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Iterative algorithms for a system of generalized variational inequalities in Hilbert spaces

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Abstract

In this paper, a new system of generalized nonlinear variational inequalities involving three operators is introduced. A three-step iterative algorithm is considered for the system of generalized nonlinear variational inequalities. Strong convergence theorems of the three-step iterative algorithm are established.

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1 Introduction

Variational inequalities are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics, transportation equilibrium and engineering sciences. There exists a vast amount of literature (see, for instance, [1–26]) on the approximation solvability of nonlinear variational inequalities as well as operator equations.

Iterative algorithms have played a central role in the approximation solvability, especially of nonlinear variational inequalities as well as of nonlinear equations, in several fields such as applied mathematics, mathematical programming, mathematical finance, control theory and optimization, engineering sciences and others. Projection methods have played a significant role in the numerical resolution of variational inequalities based on their convergence analysis. However, the convergence analysis does require some sort of strong monotonicity besides the Lipschitz continuity. There have been some recent developments where convergence analysis for projection methods under somewhat weaker conditions such as cocoercivity [28] and partial relaxed monotonicity [24] is achieved.

Recently, Chang *et al.* [17] introduced a two-step iterative algorithm for a system of nonlinear variational inequalities and established strong convergence theorems. Huang and Noor [16] introduced the so-called explicit two-step iterative algorithm for a system of nonlinear variational inequalities involving two different nonlinear operators and established strong convergence theorems.

In this paper, we consider, based on the projection method, the approximate solvability of a new system of generalized nonlinear variational inequalities involving three different nonlinear operators in the framework of Hilbert spaces. The results presented in this paper extend and improve the corresponding results announced in Huang and Noor [16], Chang *et al.* [17], Verma [24–26] and many others.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and P_C be the metric projection from H onto C.

Given nonlinear operators $T, f : C \to H$ and $g : C \to C$, we consider the problem of finding $u \in C$ such that

$$\bigl\langle g(u) - f(u) + \lambda Tu, v - g(u)\bigr\rangle \geq 0, \quad \forall v \in C, \tag{1.1}$$

where $\lambda > 0$ is a constant. The variational inequality (1.1) is called the generalized variational inequality involving three operators.

We see that an element $u \in C$ is a solution to the generalized variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping

$$I - g + P_C(f - \lambda T),$$

where I is the identity mapping. This equivalence plays an important role in this work.

If $f = g$, then the generalized variational inequality (1.1) is equivalent to the following.

Find $u \in C$ such that

$$\bigl\langle Tu, v - g(u)\bigr\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

Further, if $g = I$, then the problem (1.2) is reduced to finding $u \in C$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in C, \tag{1.3}$$

which is known as the classical variational inequality originally introduced and studied by Stampacchia [27].

Let $T : C \to H$ be a mapping. Recall the following definitions.

(1) T is said to be monotone if

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in C.$$

(2) T is called δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \delta\|x - y\|^2, \quad \forall x, y \in C.$$

This implies that

$$\|Tx - Ty\| \geq \delta\|x - y\|, \quad \forall x, y \in C,$$

that is, T is δ-expansive.

(3) T is said to be γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Clearly, every γ-cocoercive mapping A is $\frac{1}{\gamma}$-Lipschitz continuous.

(4) T is said to be relaxed γ-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

(5) T is said to be relaxed (γ, δ)-cocoercive if there exist two constants $\gamma, \delta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2 + \delta\|x - y\|^2, \quad \forall x, y \in C.$$

Let $T_i : C \times C \times C \to H$, $f_i : C \to H$ and $g_i : C \to C$ be nonlinear mappings for each $i = 1, 2, 3$. Consider a system of generalized nonlinear variational inequality (SGNVI) as follows.

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that for all $s, t, r > 0$,

$$\begin{cases} \langle sT_1(y^*, z^*, x^*) + g_1(x^*) - f_1(y^*), x - g_1(x^*)\rangle \geq 0, & \forall x \in C, \\ \langle tT_2(z^*, x^*, y^*) + g_2(y^*) - f_2(z^*), x - g_2(x^*)\rangle \geq 0, & \forall x \in C, \\ \langle rT_3(x^*, y^*, z^*) + g_3(z^*) - f_3(x^*), x - g_3(x^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.4}$$

One can easily see SGNVI (1.4) is equivalent to the following projection problem:

$$\begin{cases} g_1(x^*) = P_C(f_1(y^*) - sT_1(y^*, z^*, x^*)), & \forall s > 0, \\ g_2(y^*) = P_C(f_2(z^*) - tT_2(z^*, x^*, y^*)), & \forall t > 0, \\ g_3(z^*) = P_C(f_3(x^*) - rT_3(x^*, y^*, z^*)), & \forall r > 0. \end{cases} \tag{1.5}$$

Next, we consider some special classes of SGNVI (1.4) as follows.

(I) If $g_1 = g_2 = g_3 = I$, then SGNVI (1.4) is reduced to the following.

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that for all $s, t, r > 0$,

$$\begin{cases} \langle sT_1(y^*, z^*, x^*) + x^* - f_1(y^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle tT_2(z^*, x^*, y^*) + y^* - f_2(z^*), x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle rT_3(x^*, y^*, z^*) + z^* - f_3(x^*), x - x^*\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.6}$$

We see that the problem (1.6) is equivalent to the following projection problem:

$$\begin{cases} x^* = P_C(f_1(y^*) - sT_1(y^*, z^*, x^*)), & \forall s > 0, \\ y^* = P_C(f_2(z^*) - tT_2(z^*, x^*, y^*)), & \forall t > 0, \\ z^* = P_C(f_3(x^*) - rT_3(x^*, y^*, z^*)), & \forall r > 0. \end{cases} \tag{1.7}$$

(II) If $f_1 = f_2 = f_3 = I$, then SGNVI (1.4) is reduced to the following.

Find $(x^*, y^*, z^*) \in C \times C \times C$ such that for all $s, t, r > 0$,

$$\begin{cases} \langle sT_1(y^*, z^*, x^*) + g_1(x^*) - y^*, x - g_1(x^*)\rangle \geq 0, & \forall x \in C, \\ \langle tT_2(z^*, x^*, y^*) + g_2(y^*) - z^*, x - g_2(x^*)\rangle \geq 0, & \forall x \in C, \\ \langle rT_3(x^*, y^*, z^*) + g_3(z^*) - x^*, x - g_3(x^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.8}$$

We see that the problem (1.8) is equivalent to the following projection problem:

$$\begin{cases} g_1(x^*) = P_C[y^* - sT_1(y^*, z^*, x^*)], & \forall s > 0, \\ g_2(y^*) = P_C[z^* - tT_2(z^*, x^*, y^*)], & \forall t > 0, \\ g_3(z^*) = P_C[x^* - rT_3(x^*, y^*, z^*)], & \forall r > 0. \end{cases} \tag{1.9}$$

(III) If $g_1 = g_2 = g_3 = f_1 = f_2 = f_3 = I$, then SGNVI (1.4) is reduced to the following.
Find $(x^*, y^*, z^*) \in C \times C \times C$ such that for all $s, t, r > 0$,

$$\begin{cases} \langle sT_1(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle tT_2(z^*, x^*, y^*) + y^* - z^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle rT_3(x^*, y^*, z^*) + z^* - x^*, x - x^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.10}$$

One can easily get that the problem (1.10) is equivalent to the following projection problem:

$$\begin{cases} x^* = P_C(y^* - sT_1(y^*, z^*, x^*)), & \forall s > 0, \\ y^* = P_C(z^* - tT_2(z^*, x^*, y^*)), & \forall t > 0, \\ z^* = P_C(x^* - rT_3(x^*, y^*, z^*)), & \forall r > 0. \end{cases} \tag{1.11}$$

(IV) If T_1, T_2 and T_3 are univariate mappings, then SGNVI (1.4) is reduced to the following.
Find $(x^*, y^*, z^*) \in C \times C \times C$ such that for all $s, t, r > 0$,

$$\begin{cases} \langle sT_1 y^* + g_1(x^*) - f_1(y^*), x - g_1(x^*) \rangle \geq 0, & \forall x \in C, \\ \langle tT_2 z^* + g_2(y^*) - f_2(z^*), x - g_2(x^*) \rangle \geq 0, & \forall x \in C, \\ \langle rT_3 x^* + g_3(z^*) - f_3(x^*), x - g_3(x^*) \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.12}$$

One can easily see that the problem (1.12) is equivalent to the following projection problem:

$$\begin{cases} g_1(x^*) = P_C(f_1(y^*) - sT_1 y^*), & \forall s > 0, \\ g_2(y^*) = P_C(f_2(z^*) - tT_2 z^*), & \forall t > 0, \\ g_3(z^*) = P_C(f_3(x^*) - rT_3 x^*), & \forall r > 0. \end{cases} \tag{1.13}$$

2 Preliminaries

In this section, to study the approximate solvability of the problems (1.4), (1.6), (1.8), (1.10) and (1.12), we introduce the following three-step algorithms.

Algorithm 2.1 For any $(x_0, y_0, z_0) \in C \times C \times C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following iterative process:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(x_n - g_1(x_n) + P_C(f_1(y_n) - sT_1(y_n, z_n, x_n))), & n \geq 0, \\ g_3(z_{n+1}) = P_C(f_3(x_{n+1}) - rT_3(x_{n+1}, y_n, z_n)), & n \geq 0, \\ g_2(y_{n+1}) = P_C(f_2(z_{n+1}) - tT_2(z_{n+1}, x_n, y_n)), & n \geq 0, \end{cases}$$

where $r, s, t > 0$ are three constants and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

If $g_1 = g_2 = g_3 = I$, then Algorithm 2.1 is reduced to the following.

Algorithm 2.2 For any $(x_0, y_0, z_0) \in C \times C \times C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following iterative process:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(f_1(y_n) - sT_1(y_n, z_n, x_n)), & n \geq 0, \\ z_{n+1} = P_C(f_3(x_{n+1}) - rT_3(x_{n+1}, y_n, z_n)), & n \geq 0, \\ y_{n+1} = P_C(f_2(z_{n+1}) - tT_2(z_{n+1}, x_n, y_n)), & n \geq 0, \end{cases}$$

where $r, s, t > 0$ are three constants and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

If $f_1 = f_2 = f_3 = I$, the identity mapping, then Algorithm 2.1 is reduced to the following.

Algorithm 2.3 For any $(x_0, y_0, z_0) \in C \times C \times C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following iterative process:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_n - g_1(x_n) + P_C(y_n - sT_1(y_n, z_n, x_n))), & n \geq 0, \\ g_3(z_{n+1}) = P_C(x_{n+1} - rT_3(x_{n+1}, y_n, z_n)), & n \geq 0, \\ g_2(y_{n+1}) = P_C(z_{n+1} - tT_2(z_{n+1}, x_n, y_n)), & n \geq 0, \end{cases}$$

where $r, s, t > 0$ are three constants and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

If $f_1 = f_2 = f_3 = g_1 = g_2 = g_3 = I$, the identity mapping, then Algorithm 2.1 is reduced to the following.

Algorithm 2.4 For any $(x_0, y_0, z_0) \in C \times C \times C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following iterative process:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(y_n - sT_1(y_n, z_n, x_n)), & n \geq 0, \\ z_{n+1} = P_C(x_{n+1} - rT_3(x_{n+1}, y_n, z_n)), & n \geq 0, \\ y_{n+1} = P_C(z_{n+1} - tT_2(z_{n+1}, x_n, y_n)), & n \geq 0, \end{cases}$$

where $r, s, t > 0$ are three constants and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

(IV) If T_1, T_2 and T_3 are univariate mappings, then Algorithm 2.1 is reduced to the following.

Algorithm 2.5 For any $(x_0, y_0, z_0) \in C \times C \times C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the following iterative process:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_n - g_1(x_n) + P_C(f_1(y_n) - sT_1 y_n)), & n \geq 0, \\ g_3(z_{n+1}) = P_C(f_3(x_{n+1}) - rT_3 x_{n+1}), & n \geq 0, \\ g_2(y_{n+1}) = P_C(f_2(z_{n+1}) - tT_2 z_{n+1}), & n \geq 0, \end{cases}$$

where $r, s, t > 0$ are three constants and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In order to prove our main results, we also need the following lemma and definitions.

Lemma 2.6 [29] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\lambda_n)a_n + b_n, \quad \forall n \ge n_0,$$

where n_0 *is a nonnegative integer,* $\{\lambda_n\}$ *is a sequence in* $(0,1)$ *with* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *and* $b_n = \circ(\lambda_n)$, *then* $\lim_{n\to\infty} a_n = 0$.

Definition 2.7 A mapping $T : C \times C \times C \to H$ is said to be relaxed (γ,δ)-cocoercive if there exist constants $\gamma,\delta > 0$ such that for all $x, x' \in C$,

$$\begin{aligned}&\langle T(x,y,z) - T(x',y',z'), x - x'\rangle \\ &\quad \ge (-\gamma)\|T(x,y,z) - T(x',y',z')\|^2 + \delta\|x - x'\|^2, \quad \forall y,y',z,z' \in C.\end{aligned}$$

Definition 2.8 A mapping $T : C \times C \times C \to H$ is said to be β-Lipschitz continuous in the first variable if there exists a constant $\beta > 0$ such that for all $x, x' \in C$,

$$\|T(x,y,z) - T(x',y',z')\| \le \beta\|x - x'\|, \quad \forall y,y',z,z' \in C.$$

3 Main results

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let* $T_i : C \times C \times C \to H$ *be a relaxed* (γ_i, δ_i)*-cocoercive and* β_i*-Lipschitz continuous mapping in the first variable,* $f_i : C \to H$ *be a relaxed* (η_i, ρ_i)*-cocoercive and* λ_i*-Lipschitz continuous mapping and* $g_i : C \to C$ *be a relaxed* $(\bar{\eta}_i, \bar{\rho}_i)$*-cocoercive and* $\bar{\lambda}_i$*-Lipschitz continuous mapping for each* $i = 1,2,3$. *Suppose that* $(x^*, y^*, z^*) \in C \times C \times C$ *is a solution to the problem* (1.4). *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be the sequences generated by Algorithm* 2.1. *Assume that the following conditions are satisfied*:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $0 \le \theta_6, \theta_9 < 1$;
(c) $(\theta_1 + \theta_2)(\theta_4 + \theta_5)(\theta_7 + \theta_8) \le (1-\theta_3)(1-\theta_6)(1-\theta_9)$,

where

$$\begin{aligned}
&\theta_1 = \sqrt{1 - 2s\delta_1 + 2s\gamma_1\beta_1^2 + s^2\beta_1^2}, \qquad \theta_2 = \sqrt{1 - 2\rho_1 + \lambda_1^2 + 2\eta_1\lambda_1^2},\\
&\theta_3 = \sqrt{1 - 2\bar{\rho}_1 + \bar{\lambda}_1^2 + 2\bar{\eta}_1\bar{\lambda}_1^2}, \qquad \theta_4 = \sqrt{1 - 2t\delta_2 + 2t\gamma_2\beta_2^2 + t^2\beta_2^2},\\
&\theta_5 = \sqrt{1 - 2\rho_2 + \lambda_2^2 + 2\eta_2\lambda_2^2}, \qquad \theta_6 = \sqrt{1 - 2\bar{\rho}_2 + \bar{\lambda}_2^2 + 2\bar{\eta}_2\bar{\lambda}_2^2},\\
&\theta_7 = \sqrt{1 - 2r\delta_3 + 2r\gamma_3\beta_3^2 + r^2\beta_3^2}, \qquad \theta_8 = \sqrt{1 - 2\rho_3 + \lambda_3^2 + 2\eta_3\lambda_3^2},
\end{aligned}$$

and

$$\theta_9 = \sqrt{1 - 2\bar{\rho}_3 + \bar{\lambda}_3^2 + 2\bar{\eta}_3\bar{\lambda}_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

Proof In view of (x^*, y^*, z^*) being a solution to the problem (1.4), we see that

$$\begin{cases} x^* = (1-\alpha_n)x^* + \alpha_n(x^* - g_1(x^*) + P_C(f_1(y^*) - sT_1(y^*, z^*, x^*))), & n \geq 0, \\ g_3(z^*) = P_C(f_3(x^*) - rT_3(x^*, y^*, z^*)), & n \geq 0, \\ g_2(y^*) = P_C(f_2(z^*) - tT_2(z^*, x^*, y^*)), & n \geq 0. \end{cases}$$

It follows from Algorithm (2.1) that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad = \|(1-\alpha_n)x_n + \alpha_n\big(x_n - g_1(x_n) + P_C\big(f_1(y_n) - sT_1(y_n, z_n, x_n)\big)\big) - x^*\| \\ &\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|\big(x_n - g_1(x_n) + P_C\big(f_1(y_n) - sT_1(y_n, z_n, x_n)\big)\big) \\ &\qquad - \big(x^* - g_1\big(x^*\big) + P_C\big(f_1\big(y^*\big) - sT_1\big(y^*, z^*, x^*\big)\big)\big)\| \\ &\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - \big(g_1(x_n) - g_1\big(x^*\big)\big)\| + \alpha_n\|f_1(y_n) - f_1\big(y^*\big) \\ &\qquad - s\big(T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big)\big)\| \\ &\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - \big(g_1(x_n) - g_1\big(x^*\big)\big)\| \\ &\qquad + \alpha_n\|y_n - y^* - \big(f_1(y_n) - f_1\big(y^*\big)\big)\| \\ &\qquad + \alpha_n\|\big(y_n - y^*\big) - s\big(T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big)\big)\|. \end{aligned} \tag{3.1}$$

By the assumption that T_1 is relaxed (γ_1, r_1)-cocoercive and β_1-Lipschitz continuous in the first variable, we obtain that

$$\begin{aligned} &\|\big(y_n - y^*\big) - s\big(T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big)\big)\|^2 \\ &\quad = \|y_n - y^*\|^2 - 2s\langle T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big), y_n - y^*\rangle \\ &\qquad + s^2\|T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big)\|^2 \\ &\quad \leq \|y_n - y^*\|^2 - 2s\big((-\gamma_1)\|T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big)\|^2 + \delta_1\|y_n - y^*\|^2\big) \\ &\qquad + s^2\|T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big)\|^2 \\ &\quad = (1 - 2s\delta_1)\|y_n - y^*\|^2 + \big(2s\gamma_1 + s^2\big)\|T_1(y_n, z_n, x_n) - T_1\big(y^*, z^*, x^*\big)\|^2 \\ &\quad \leq (1 - 2s\delta_1)\|y_n - y^*\|^2 + \big(2s\gamma_1 + s^2\big)\beta_1^2\|y_n - y^*\|^2 \\ &\quad = \theta_1^2\|y_n - y^*\|^2, \end{aligned} \tag{3.2}$$

where $\theta_1 = \sqrt{1 - 2s\delta_1 + 2s\gamma_1\beta_1^2 + s^2\beta_1^2}$. On the other hand, it follows from the assumption that f_1 is relaxed (η_1, ρ_1)-cocoercive and λ_1-Lipschitz continuous that

$$\begin{aligned} &\|y_n - y^* - \big(f_1(y_n) - f_1\big(y^*\big)\big)\|^2 \\ &\quad = \|y_n - y^*\|^2 - 2\langle f_1(y_n) - f_1\big(y^*\big), y_n - y^*\rangle + \|f_1(y_n) - f_1\big(y^*\big)\|^2 \\ &\quad \leq \|y_n - y^*\|^2 - 2\big((-\eta_1)\|f_1(y_n) - f_1\big(y^*\big)\|^2 + \rho_1\|y_n - y^*\|^2\big) + \lambda_1^2\|y_n - y^*\|^2 \\ &\quad = \big(1 - 2\rho_1 + \lambda_1^2\big)\|y_n - y^*\|^2 + 2\eta_1\|f_1(y_n) - f_1\big(y^*\big)\|^2 \\ &\quad \leq \theta_2^2\|y_n - y^*\|^2, \end{aligned} \tag{3.3}$$

where $\theta_2 = \sqrt{1 - 2\rho_1 + \lambda_1^2 + 2\eta_1\lambda_1^2}$. In a similar way, we can obtain that

$$\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \le \theta_3 \|x_n - x^*\|, \tag{3.4}$$

where $\theta_3 = \sqrt{1 - 2\bar{\rho}_1 + \bar{\lambda}_1^2 + 2\bar{\eta}_1\bar{\lambda}_1^2}$. Substituting (3.2), (3.3) and (3.4) into (3.1), we arrive at

$$\|x_{n+1} - x^*\| \le [1 - \alpha_n(1 - \theta_3)]\|x_n - x^*\| + \alpha_n(\theta_1 + \theta_2)\|y_n - y^*\|. \tag{3.5}$$

Next, we estimate $\|y_n - y^*\|$. From Algorithm 2.1, we see that

$$\begin{aligned} \|g_2(y_{n+1}) - g_2(y^*)\| &= \|P_C(f_2(z_{n+1}) - tT_2(z_{n+1}, x_n, y_n)) \\ &\quad - P_C(f_2(z^*) - tT_2(z^*, x^*, y^*))\| \\ &\le \|(f_2(z_{n+1}) - tT_2(z_{n+1}, x_n, y_n)) - (f_2(z^*) - tT_2(z^*, x^*, y^*))\| \\ &\le \|(z_{n+1} - z^*) - (f_2(z_{n+1}) - f_2(z^*))\| \\ &\quad + \|(z_{n+1} - z^*) - t(T_2(z_{n+1}, x_n, y_n) - T_2(z^*, x^*, y^*))\|. \end{aligned} \tag{3.6}$$

By the assumption that T_2 is relaxed (γ_2, r_2)-cocoercive and β_2-Lipschitz continuous in the first variable, we obtain that

$$\begin{aligned} &\|(z_{n+1} - z^*) - t(T_2(z_{n+1}, x_n, y_n) - T_2(z^*, x^*, y^*))\|^2 \\ &\quad = \|z_{n+1} - z^*\|^2 - 2t\langle T_2(z_{n+1}, x_n, y_n) - T_2(z^*, x^*, y^*), z_{n+1} - z^*\rangle \\ &\qquad + t^2\|T_2(z_{n+1}, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\ &\quad \le \|z_{n+1} - z^*\|^2 - 2t((-\gamma_2)\|T_2(z_{n+1}, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 + \delta_2\|z_{n+1} - z^*\|^2) \\ &\qquad + t^2\|T_2(z_{n+1}, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\ &\quad = (1 - 2t\delta_2)\|z_{n+1} - z^*\|^2 + (2t\gamma_2 + t^2)\|T_2(z_{n+1}, x_n, y_n) - T_2(z^*, x^*, y^*)\|^2 \\ &\quad \le (1 - 2t\delta_2)\|z_{n+1} - z^*\|^2 + (2t\gamma_2 + t^2)\beta_2^2\|z_{n+1} - z^*\|^2 \\ &\quad = \theta_4^2\|z_{n+1} - z^*\|^2, \end{aligned} \tag{3.7}$$

where $\theta_4 = \sqrt{1 - 2t\delta_2 + 2t\gamma_2\beta_2^2 + t^2\beta_2^2}$. It follows from the assumption that f_2 is relaxed (η_2, ρ_2)-cocoercive and λ_2-Lipschitz continuous that

$$\begin{aligned} &\|(z_{n+1} - z^*) - (f_2(z_{n+1}) - f_2(z^*))\|^2 \\ &\quad = \|z_{n+1} - z^*\|^2 - 2\langle f_2(z_{n+1}) - f_2(z^*), z_n - z^*\rangle + \|f_2(z_{n+1}) - f_2(z^*)\|^2 \\ &\quad \le \|z_{n+1} - z^*\|^2 - 2((-\eta_2)\|f_2(z_{n+1}) - f_2(z^*)\|^2 + \rho_2\|z_{n+1} - z^*\|^2) + \lambda_2^2\|z_{n+1} - z^*\|^2 \\ &\quad = (1 - 2\rho_2 + \lambda_2^2)\|z_{n+1} - z^*\|^2 + 2\eta_2\|f_2(z_{n+1}) - f_2(z^*)\|^2 \\ &\quad = \theta_5^2\|z_{n+1} - z^*\|^2, \end{aligned} \tag{3.8}$$

where $\theta_5 = \sqrt{1 - 2\rho_2 + \lambda_2^2 + 2\eta_2\lambda_2^2}$. Substituting (3.7) and (3.8) into (3.6), we see that

$$\|g_2(y_{n+1}) - g_2(y^*)\| \le (\theta_4 + \theta_5)\|z_{n+1} - z^*\|. \tag{3.9}$$

On the other hand, we have

$$\|y_{n+1} - y^*\| \leq \|y_{n+1} - y^* - (g_2(y_{n+1}) - g_2(y^*))\| + \|g_2(y_{n+1}) - g_2(y^*)\|. \tag{3.10}$$

From the proof of (3.8), we arrive at

$$\|(y_{n+1} - y^*) - (g_2(y_{n+1}) - g_2(y^*))\| \leq \theta_6 \|y_{n+1} - y^*\|, \tag{3.11}$$

where $\theta_6 = \sqrt{1 - 2\bar{\rho}_2 + \bar{\lambda}_2^2 + 2\bar{\eta}_2\bar{\lambda}_2^2}$. Substituting (3.9) and (3.11) into (3.10), we see that

$$\|y_{n+1} - y^*\| \leq \theta_6 \|y_{n+1} - y^*\| + (\theta_4 + \theta_5)\|z_{n+1} - z^*\|.$$

It follows from the condition (b) that

$$\|y_{n+1} - y^*\| \leq \frac{\theta_4 + \theta_5}{1 - \theta_6} \|z_{n+1} - z^*\|.$$

That is,

$$\|y_n - y^*\| \leq \frac{\theta_4 + \theta_5}{1 - \theta_6} \|z_n - z^*\|. \tag{3.12}$$

Finally, we estimate $\|z_n - z^*\|$. It follows from Algorithm 2.1 that

$$\begin{aligned} &\|g_3(z_{n+1}) - g_3(z^*)\| \\ &\quad = \|P_C(f_3(x_{n+1}) - rT_3(x_{n+1}, y_n, z_n)) - P_C[f_3(x^*) - rT_3(x^*, y^*, z^*)]\| \\ &\quad \leq \|(x_{n+1} - x^*) - (f_3(x_{n+1}) - f(x^*))\| \\ &\qquad + \|(x_{n+1} - x^*) - r(T_3(x_{n+1}, y_n, z_n) - T_3(x^*, y^*, z^*))\|. \end{aligned} \tag{3.13}$$

In a similar way, we can show that

$$\|(x_{n+1} - x^*) - r(T_3(x_{n+1}, y_n, z_n) - T_3(x^*, y^*, z^*))\| \leq \theta_7 \|x_{n+1} - x^*\| \tag{3.14}$$

and

$$\|(x_{n+1} - x^*) - (f_3(x_{n+1}) - f_3(x^*))\| \leq \theta_8 \|x_{n+1} - x^*\|, \tag{3.15}$$

where $\theta_7 = \sqrt{1 - 2r\delta_3 + 2r\gamma_3\beta_3^2 + r^2\beta_3^2}$ and $\theta_8 = \sqrt{1 - 2\rho_3 + \lambda_3^2 + 2\eta_3\lambda_3^2}$. Substituting (3.14) and (3.15) into (3.13), we arrive at

$$\|g_3(z_{n+1}) - g_3(z^*)\| \leq (\theta_7 + \theta_8)\|x_{n+1} - x^*\|. \tag{3.16}$$

Note that

$$\|z_{n+1} - z^*\| \leq \|z_{n+1} - z^* - (g_3(z_{n+1}) - g_3(z^*))\| + \|g_3(z_{n+1}) - g_3(z^*)\|. \tag{3.17}$$

On the other hand, we have

$$\|z_{n+1} - z^* - (g_3(z_{n+1}) - g_3(z^*))\| \le \theta_9 \|z_{n+1} - z^*\|, \tag{3.18}$$

$\theta_9 = \sqrt{1 - 2\bar{\rho}_3 + \bar{\lambda}_3^2 + 2\bar{\eta}_3\bar{\lambda}_3^2}$. Substituting (3.16) and (3.18) into (3.17), we arrive at

$$\|z_{n+1} - z^*\| \le \theta_9 \|z_{n+1} - z^*\| + (\theta_7 + \theta_8)\|x_{n+1} - x^*\|.$$

It follows from the condition (b) that

$$\|z_{n+1} - z^*\| \le \frac{\theta_7 + \theta_8}{1 - \theta_9}\|x_{n+1} - x^*\|.$$

That is,

$$\|z_n - z^*\| \le \frac{\theta_7 + \theta_8}{1 - \theta_9}\|x_n - x^*\|. \tag{3.19}$$

Combining (3.5), (3.12) with (3.19), we obtain that

$$\|x_{n+1} - x^*\| \le \left(1 - \alpha_n\left(1 - \theta_3 - (\theta_1 + \theta_2)\frac{\theta_4 + \theta_5}{1 - \theta_6}\frac{\theta_7 + \theta_8}{1 - \theta_9}\right)\right)\|x_n - x^*\|.$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$ and the condition (c), we can conclude the desired conclusion easily from Lemma 2.6. This completes the proof. □

Remark 3.2 Theorem 3.1 includes the corresponding results in Huang and Noor [16] Chang *et al.* [17], and Verma [24–26] as special cases.

From Theorem 3.1, we can get the following results immediately.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \times C \times C \to H$ be a relaxed (γ_i, δ_i)-cocoercive and β_i-Lipschitz continuous mapping in the first variable and $f_i : C \to H$ be a relaxed (η_i, ρ_i)-cocoercive and λ_i-Lipschitz continuous mapping for each $i = 1, 2, 3$. Suppose that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution to the problem* (1.6). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm* 2.2. *Assume that the following conditions are satisfied*:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $(\theta_1 + \theta_2)(\theta_4 + \theta_5)(\theta_7 + \theta_8) \le 1$,

where

$$\theta_1 = \sqrt{1 - 2s\delta_1 + 2s\gamma_1\beta_1^2 + s^2\beta_1^2}, \qquad \theta_2 = \sqrt{1 - 2\rho_1 + \lambda_1^2 + 2\eta_1\lambda_1^2},$$
$$\theta_4 = \sqrt{1 - 2t\delta_2 + 2t\gamma_2\beta_2^2 + t^2\beta_2^2}, \qquad \theta_5 = \sqrt{1 - 2\rho_2 + \lambda_2^2 + 2\eta_2\lambda_2^2},$$

and

$$\theta_7 = \sqrt{1 - 2r\delta_3 + 2r\gamma_3\beta_3^2 + r^2\beta_3^2}, \qquad \theta_8 = \sqrt{1 - 2\rho_3 + \lambda_3^2 + 2\eta_3\lambda_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^*, y^* and z^*, respectively.

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \times C \times C \to H$ be a relaxed (γ_i, δ_i)-cocoercive and β_i-Lipschitz continuous mapping in the first variable and $g_i : C \to C$ be a relaxed $(\bar{\eta}_i, \bar{\rho}_i)$-cocoercive and $\bar{\lambda}_i$-Lipschitz continuous mapping for each $i = 1, 2, 3$. Suppose that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution to the problem* (1.8). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm* 2.3. *Assume that the following conditions are satisfied*:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $0 \leq \theta_6, \theta_9 < 1$;
(c) $\theta_1\theta_4\theta_7 \leq (1 - \theta_3)(1 - \theta_6)(1 - \theta_9)$,

where

$$\theta_1 = \sqrt{1 - 2s\delta_1 + 2s\gamma_1\beta_1^2 + s^2\beta_1^2}, \qquad \theta_3 = \sqrt{1 - 2\bar{\rho}_1 + \bar{\lambda}_1^2 + 2\bar{\eta}_1\bar{\lambda}_1^2},$$
$$\theta_4 = \sqrt{1 - 2t\delta_2 + 2t\gamma_2\beta_2^2 + t^2\beta_2^2}, \qquad \theta_6 = \sqrt{1 - 2\bar{\rho}_2 + \bar{\lambda}_2^2 + 2\bar{\eta}_2\bar{\lambda}_2^2},$$

and

$$\theta_7 = \sqrt{1 - 2r\delta_3 + 2r\gamma_3\beta_3^2 + r^2\beta_3^2}, \qquad \theta_9 = \sqrt{1 - 2\bar{\rho}_3 + \bar{\lambda}_3^2 + 2\bar{\eta}_3\bar{\lambda}_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^*, y^* and z^*, respectively.

Corollary 3.5 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \times C \times C \to H$ be a relaxed (γ_i, δ_i)-cocoercive and β_i-Lipschitz continuous mapping in the first variable for each $i = 1, 2, 3$. Suppose that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution to the problem* (1.10). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm* 2.4. *Assume that the following conditions are satisfied*:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $\theta_1\theta_4\theta_7 \leq 1$,

where

$$\theta_1 = \sqrt{1 - 2s\delta_1 + 2s\gamma_1\beta_1^2 + s^2\beta_1^2}, \qquad \theta_4 = \sqrt{1 - 2t\delta_2 + 2t\gamma_2\beta_2^2 + t^2\beta_2^2},$$

and

$$\theta_7 = \sqrt{1 - 2r\delta_3 + 2r\gamma_3\beta_3^2 + r^2\beta_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^*, y^* and z^*, respectively.

Corollary 3.6 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to H$ be a relaxed (γ_i, δ_i)-cocoercive and β_i-Lipschitz continuous mapping, $f_i : C \to H$ be a relaxed (η_i, ρ_i)-cocoercive and λ_i-Lipschitz continuous mapping and $g_i : C \to C$ be a relaxed $(\bar{\eta}_i, \bar{\rho}_i)$-cocoercive and $\bar{\lambda}_i$-Lipschitz continuous mapping for each $i = 1, 2, 3$. Suppose that $(x^*, y^*, z^*) \in C \times C \times C$ is a solution to the problem* (1.12). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences generated by Algorithm* 2.5. *Assume that the following conditions are satisfied*:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $0 \leq \theta_6, \theta_9 < 1$;
(c) $(\theta_1 + \theta_2)(\theta_4 + \theta_5)(\theta_7 + \theta_8) \leq (1 - \theta_3)(1 - \theta_6)(1 - \theta_9)$,

where

$$\theta_1 = \sqrt{1 - 2s\delta_1 + 2s\gamma_1\beta_1^2 + s^2\beta_1^2}, \qquad \theta_2 = \sqrt{1 - 2\rho_1 + \lambda_1^2 + 2\eta_1\lambda_1^2},$$
$$\theta_3 = \sqrt{1 - 2\bar{\rho}_1 + \bar{\lambda}_1^2 + 2\bar{\eta}_1\bar{\lambda}_1^2}, \qquad \theta_4 = \sqrt{1 - 2t\delta_2 + 2t\gamma_2\beta_2^2 + t^2\beta_2^2},$$
$$\theta_5 = \sqrt{1 - 2\rho_2 + \lambda_2^2 + 2\eta_2\lambda_2^2}, \qquad \theta_6 = \sqrt{1 - 2\bar{\rho}_2 + \bar{\lambda}_2^2 + 2\bar{\eta}_2\bar{\lambda}_2^2},$$
$$\theta_7 = \sqrt{1 - 2r\delta_3 + 2r\gamma_3\beta_3^2 + r^2\beta_3^2}, \qquad \theta_8 = \sqrt{1 - 2\rho_3 + \lambda_3^2 + 2\eta_3\lambda_3^2},$$

and

$$\theta_9 = \sqrt{1 - 2\bar{\rho}_3 + \bar{\lambda}_3^2 + 2\bar{\eta}_3\bar{\lambda}_3^2}.$$

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* x^*, y^* *and* z^*, *respectively.*

Competing interests
The author declares that they have no competing interests.

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