\tau - \{\varnothing\}, U \cap \Phi_G(X) = U \cap \Phi_G(U \cap X)$ and implies $U = U \cap \Phi_G(U)$. Thus, $\Phi_G(U) \supseteq U$. □

Corollary 3.14. Let A be a subset of a grill N-topological space $(X, N\tau, G)$. If $U \in N\tau$, then $U \cap N\tau_G\text{-}cl(A) \subseteq N\tau_G\text{-}cl(U \cap A)$.

Proof. Since $U \in N\tau$ and by Corollary 3.12, we obtain $U \cap N\tau_G\text{-}cl(A) = (U \cap A) \cup (U \cap \Phi_G(A)) \subseteq (U \cap A) \cup \Phi_G(U \cap A) = N\tau_G\text{-}cl(U \cap A)$. □

4. Generalized Open Sets in Grill N-Topological Spaces

In this section we introduce some weak forms of open sets in grill N-topological spaces and also we discuss the relationships between them.

Definition 4.1. Let $(X, N\tau, G)$ be a grill N-topological space and $A \subseteq X$. Then A is said to be

(i) ${}_GN\tau$-open if $A \subseteq N\tau\text{-}int(\Phi_G(A))$.
(ii) ${}_GN\tau$-α open if $A \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(A)))$.
(iii) ${}_GN\tau$-semi open if $A \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A))$.
(iv) ${}_GN\tau$-pre open if $A \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(A))$.
(v) ${}_GN\tau$-β open if $A \subseteq N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A)))$.

The set of all ${}_GN\tau$-open (resp. ${}_GN\tau$-α open, ${}_GN\tau$-semi open, ${}_GN\tau$-pre open, ${}_GN\tau$-β open) sets in a grill N-topological space $(X, N\tau, G)$ is denoted by ${}_GN\tau O(X)$ (resp. ${}_GN\tau\alpha O(X)$, ${}_GN\tau SO(X)$, ${}_GN\tau PO(X)$, ${}_GN\tau\beta O(X)$). The complements of ${}_GN\tau$-open (resp. ${}_GN\tau$-α open, ${}_GN\tau$-semi open, ${}_GN\tau$-pre open, ${}_GN\tau$-β open) sets in a grill N-topological space $(X, N\tau, G)$ are called their respective closed sets and the set of all ${}_GN\tau$-closed (resp. ${}_GN\tau$-α closed, ${}_GN\tau$-semi closed, ${}_GN\tau$-pre closed, ${}_GN\tau$-β closed) sets in a grill N-topological space $(X, N\tau, G)$ is denoted by ${}_GN\tau C(X)$ (resp. ${}_GN\tau\alpha C(X)$, ${}_GN\tau SC(X)$, ${}_GN\tau PC(X)$, ${}_GN\tau\beta C(X)$). For $N = 1$, then we take ${}_G1\tau O(X)$ (resp. ${}_G\alpha O(X)$, ${}_GSO(X)$, ${}_GPO(X)$, ${}_G\beta O(X)$). For $N = 2$, then we take ${}_G2\tau O(X)$ (resp. ${}_G2\tau\alpha O(X)$, ${}_G2\tau SO(X)$, ${}_G2\tau PO(X)$, ${}_G2\tau\beta O(X)$) and so on.

We observe that part (iii) of the next theorem is analogous to the 1985 topological space result of Reilly and Vamanamurthy [12].

Theorem 4.2. *Let A be a subset of a grill N-topological space $(X, N\tau, G)$.*

(i) *If A is $N\tau$-open, then A is ${}_GN\tau$-α open.*
(ii) *If A is ${}_GN\tau$-open, then A is ${}_GN\tau$-pre open.*
(iii) *A is ${}_GN\tau$-α open if and only if it is ${}_GN\tau$-semi open and ${}_GN\tau$-pre open.*
(iv) *If A is ${}_GN\tau$-semi open, then A is ${}_GN\tau$-β open.*
(v) *If A is ${}_GN\tau$-pre open, then A is ${}_GN\tau$-β open.*

Proof. Here we prove part (iii) only, and note that the remaining parts have similar proofs.
(iii). Since A is ${}_GN\tau$-α-open, then $A \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(A))) \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(A))$ and $A \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(A))) \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A))$. On the other hand, since A is ${}_GN\tau$-semi open and ${}_GN\tau$-pre open, then $A \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(A)) \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(N\tau_G\text{-}cl(N\tau\text{-}int(A)))) \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(A)))$. □

The following examples show that the converse of the above theorem need not be true, that ${}_GN\tau$-open sets and $N\tau$-open sets are independent, and that ${}_GN\tau$-semi open sets and ${}_GN\tau$-pre open sets are independent.

Example 4.3. Let $N=5$ and $X=\{a,b,c\}$ and consider $\tau_1O(X)=\{\varnothing,X,\{a\}\}$, $\tau_2O(X)=\{\varnothing,X,\{b\}\}$, $\tau_3O(X)=\{\varnothing,X,\{a,b\}\}$, $\tau_4O(X)=\{\varnothing,X,\{a\},\{a,b\}\}$ and $\tau_5O(X)=\{\varnothing,X,\{b\},\{a,b\}\}$. Then $5\tau O(X)=\{\varnothing,X,\{a\},\{b\},\{a,b\}\}$ is a 5τ-topology and consider the grill $G=\{\{a\},\{a,b\},\{a,c\},X\}$. Thus $(X,5\tau,G)$ is a grill 5-topological space. Here the set $\{b\}$ is a 5τ-open but not a ${}_G5\tau$-open set and the set $\{b,c\}$ is ${}_G5\tau$-β-open but not ${}_G5\tau$-semi open, not ${}_G5\tau$-pre open and not ${}_G5\tau$-α open. Also the set $\{a,c\}$ is ${}_G5\tau$-semi open but not ${}_G5\tau$-pre open and not ${}_G5\tau$-α open and the set $\{a,b\}$ is ${}_G5\tau$-pre open but not ${}_G5\tau$-open.

Example 4.4. Let $N=3$ and $X=\{a,b,c,d\}$ and consider $\tau_1O(X)=\{\varnothing,X\}$, $\tau_2O(X)=\{\varnothing,X,\{a\}\}$ and $\tau_3O(X)=\{\varnothing,X,\{a,b\}\}$. Then $3\tau O(X)=\{\varnothing,X,\{a\},\{a,b\}\}$ is a 3τ-topology and consider the grill $G=\{\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X\}$. Thus $(X,3\tau,G)$ is a grill tritopological space. Here the set $\{a,b,c\}$ is ${}_G3\tau$-open and ${}_G3\tau$-α open but not a 3τ-open set. Also if $N=2$ and $X=\{a,b,c\}$, and consider $\tau_1O(X)=\{\varnothing,X,\{a\}\}$ and $\tau_2O(X)=\{\varnothing,X\}$. Then $2\tau O(X)=\{\varnothing,X,\{a\}\}$ is a 2τ-topology and consider the grill $G=\{\{a,b\},X\}$. Thus $(X,2\tau,G)$ is a grill bitopological space. Here the set $\{a,b\}$ is ${}_G2\tau$-pre open but not ${}_G2\tau$-semi open and not ${}_G2\tau$-α open.

Theorem 4.5. *In a grill N-topological space $(X,N\tau,G)$, the following are true:*

(i) *If $N\tau-\{\varnothing\}\subseteq G$, then every $N\tau$-open set is ${}_GN\tau$-open.*
(ii) *If $A\subseteq X$ is ${}_GN\tau$-open and $N\tau_G$-closed, then A is $N\tau$-open.*
(iii) *If $A\subseteq X$ is ${}_GN\tau$-closed, then $\Phi_G(N\tau\text{-}int(A))\subseteq N\tau\text{-}cl(N\tau\text{-}int(A))\subseteq A$.*

Proof.

(i) Let A be a $N\tau$-open set and by Corollary 3.13, $A=N\tau\text{-}int(A)\subseteq N\tau\text{-}int(\Phi_G(A))$.
(ii) Since A is $N\tau_G$-closed, $A=N\tau_G\text{-}cl(A)=A\cup\Phi_G(A)\Rightarrow\Phi_G(A)\subseteq A$ and since A is ${}_GN\tau$-open, $A\subseteq N\tau\text{-}int(\Phi_G(A))\subseteq N\tau\text{-}int(A)$. Thus $A=N\tau\text{-}int(A)$.
(iii) Assume A is ${}_GN\tau$-closed, $X-A$ is ${}_GN\tau$-open and $X-A\subseteq N\tau\text{-}int(\Phi_G(X-A))\subseteq\Phi_G(X-A)$. Then by Theorem 3.5, $\Phi_G(X-A)=N\tau\text{-}cl(X-A)=X-N\tau\text{-}int(A)$ and now $X-A\subseteq N\tau\text{-}int(\Phi_G(X-A))\subseteq N\tau\text{-}int(X-N\tau\text{-}int(A))\subseteq X-N\tau\text{-}cl(N\tau\text{-}int(A))$. Thus $N\tau\text{-}cl(N\tau\text{-}int(A))\subseteq A$ and also $\Phi_G(N\tau\text{-}int(A))\subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A))\subseteq N\tau\text{-}cl(N\tau\text{-}int(A))\subseteq A$.

□

Theorem 4.6. *Let $(X,N\tau,G)$ be a grill N-topological space and Ω be an index set.*

(i) *If $\{A_i\}_{i\in\Omega}\in{}_GN\tau O(X)$, then $\bigcup_{i\in\Omega}A_i\in{}_GN\tau O(X)$.*
(ii) *If $\{A_i\}_{i\in\Omega}\in{}_GN\tau\alpha O(X)$, then $\bigcup_{i\in\Omega}A_i\in{}_GN\tau\alpha O(X)$.*
(iii) *If $\{A_i\}_{i\in\Omega}\in{}_GN\tau SO(X)$, then $\bigcup_{i\in\Omega}A_i\in{}_GN\tau SO(X)$.*

(iv) *If* $\{A_i\}_{i\in\Omega} \in {}_GN\tau PO(X)$, *then* $\bigcup_{i\in\Omega} A_i \in {}_GN\tau PO(X)$.
(v) *If* $\{A_i\}_{i\in\Omega} \in {}_GN\tau\beta O(X)$, *then* $\bigcup_{i\in\Omega} A_i \in {}_GN\tau\beta O(X)$.

Proof. We prove part (v) only, and note that the remaining parts have similar proofs.
(v). Assume $\{A_i\}_{i\in\Omega} \in {}_GN\tau\beta O(X)$, then for each $i \in \Omega, A_i \subseteq N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A_i))) \Rightarrow \bigcup_{i\in\Omega} A_i \subseteq \bigcup_{i\in\Omega}(N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A_i)))) = N\tau\text{-}cl(\bigcup_{i\in\Omega}(N\tau\text{-}int(N\tau_G\text{-}cl(A_i)))) \subseteq N\tau\text{-}cl(N\tau\text{-}int(\bigcup_{i\in\Omega}(N\tau_G\text{-}cl(A_i)))) \subseteq N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(\bigcup_{i\in\Omega} A_i)))$. This shows that $\bigcup_{i\in\Omega} A_i \in {}_GN\tau\beta O(X)$. □

Theorem 4.7. *Let* $(X, N\tau, G)$ *be a grill* N*-topological space and* $A, B \subseteq X$, *then the following statements are true:*

(i) *If* $A \in {}_GN\tau SO(X)$ *and* $B \in {}_GN\tau\alpha O(X)$, *then* $A \cap B \in {}_GN\tau SO(X)$.
(ii) *If* $A \in {}_GN\tau PO(X)$ *and* $B \in {}_GN\tau\alpha O(X)$, *then* $A \cap B \in {}_GN\tau PO(X)$.

Proof. Here we prove part (i) only, and note that part (ii) has a similar proof.
(i) Since $A \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A))$, $B \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(A)))$ and by Corollary 3.14, $A \cap B \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A)) \cap N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(B))) \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A) \cap N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(B)))) \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A) \cap N\tau_G\text{-}cl(N\tau\text{-}int(B))) \subseteq N\tau_G\text{-}cl(N\tau_G\text{-}cl(N\tau\text{-}int(A \cap B)))$. This shows that $A \cap B \in {}_GN\tau SO(X)$. □

Lemma 4.8. *Let* $(X, N\tau, G)$ *be a grill* N*-topological space and* $A, B \subseteq X$, *then the following statements are true:*

(i) *If* $A \in {}_GN\tau SO(X)$ *and* $B \in N\tau O(X)$, *then* $A \cap B \in {}_GN\tau SO(X)$.
(ii) *If* $A \in {}_GN\tau PO(X)$ *and* $B \in N\tau O(X)$, *then* $A \cap B \in {}_GN\tau PO(X)$.

Example 4.9. Let $N = 5$ and $X = \{a, b, c\}$ and consider $\tau_1 O(X) = \{\varnothing, X, \{a\}\}$, $\tau_2 O(X) = \{\varnothing, X, \{b\}\}$, $\tau_3 O(X) = \{\varnothing, X, \{a, b\}\}$, $\tau_4 O(X) = \{\varnothing, X, \{a\}, \{a, b\}\}$ and $\tau_5 O(X) = \{\varnothing, X, \{b\}, \{a, b\}\}$. Then $5\tau O(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ is a 5τ-topology and consider the grill $G = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Thus $(X, 5\tau, G)$ is a grill 5-topological space. The sets $\{a, c\}$ and $\{b, c\}$ are ${}_G5\tau$-open (resp. ${}_G5\tau$-pre open, ${}_G5\tau$-β open) sets but their intersection $\{c\}$ is not a ${}_G5\tau$-open (resp. ${}_G5\tau$-pre open, ${}_G5\tau$-β open) set. In the same 5τ-topology, consider the maximal grill $G = P(X) - \{\varnothing\}$. Thus $(X, 5\tau, G)$ is a grill 5-topological space. The set $\{a, c\}$ and $\{b, c\}$ are ${}_G5\tau$-semi open sets but their intersection $\{c\}$ is not a ${}_G5\tau$-semi-open set.

Theorem 4.10. *Let* $(X, N\tau, G)$ *be a grill* N*-topological space and* $A, B \in {}_GN\tau\alpha O(X)$, *then* $A \cap B \in {}_GN\tau\alpha O(X)$.

Proof. Since $A, B \in {}_GN\tau\alpha O(X)$, then by using Theorem 4.2 and Theorem 4.7 we get $A \cap B \in {}_GN\tau SO(X)$, $A \cap B \in {}_GN\tau PO(X)$, and therefore $A \cap B \in {}_GN\tau\alpha O(X)$. □

Theorem 4.11. *In a grill N-topological space $(X, N\tau, G)$, the family ${}_GN\tau\alpha O(X)$ is a topology and $N\tau O(X) \subseteq {}_GN\tau\alpha O(X)$.*

Proof. Clearly, $\varnothing, X \in {}_GN\tau\alpha O(X)$. The desired result follows from the Theorem 4.2, Theorem 4.6 and Theorem 4.10. □

Theorem 4.12. *Let $(X, N\tau, G)$ be a grill N-topological space. Then $A \subseteq X$ is*

(i) *${}_GN\tau$-semi open if and only if $N\tau_G\text{-}cl(A) = N\tau_G\text{-}cl(N\tau\text{-}int(A))$.*
(ii) *${}_GN\tau$-pre open, then $N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A))) = N\tau\text{-}cl(A)$.*

Proof.

(i) Assume that A is ${}_GN\tau$-semi open, then $A \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A)) \Rightarrow N\tau_G\text{-}cl(A) \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A)) \subseteq N\tau_G\text{-}cl(A)$. Thus $N\tau_G\text{-}cl(A) = N\tau_G\text{-}cl(N\tau\text{-}int(A))$. Converse is obvious, since $A \subseteq N\tau_G\text{-}cl(A)$.
(ii) Assume that A is ${}_GN\tau$-pre open, then $A \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(A)) \Rightarrow N\tau\text{-}cl(A) \subseteq N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A))) \subseteq N\tau\text{-}cl(N\tau_G\text{-}cl(A)) = N\tau\text{-}cl(A \cup \Phi_G(A)) = N\tau\text{-}cl(A) \cup N\tau\text{-}cl(\Phi_G(A)) = N\tau\text{-}cl(A) \cup \Phi_G(A) \subseteq N\tau\text{-}cl(A)$. Thus $N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A))) = N\tau\text{-}cl(A)$.

□

Theorem 4.13. *Let $(X, N\tau, G)$ be a grill N-topological space and $A \subseteq X$.*

(i) *Then A is ${}_GN\tau$-semi open if and only if there exists a $U \in N\tau$ such that $U \subseteq A \subseteq N\tau_G\text{-}cl(U)$.*
(ii) *If A is a ${}_GN\tau$-semi open and $A \subseteq B \subseteq N\tau_G\text{-}cl(A)$, then B is ${}_GN\tau$-semi open.*

Proof.

(i) Since A is ${}_GN\tau$-semi open, then $A \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A))$. Take $U = N\tau\text{-}int(A)$. Then we have $U \subseteq A \subseteq N\tau_G\text{-}cl(U)$. On the other hand, assume $U \subseteq A \subseteq N\tau_G\text{-}cl(U)$ for some $U \in N\tau$. Since $U \subseteq A$, then $U \subseteq N\tau\text{-}int(A) \Rightarrow N\tau_G\text{-}cl(U) \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A))$. Thus $A \subseteq N\tau_G\text{-}cl(N\tau\text{-}int(A))$.
(ii) Since A is ${}_GN\tau$-semi open, then there exists a $U \in N\tau$ such that $U \subseteq A \subseteq N\tau_G\text{-}cl(U)$. Then $U \subseteq A \subseteq B \subseteq N\tau_G\text{-}cl(A) \subseteq N\tau_G\text{-}cl(N\tau_G\text{-}cl(U)) = N\tau_G\text{-}cl(U)$. By part(i), we have B is ${}_GN\tau$-semi open.

□

Theorem 4.14. *Let $(X, N\tau, G)$ be a grill N-topological space and $A \subseteq X$.*

(i) *If A is ${}_GN\tau$-α closed, then $N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A))) \subseteq A$.*
(ii) *If A is ${}_GN\tau$-semi closed, then $N\tau\text{-}int(N\tau_G\text{-}cl(A)) \subseteq A$.*
(iii) *If A is ${}_GN\tau$-pre closed, then $N\tau_G\text{-}cl(N\tau\text{-}int(A)) \subseteq A$.*
(iv) *If A is ${}_GN\tau$-β-closed, then $N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(A))) \subseteq A$.*

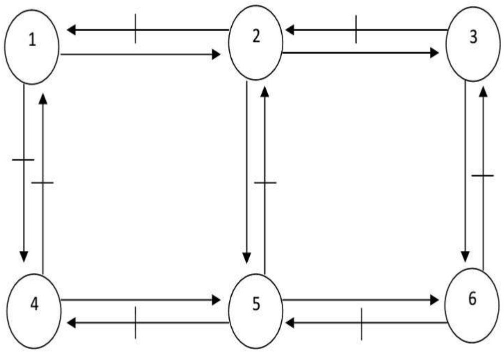
Proof. (i). Assume A is ${}_GN\tau$-α closed, then $X - A$ is ${}_GN\tau$-α open and implies $X - A \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(N\tau\text{-}int(X - A))) \subseteq N\tau\text{-}int(N\tau_G\text{-}cl(X - N\tau\text{-}cl(A))) \subseteq N\tau\text{-}int(N\tau\text{-}cl(X - N\tau_G\text{-}cl(A))) \subseteq N\tau\text{-}int(X - N\tau\text{-}int(N\tau_G\text{-}cl(A))) \subseteq X - N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A)))$. Thus $N\tau\text{-}cl(N\tau\text{-}int(N\tau_G\text{-}cl(A))) \subseteq A$. Similarly we can prove the remaining parts. □

The proof of the next theorem is straightforward.

Theorem 4.15. *Let A be a subset of a grill N-topological space $(X, N\tau, G)$.*

(i) *If A is $N\tau$-closed, then A is ${}_GN\tau$-α closed.*
(ii) *If A is ${}_GN\tau$-closed, then A is ${}_GN\tau$-pre closed.*
(iii) *A is ${}_GN\tau$-α closed if and only if it is ${}_GN\tau$-semi closed and ${}_GN\tau$-pre closed.*
(iv) *If A is ${}_GN\tau$-semi closed, then A is ${}_GN\tau$-β closed.*
(v) *If A is ${}_GN\tau$-pre closed, then A is ${}_GN\tau$-β closed.*

Remark 4.16. From the above theorems, lemmas and examples we have the following diagram. We depict by arrow the implications between the classes of generalized open sets.



(1) $N\tau$-open, (2) ${}_GN\tau$-α open, (3) ${}_GN\tau$-semi open, (4) ${}_GN\tau$-open, (5) ${}_GN\tau$-pre open, (6) ${}_GN\tau$-β open.

Conclusion

A set is merely an amorphous collection of elements, without coherence or form. When some kind of algebraic or geometric structure is imposed on a set, so that its elements are organized into a systematic whole, then it becomes a space. This paper is an attempt to provide a rigorous definition of generalized open sets of grill topologies on a non empty set, and to establish their properties with suitable examples. The grill N-topological concepts can be extended to other applicable research areas of topology such as Nano topology, Fuzzy topology, Intuitionistic topology, Digital topology and so on.

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