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Stability of weak solutions to obstacle problem in Clifford analysis

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Abstract

This paper is concerned with the regularity properties of weak solutions to the obstacle problem for Clifford-valued functions. Our main results are a global reverse Hölder inequality and stability of the weak solutions to the obstacle problem.

Keywords: Clifford-valued function; obstacle problem; higher integrability; stability

1 Introduction

Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ with the relation $e_i e_j + e_j e_i = -2\delta_{ij}$. For $k = 0, 1, \ldots, n$, we denote by $C\ell_n^k = C\ell_n^k(\mathbb{R}^n)$ the linear space of all k-multivectors, spanned by the reduced products $e_I = e_{i_1} e_{i_2} \cdots e_{i_k}$, corresponding to all ordered k-tuples $I = (i_1, i_2, \ldots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Thus, Clifford algebra $C\ell_n = \bigoplus_k C\ell_n^k$ is a graded algebra, especially $C\ell_n^0 = \mathbb{R}$ and $C\ell_n^1 = \mathbb{R}^n$. $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset C\ell_n^3 \subset \cdots$ is an increasing chain, where $\mathbb{H}$ is the Hamilton's algebra of quaternions. For $u \in C\ell_n$, u can be written as

$$u = \sum_I u_I e_I = \sum_{1 \le i_1 < \cdots < i_k \le n} u_{i_1, \ldots, i_k} e_{i_1} \cdots e_{i_k},$$

where $1 \le k \le n$.

The norm of $u \in C\ell_n$ is given by $|u| = (\sum_I u_I^2)^{1/2}$. Clifford conjugation $\overline{e_{\alpha_1} \cdots e_{\alpha_k}} = (-1)^k e_{\alpha_k} \cdots e_{\alpha_1}$. For $u = \sum_I u_I e_I \in C\ell_n$, $v = \sum_J v_J e_J \in C\ell_n$,

$$\langle u, v \rangle = \left\langle \sum_I u_I e_I, \sum_J v_J e_J \right\rangle = \sum_I u_I v_I$$

defines the corresponding inner product on $C\ell_n$. Denote by $\mathrm{Sc}(u)$ the scalar part of u, the coefficient of the element e_0, and we also have $\langle u, v \rangle = \mathrm{Sc}(\bar{u}v)$.

The Dirac operator used in this paper is given by

$$D = \sum_{j=i}^n e_j \frac{\partial}{\partial x_j}. \tag{1}$$

Throughout this paper, we write $\mathcal{D}'(\Omega, C\ell_n)$ for the space of Clifford-valued functions whose coefficients are Schwartz distributions on Ω. For $p > 0$, we write $L^p(\Omega, C\ell_n^k)$ for the space of Clifford-valued functions u, whose coefficients belong to the usual $L^p(\Omega)$ space.

It is a reflexive Banach space endowed with the norm

$$\|u\|_p = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}.$$

Also, $W^{1,p}(\Omega, C\ell_n^k)$ is the Sobolev space of Clifford-valued functions u, whose coefficients belong to $W^{1,p}(\Omega)$. It is a reflexive Banach space endowed with the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p,$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$, $\frac{\partial u}{\partial x_i} = \sum_I \frac{\partial u_I}{\partial x_i} e_I$.

Let Ω be a bounded domain. This paper is concerned with the A-Dirac equation

$$DA_p(x, Du) = 0, \tag{2}$$

where $1 < p < \infty$, $A_p : \Omega \times C\ell_n \to C\ell_n$ preserves the grading of the Clifford algebra and satisfies the following conditions for some constants $0 < a \le b < \infty$:

(i) The mapping $x \to A_p(x, \xi)$ is measurable for all $\xi \in C\ell_n^k$.
(ii) The mapping $\xi \to A_p(x, \xi)$ is continuous for a.e. $x \in \Omega$.
(iii) $|A_p(x, \xi)| \le b|\xi|^{p-1}$.
(iv) $\langle A_p(x, \xi) - A_p(x, \eta), \xi - \eta \rangle \ge a|\xi - \eta|^p$ for all $x \in \Omega$ with $\xi \ne \eta$.

Definition 1.1 [1] A Clifford-valued function $u \in W^{1,p}(\Omega, C\ell_n^k)$ for $k = 0, 1, \ldots, n$, is a weak solution to equation (2) if for all $\varphi \in W^{1,p}(\Omega, C\ell_n^k)$ with compact support, we have

$$\int_\Omega \overline{A_p(x, Du)} D\varphi = 0. \tag{3}$$

In [1, 2], Nolder explained how quasi-linear elliptic equations

$$-\operatorname{div} A_p(x, \nabla u) = 0 \tag{4}$$

arise as components of Dirac systems (2) and discussed some properties of the weak solutions to the scalar parts of equations, such as the Caccioppoli estimate and the removability theorem. In [3], Heinonen *et al.* studied the quasi-linear elliptic equations (4) by means of potential theory systematically. Many other mathematicians also work on properties of solutions to equations (4), such as the regularity, stability, convergence and so on, see [4–9].

This paper is organized as follows. In Section 2, some preliminary results about Clifford-valued functions are presented. In Section 3, higher integrability of weak solutions to obstacle problem for Clifford-valued functions are obtained. Section 4 is concerned with the stability of the weak solutions to obstacle problem. For other works about Clifford analysis and Dirac equations, see [1, 2, 10–12].

2 Preliminary results

Lemma 2.1 $C_0^\infty(\Omega, C\ell_n^k)$ *is dense in* $W_0^{1,p}(\Omega, C\ell_n^k)$ *for* $1 \le p < \infty$.

Proof Let $u = \sum_I u_I(x)e_I \in W_0^{1,p}(\Omega, C\ell_n^k)$, then $u_I \in W_0^{1,p}(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we can find a sequence $\{u_I^k\} \subset C_0^\infty(\Omega)$ converges to u_I in $W_0^{1,p}(\Omega)$. Let $u_k = \sum_I u_I^k e_I$, then $\{u_k\} \subset C_0^\infty(\Omega, C\ell_n^k)$.

$$\int_\Omega |u_k - u|^p \, dx + \int_\Omega |\nabla u_k - \nabla u|^p \, dx$$

$$\leq \int_\Omega \left(\sum_I |u_I^k - u_I|^2\right)^{\frac{p}{2}} dx + \int_\Omega \left(\sum_{i=1}^n \left|\frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{p}{2}} dx$$

$$\leq c(n,p) \sum_I \int_\Omega |u_I^k - u_I|^p \, dx + \int_\Omega \left(\sum_{i=1}^n \sum_I \left(\frac{\partial u_I^k}{\partial x_i} - \frac{\partial u_I}{\partial x_i}\right)^2\right)^{\frac{p}{2}} dx$$

$$\leq c_1(n,p) \sum_I \int_\Omega |u_I^k - u_I|^p \, dx + c_2(n,p) \sum_I \int_\Omega |\nabla u_I^k - \nabla u_I|^p \, dx.$$

Since $\{u_I^k\}$ converges to u_I in $W_0^{1,p}(\Omega)$. Then $\int_\Omega |u_I^k - u_I|^p \, dx \to 0$, $\int_\Omega |\nabla u_I^k - \nabla u_I| \, dx \to 0$. Thus,

$$\int_\Omega |u_k - u|^p \, dx + \int_\Omega |\nabla u_k - \nabla u|^p \, dx \to 0 \quad (k \to \infty).$$

This means u_k converges to u in $W_0^{1,p}(\Omega, C\ell_n^k)$. $\square$

Here, we denote the Grassmann algebra $\bigwedge = \bigoplus_k \bigwedge^k$, d is the exterior derivative operator, and d^* is the formal adjoint operator. For more details about differential form, see [1]. From [13], we know that the linear mapping

$$\lambda : e_{i_1} \wedge \cdots \wedge e_{i_k} \to e_{i_1} \cdots e_{i_k} \quad (k \geq 0)$$

to all of $\bigwedge(\mathbb{R}^n)$ defines a linear isomorphism form $\bigwedge(\mathbb{R}^n)$ onto $C\ell_n(\mathbb{R}^n)$ independently of the choice of orthonormal basis $\{e_j\}$ for $\mathbb{R}^n$.

For a Clifford-valued function u, we write $u^\sharp$ for $\lambda^{-1}u$. Via λ, the operator $d + d^*$ is mapped to D, it means

$$(Du)^\sharp = du^\sharp + d^*u^\sharp. \tag{5}$$

Then, if $u \in \mathcal{D}'(\Omega, C\ell_n^k)$, there exists a constant C, such that

$$\max\{|du^\sharp|, |d^*u^\sharp|\} \leq C|Du|. \tag{6}$$

In [4], Iwaniec gave the following Poincaré inequality for differential forms.

Theorem 2.2 *For each $u^\sharp \in C_0^\infty(\Omega, \bigwedge^k)$, there exits a constant $C(n,p), 1 < p < \infty$ such that*

$$\|\nabla u^\sharp\|_p \leq C(n,p)(\|du^\sharp\|_p + \|d^*u^\sharp\|_p). \tag{7}$$

Combining (6), (7) and Lemma 2.1, we have the following Poincaré-Sobolev inequality.

Proposition 2.3 *For each* $u \in W_0^{1,p}(\Omega, C\ell_n^k)$, $1 < p < \infty$, *there exists a constant C, such that*

$$\int_\Omega |\nabla u|^p \leq c(n,p) \int_\Omega |Du|^p \, dx. \tag{8}$$

Using the Poincaré inequality of the real function and $|\nabla|u|| \leq c(n)|\nabla u|$, we have

Corollary 2.4 *For each* $u \in W_0^{1,p}(\Omega, C\ell_n^k)$, $1 < p < n$, *there exists a constant C such that*

$$\left(\int_\Omega |u|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{np}} \leq C \left(\int_\Omega |Du|^p \, dx \right)^{\frac{1}{p}}. \tag{9}$$

Similar to the process of the Ponicaré inequality for differential forms in [14], the following inequalities for Clifford-valued function can be obtained.

Corollary 2.5 *Let* $u \in W_0^{1,s}(\Omega, C\ell_n^k)$, $1 < s < \infty$, *then, there exists a constant C such that*

$$\left(\fint_\Omega |u|^s \, dx \right)^{\frac{1}{s}} \leq C|\Omega|^{\frac{1}{n}} \left(\fint_\Omega |Du|^{\frac{ns}{n+s-1}} \, dx \right)^{\frac{n+s-1}{ns}}. \tag{10}$$

Proof If $1 < s < \frac{n}{n-1}$, then $\frac{ns}{n-1} \geq \{s, \frac{n}{n-1}\}$. From the Hölder inequality and Corollary 2.4, we have

$$\left(\fint_\Omega |u|^s \, dx \right)^{\frac{1}{s}} \leq \left(\fint_\Omega |u|^{\frac{ns}{n-1}} \, dx \right)^{\frac{n-1}{ns}}$$

$$\leq C|\Omega|^{\frac{1}{n}} \left(\fint_\Omega |Du|^{\frac{ns}{n+s-1}} \, dx \right)^{\frac{n+s-1}{ns}}.$$

If $s > \frac{n}{n-1}$, then

$$\left(\int_\Omega |u|^s \, dx \right)^{\frac{1}{s}} \leq C \left(\int_\Omega |Du|^{\frac{ns}{n+s}} \, dx \right)^{\frac{n+s}{ns}}$$

$$\leq C|\Omega|^{\frac{n+s}{ns}} \left(\int_\Omega |Du|^{\frac{ns}{n+s-1}} \, dx \right)^{\frac{n+s-1}{ns}}.$$

Thus, we have

$$\left(\fint_\Omega |u|^s \, dx \right)^{\frac{1}{s}} \leq C|\Omega|^{\frac{1}{n}} \left(\fint_\Omega |Du|^{\frac{ns}{n+s-1}} \, dx \right)^{\frac{n+s-1}{ns}}. \qquad \square$$

3 Higher integrability

In this section, we will prove the higher integrability of the weak solutions to the obstacle problem $\mathcal{K}_\psi^{f,p}(\Omega, C\ell_n^k)$. Here, we assume that the complement of Ω satisfies the measure density condition, it means that there exists a positive constant $C > 0$ such that

$$|\Omega^c \cap B| \geq C|B|, \tag{11}$$

where B is a ball in $\mathbb{R}^n$. In order to prove the higher integrability, we need the following Gehring's lemma, which appeared in [5].

Lemma 3.1 *Let B_0 be a ball in $\mathbb{R}^n$, and let $g, h \in L^p(B_0)$ be a nonnegative function satisfying*

$$\fint_B g^p \, dx \le c\left[\left(\fint_{2B} g \, dx\right)^p + \fint_{2B} h^q \, dx\right] \tag{12}$$

for all balls $B \subset 2B \subset B_0$, $1 < p < \infty$. Then, for each $0 < \sigma < 1$, $p < s < p + \frac{p-1}{10^{n-p}4^n k^p}$, we have

$$\left(\fint_{\sigma B_0} g^s \, dx\right)^{\frac{1}{s}} \le \frac{100^n}{\sigma^{\frac{n}{s}}(1-\sigma)^{\frac{n}{p}}}\left[\left(\fint_{B_0} g^p \, dx\right)^{\frac{1}{p}} + \left(\fint_{B_0} h^s \, dx\right)^{\frac{1}{s}}\right]. \tag{13}$$

Definition 3.2 For Clifford-valued functions $a = \sum_I a_I(x)e_I$ and $b = \sum_I b_I(x)e_I$, we say that $a \ge b$ in Ω if $a_I(x) \ge b_I(x)$ for a.e. $x \in \Omega$ and all ordered tuple I.

Let $\psi = \sum_I \psi_I e_I$ be a Clifford-valued function in $W^{1,p}(\Omega, C\ell_n^k)$, where $\psi_I : \Omega \to [-\infty, +\infty)$. $f \in W^{1,p}(\Omega, C\ell_n^k)$ is a function, which gives the boundary. We consider the obstacle problem

$$\mathcal{K}_\psi^{f,p}(\Omega, C\ell_n^k) = \left\{v \in W^{1,p}(\Omega, C\ell_n^k) : v - f \in W_0^{1,p}(\Omega, C\ell_n^k), v \ge \psi \text{ a.e. } \Omega\right\} \tag{14}$$

for Clifford-valued functions. To avoid trivialities, we always assume that the set $\mathcal{K}_\psi^{f,p}(\Omega, C\ell_n^k)$ is not empty.

Definition 3.3 We say that a function $u \in W_{\text{loc}}^{1,p}(\Omega, C\ell_n^k)$ is a weak solution to the obstacle problem $\mathcal{K}_\psi^{f,p}(\Omega, C\ell_n^k)$, if

$$\int_\Omega \langle A_p(x, Du), Dv - Du \rangle \, dx \ge 0, \tag{15}$$

whenever $v \in \mathcal{K}_\psi^{f,p}(\Omega, C\ell_n^k)$.

Remark 3.4 If u is a weak solution to obstacle problem (15), then u holds for the scalar part of (3), *i.e.*,

$$\int_\Omega \text{Sc}\left(\overline{A(x, Du)}D\varphi\right) dx = 0. \tag{16}$$

Remark 3.5 We assume $f \ge \psi$ in Ω. Indeed, denote $f = \sum_I f_I e_I$, $\psi = \sum_I \psi_I e_I$. Then, since $u - f \in W_0^{1,p}(\Omega, C\ell_n^k)$,

$$0 \le (\psi_I - f_I)^+ \le (u - f_I)^+.$$

Thus, we have $(\psi_I - f_I)^+ \in W_0^{1,p}(\Omega)$ and $f_I^1 = (\psi_I - f_I)^+ + f_I \ge \psi_I$ in Ω. Let $f = \sum_I f_I^1 e_I$, then $f \ge \psi$ in Ω.

Theorem 3.6 *Suppose that the complement of Ω satisfies the measure density condition (11), and let $u \in W_{\text{loc}}^{1,p}(\Omega, C\ell_n^k)$ be a weak solution to the obstacle problem (14), where $f \in$*

$W^{1,p+\delta}(\Omega, C\ell_n^k)$ *for some* $\delta > 0$. *Then, there exists a positive number* $\varepsilon_0 > 0$ *and a constant* C, *independent of* u, *such that* $|Du| \in L^{p+\varepsilon}(\Omega)$, *whenever* $0 < \varepsilon < \varepsilon_0$, *and*

$$\int_\Omega |Du|^{p+\varepsilon}\, dx \le C\left[\left(\int_\Omega |Du|^p\, dx\right)^{1+\frac{\varepsilon}{p}} + \int_\Omega |Df|^{p+\varepsilon}\, dx\right]. \tag{17}$$

Proof First, let B_0 be a ball with $\Omega \subset \frac{1}{2}B_0$, $2B \subset B_0$ be a ball such that $\Omega \cap B \neq \emptyset$ and $\eta \in C_0^\infty(2B)$ be the cut-off function such that $\eta|_{\overline{B}} = 1$, $0 \le \eta \le 1$, $|\nabla \eta| \le \frac{c(n)}{|B|^{\frac{1}{n}}}$. Denote by $D = 2B \cap \Omega$, now we test (15) with $v = u - \eta^p(u - f)$. Since $u - f \in W_0^{1,p}(\Omega, C\ell_n^k)$, then $v - f \in W_0^{1,p}(\Omega, C\ell_n^k)$. For any I, we get

$$v_I - \psi_I = (1 - \eta^p)(u_I - \psi_I) + \eta^p(f_I - \psi_I) \ge 0$$

a.e. Ω. Thus, v is an admissible test function, and we have

$$\begin{aligned}
0 &\le \int_\Omega \langle A_p(x, Du), Dv - Du\rangle\, dx \\
&\le \int_D \langle A_p(x, Du), -\eta^p Du + \eta^p Df - p\eta^{p-1} D\eta(u - f)\rangle\, dx.
\end{aligned} \tag{18}$$

Since A satisfies (iii), (iv), combining the Hölder inequality and the Young inequality, it follows from (18) that

$$\begin{aligned}
a \int_D \eta^p |Du|^p\, dx &\le b \int_D \eta^p |Du|^{p-1}|Df|\, dx + p \int_D |Du|^{p-1}\eta^{p-1}|D\eta||u - f|\, dx \\
&\le b\left(\int_D \eta^p |Du|^p\, dx\right)^{\frac{p-1}{p}}\left(\int_D \eta^p |Df|^p\, dx\right)^{\frac{1}{p}} \\
&\quad + bp\left(\int_D \eta^p |Du|^p\, dx\right)^{\frac{p-1}{p}}\left(\int_D \eta^p |u - f|^p\, dx\right)^{\frac{1}{p}} \\
&\le \frac{b\epsilon_1(p-1)}{p}\int_D \eta^p |Du|^p\, dx + bc(p, \epsilon_1)\int_D \eta^p |Df|^p\, dx \\
&\quad + \frac{b\epsilon_2(p-1)}{p}\int_D \eta^p |Du|^p\, dx + bc(p, \epsilon_2)\int_D |D\eta|^p |u - f|^p\, dx.
\end{aligned}$$

Choose ϵ_1, ϵ_2 such that

$$\frac{b\epsilon_1(p-1)}{p} + \frac{b\epsilon_2(p-1)}{p} \le \frac{a}{2}.$$

Since $u - f \in W_0^{1,p}(\Omega, C\ell_n^k)$, using Lemma 2.5, we have

$$\begin{aligned}
\int_{B \cap \Omega} |Du|^p\, dx &\le c_1 \int_{2B \cap \Omega} |Df|^p\, dx + \frac{c_2}{|B|^{\frac{1}{n}}}\int_{2B \cap \Omega} |u - f|^p\, dx \\
&\le c_1 \int_{2B \cap \Omega} |Df|^p\, dx + \frac{c_2}{|B|^{\frac{1}{n}}}|2B \cap \Omega|^{\frac{p}{n}}\left(\int_{2B \cap \Omega} |Du - Df|^{\frac{p}{\theta}}\, dx\right)^\theta,
\end{aligned} \tag{19}$$

where $\theta = \frac{n+p-1}{n} > 1$. By means of condition (11), we have $|2B \cap \Omega| \leq (1 - C)|2B|$, then (19) becomes

$$\frac{1}{|B|}\int_{B\cap\Omega}|Du|^p\,dx \leq c_3\frac{1}{|2B|}\int_{2B\cap\Omega}|Df|^p\,dx + c_4\frac{1}{|2B|^\theta}\left(\int_{2B\cap\Omega}|Du|^{\frac{p}{\theta}}+|Df|^{\frac{p}{\theta}}\,dx\right)^\theta.\quad(20)$$

Let

$$g(x) = \begin{cases} |Du|^{\frac{p}{\theta}}, & \text{if } x \in \Omega, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h(x) = \begin{cases} |Df|^{\frac{p}{\theta}}, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the following reverse Hölder estimate

$$\fint_B g^\theta\,dx \leq c_3 \fint_{2B} h^\theta\,dx + c_4\left(\fint_{2B} g + h\,dx\right)^\theta$$
$$\leq c_5\left(\fint_{2B} g\,dx\right)^\theta + c_6\fint_{2B} h^\theta\,dx \quad(21)$$

holds when $2B \subset B_0$. According to Lemma 3.1, (21) becomes

$$\left(\fint_B |g|^{\theta(1+\frac{\varepsilon}{p})}\,dx\right)^{\frac{1}{\theta(1+\frac{\varepsilon}{p})}} \leq C\left[\left(\fint_{B_0}|g|^\theta\,dx\right)^{\frac{1}{\theta}} + \left(\fint_{B_0}h^{\theta(1+\frac{\varepsilon}{p})}\,dx\right)^{\frac{1}{\theta(1+\frac{\varepsilon}{p})}}\right]$$
$$\leq C\left[\left(\fint_\Omega|g|^\theta\,dx\right)^{\frac{1}{\theta}} + \left(\fint_\Omega h^{\theta(1+\frac{\varepsilon}{p})}\,dx\right)^{\frac{1}{\theta(1+\frac{\varepsilon}{p})}}\right].$$

That is,

$$\int_B |Du|^{p+\varepsilon}\,dx \leq C\left[\left(\int_\Omega|Du|^p\,dx\right)^{1+\frac{\varepsilon}{p}} + \int_\Omega|Df|^{p+\varepsilon}\,dx\right].$$

Since Ω is a bounded domain, $\overline{\Omega}$ can be covered by a finite number of balls such that the previous inequality holds, then the estimates (17) follows immediately. □

4 Stability

In this section, we will show that the weak solutions to obstacle problem are stable under some suitable assumptions.

Suppose that $\{p_i\}$ is a sequence such that $p_i \to p$ and for each $i = 0, 1, \ldots$, there is an operator A_{p_i} satisfying (i)-(iv) for $p = p_i$ and for a.e. $x \in \Omega$,

$$A_{p_i}(x,\xi) \to A_p(x,\xi) \quad(22)$$

uniformly on compact subset of $\mathbb{R}^n$. We consider the obstacle problem

$$\mathcal{K}_{\psi}^{f,p_i}\left(\Omega, C\ell_n^k\right) = \left\{v \in W^{1,p_i}\left(\Omega, C\ell_n^k\right) : v - f \in W_0^{1,p_i}\left(\Omega, C\ell_n^k\right), v \geq \psi \text{ a.e. } \Omega\right\}. \tag{23}$$

Assume that A_{p_i} preserves the grading of the Clifford algebra and converges to $A_p(x, \xi)$ uniformly on compact subset of $\mathbb{R}^n$. Suppose that $u_i \in W^{1,p_i}(\Omega, C\ell_n^k)$ is the weak solution to the following $\mathcal{K}_{\psi}^{f,p_i}(\Omega, C\ell_n^k)$-obstacle problem

$$\int_{\Omega} \langle A_{p_i}(x, Du_i), Dv - Du_i \rangle \, dx \geq 0 \tag{24}$$

for each $v \in \mathcal{K}_{\psi}^{f,p_i}(\Omega, C\ell_n^k)$ and $u_0 \in W^{1,p}(\Omega, C\ell_n^k)$ is the weak solution to

$$\int_{\Omega} \langle A_p(x, Du_0), Dv - Du_0 \rangle \, dx \geq 0 \tag{25}$$

for all $v \in \mathcal{K}_{\psi}^{f,p}(\Omega, C\ell_n^k)$.

Definition 4.1 We call a sequence $u_j \in L^p(\Omega, C\ell_n)$ converges weakly in $L^p(\Omega, C\ell_n)$ to u if

$$\int_{\Omega} v u_j \, dx \to \int_{\Omega} v u \, dx, \tag{26}$$

whenever $v \in L^{\frac{p}{p-1}}(\Omega, C\ell_n)$. Denote it by $u_j \rightharpoonup u$.

Remark 4.2 Suppose that u_i weakly converges to u in $L^p(\Omega, C\ell_n^k)$, then

$$\int_{\Omega} \langle v, u_i \rangle \, dx \to \int_{\Omega} \langle v, u \rangle \, dx, \tag{27}$$

whenever $v \in W^{1,p}(\Omega, C\ell_n^k)$.

Now, we start with the main result of this section.

Theorem 4.3 *Suppose that the complement of Ω satisfies the measure density condition* (11). *Let p_i, $A_{p_i}(x, \xi)$, $A_p(x, \xi)$ be defined as described above. Let the boundary value function f be in $W^{1,p+\tau}(\Omega, C\ell_n^k)$ for some $\tau > 0$. Then, there is a small number δ_0 such that the sequence $\{u_i\}$ of weak solutions to the obstacle problem* (24) *has a subsequence, which converges to the weak solution u_0 of* (25) *in $W^{1,p+\delta}(\Omega, C\ell_n^k)$ for any $\delta < \delta_0$.*

Before proving Theorem 4.3, we need the following lemmas.

Lemma 4.4 [6] *Suppose that a bounded open set Ω satisfies* (11). *Let $p_i > 1$ and $u_i \in W^{1,p_i}(\Omega)$, $i = 0, 1, \ldots$, and suppose that $p_i \to p$ and $u_i \to u_0$ a.e in Ω. If $\theta \in W^{1,p+\epsilon}(\Omega)$, $\epsilon > 0$ and if $u_i - \theta \in W_0^{1,p_i}(\Omega)$ with*

$$\int_{\Omega} \left|\nabla(u_i - \theta)\right|^{p_i} \, dx \leq M \tag{28}$$

$i = 1, 2, \ldots$, where $M < \infty$ is independent of i, then $u_0 \in W_0^{1,p}(\Omega)$.

Lemma 4.5 *Let u_i be described in Theorem* 4.3. *There exists a constant $\gamma_0 > 0$ such that for each $\gamma \in [0, \gamma_0]$,*

$$\int_\Omega |Du_i|^{p+\frac{\gamma}{2}}\,dx \le C, \tag{29}$$

where i is sufficiently large and C is independent of i.

Proof By virtue of Remark 3.5, we can use f as a test function in (24), so that

$$\int_\Omega \langle A_{p_i}(x, Du_i), Df - Du_i \rangle\,dx \ge 0.$$

From the structure conditions on A_i and the Hölder inequality, it follows

$$a \int_\Omega |Du_i|^{p_i}\,dx \le b \left(\int_\Omega |Du_i|^{p_i}\,dx \right)^{\frac{p_i-1}{p_i}} \left(\int_\Omega |Df|^{p_i}\,dx \right)^{\frac{1}{p_i}}.$$

Then we have

$$\int_\Omega |Du_i|^{p_i}\,dx \le C \int_\Omega |Df|^{p_i}\,dx. \tag{30}$$

By the Hölder inequality and Young's inequality, (30) becomes

$$\int_\Omega |Du_i|^{p_i}\,dx \le C \left(|\Omega| + \int_\Omega |Df|^{p+\sigma}\,dx \right), \tag{31}$$

when i is sufficiently large and $\sigma > 0$. According to Theorem 3.6, there exists a constant $\gamma_0 > 0$ such that for each $\gamma \in [0, \gamma_0]$,

$$\left(\int_\Omega |Du_i|^{p_i+\gamma}\,dx \right)^{\frac{p_i}{p_i+\gamma}}$$
$$\le C \left[\int_\Omega |Du_i|^{p_i}\,dx + \left(\int_\Omega |Df|^{p_i+\gamma}\,dx \right)^{\frac{p_i}{p_i+\gamma}} \right]. \tag{32}$$

Since $p_i \to p$, choose i sufficiently large such that $p - \frac{\gamma}{2} \le p_i \le p + \frac{\gamma}{2}$. Then $p + \frac{\gamma}{2} \le p + \gamma$. So, there exists a constant C, such that

$$\int_\Omega |Du_i|^{p+\frac{\gamma}{2}}\,dx \le C. \qquad \square$$

Proof of Theorem 4.3 Let $\kappa = \frac{n}{n-1}$. Since $u_i - f \in W_0^{1,p}(\Omega, C\ell_n^k)$, we choose i so large that $p + \frac{\gamma}{2} \le \kappa p_i$ and $p_i \le p + \frac{\gamma}{2}$, using Lemma 2.4, we get

$$\|u_i - f\|_{p+\frac{\gamma}{2}} \le C \|u_i - f\|_{\kappa p_i}$$
$$\le C \|u_i - f\|_{\frac{np_i}{n-p_i}}$$
$$\le C \|Du_i - Df\|_{p_i}$$
$$\le C \left(\|Du_i\|_{p+\frac{\gamma}{2}} + \|Df\|_{p+\frac{\gamma}{2}} \right).$$

Implied by the Minkowski inequality that

$$
\|u_i\|_{p+\frac{\gamma}{2}} \leq \|u_i - f\|_{p+\frac{\gamma}{2}} + \|f\|_{p+\frac{\gamma}{2}}
$$
$$
\leq C\big(\|Du_i\|_{p+\frac{\gamma}{2}} + \|Df\|_{p+\frac{\gamma}{2}} + \|f\|_{p+\frac{\gamma}{2}}\big).
$$

Then it follows from Lemma 4.5 that

$$
\|u_i\|_{W^{1,p+\frac{\gamma}{2}}(\Omega,\,C\ell_n^k)} \leq C. \tag{33}
$$

Write $\delta = \frac{\gamma}{2}$, $u_i = \sum_I u_I^i e_I$, according to (33), we have $u_i \in W^{1,p+\delta}(\Omega, C\ell_n^k)$. So, we can extract a subsequence, still denote by u_I^i, such that

$$
\begin{cases}
u_I^i \rightharpoonup u_I, & \text{in } W^{1,p+\delta}(\Omega), \\
u_I^i \to u_I, & \text{in } L^{p+\delta}(\Omega), \\
u_I^i \to u_I, & \text{pointwise a.e. in } \Omega.
\end{cases} \tag{34}
$$

Then $\nabla u_I^i \rightharpoonup \nabla u_I$. Let $u = \sum_I u_I e_I$, then $u_i \to u$ in $L^{p+\delta}(\Omega)$; $u_i \to u$ pointwise a.e. in Ω. Since $\nabla u_I^i \rightharpoonup \nabla u_I$, for each $j = 1, 2, \ldots, n$, we have

$$
\int_\Omega y_\alpha \frac{\partial u_I^i}{\partial x_j}\, dx \to \int_\Omega y_\alpha \frac{\partial u_I}{\partial x_j}\, dx,
$$

whenever $y_\alpha \in W^{1,p}(\Omega)$. Then, $\int_\Omega y Du_i\, dx \to \int_\Omega y Du\, dx$ for each $y \in W^{1,p}(\Omega, C\ell_n)$. This yields $Du_i \rightharpoonup Du$ in $L^p(\Omega, C\ell_n)$.

The next stage is to extract a further subsequence, so that $Du_i \to Du$ pointwise almost everywhere in Ω.

Because of $u_i - f \in W_0^{1,p_i}(\Omega, C\ell_n^k)$ and $u_i - f \to u - f$, we get $u - f \in W_0^{1,p_i}(\Omega, C\ell_n^k)$ a.e. in Ω, that is to say, $u \in \mathcal{K}_\psi^{f,p_i}(\Omega, C\ell_n^k)$. So,

$$
\int_\Omega \langle A_{p_i}(x, Du_i), Du - Du_i \rangle\, dx \geq 0. \tag{35}
$$

Moreover, by the convergence assumption, we obtain

$$
A_{p_i}(x, Du) \to A_p(x, Du)
$$

almost everywhere in Ω.

From this, if follows that $u_i \to u$ in $W^{1,p+\delta'}(\Omega, C\ell_n^k)$ for all $\delta' < \delta$.

At last, we will show that $u = u_0$. Using Lemma (4.4), we get $u - f \in W_0^{1,p}(\Omega, C\ell_n^k)$, then $u \in \mathcal{K}_\psi^{f,p}(\Omega, C\ell_n^k)$. This yields

$$
\int_\Omega \langle A_p(x, Du_0), Du - Du_0 \rangle\, dx \geq 0. \tag{36}
$$

On the other hand,

$$
\int_\Omega \langle A_{p_i}(x, Du_i), Du_0 - Du_i \rangle\, dx \geq 0.
$$

Let $i \to \infty$, since $u_i \to u$ in $W^{1,p}(\Omega, C\ell_n^k)$, we get

$$\int_{\Omega} \langle A_p(x, Du), Du_0 - Du \rangle \, dx \geq 0. \tag{37}$$

Combining (36) and (37), we have

$$0 \leq \int_{\Omega} \langle A_p(x, Du) - A_p(x, Du_0), Du - Du_0 \rangle \, dx \leq 0.$$

It follows that $u = u_0$. This proof is completed. $\qquad\square$

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