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Sums of products of the degenerate Euler numbers

Ming Wu¹ and Hao Pan^{2*}

*Correspondence:

haopan1979@gmail.com

²Department of Mathematics,
Nanjing University, Nanjing, 210093,
People's Republic of China
Full list of author information is
available at the end of the article**Abstract**

The paper focuses on the degenerate Euler numbers, the degenerate Euler polynomials and the degenerate Bernoulli polynomials. By adopting the method of recurrences, two explicit expressions have been established for sums of products of the degenerate Euler polynomials and the degenerate Bernoulli polynomials. As a special case of the degenerate Euler polynomials, an expression can be obtained for

$$\sum_{\substack{j_1+j_2+\dots+j_N=n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} \varepsilon_{j_1}(\lambda) \varepsilon_{j_2}(\lambda) \cdots \varepsilon_{j_N}(\lambda).$$

MSC: Primary 11B68; secondary 11B65; 11B73**Keywords:** degenerate Euler number; degenerate Euler polynomial; degenerate Bernoulli polynomial

1 Introduction

The Bernoulli numbers B_n are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The study of the Bernoulli numbers has a very long history. In fact, Euler had found that

$$\sum_{k=0}^n \binom{n}{k} B_k B_{n-k} = -(n-1)B_n - nB_{n-1} \quad (1.1)$$

for any $n \geq 1$.

In 1996, Dilcher [1] generalized (1.1) to the sums of products of N Bernoulli numbers:

$$\begin{aligned} & \sum_{\substack{j_1+j_2+\dots+j_N=n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} B_{j_1} B_{j_2} \cdots B_{j_N} \\ &= N \binom{n}{N} \sum_{j=0}^{N-1} (-1)^{N-1-j} s(N, N-j) \cdot \frac{B_{n-j}}{n-j}, \end{aligned} \quad (1.2)$$

where $\binom{n}{j_1, j_2, \dots, j_N}$ are the multinomial coefficients defined by

$$\binom{n}{j_1, j_2, \dots, j_N} = \frac{n!}{j_1! j_2! \cdots j_N!}$$

and $s(n, k)$ are the Stirling numbers of the first kind (see, e.g., [2]). Furthermore, Dilcher also extended (1.2) to the Bernoulli polynomials.

On the other hand, in [3, 4], Carlitz defined the degenerate Bernoulli numbers $\beta_n(\lambda)$ as

$$\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{t^n}{n!}. \tag{1.3}$$

Carlitz showed that $\beta_n(\lambda)$ is a polynomial in λ . And the explicit formula for $\beta_n(\lambda)$ was obtained by Howard [5]. Note that $(1 + \lambda t)^{1/\lambda}$ tends to e^t as $\lambda \rightarrow 0$. So we have $\beta_n(0) = B_n$. Furthermore, we know that $\lambda^{-m} \beta_m(\lambda)$ tends to $m!b_m$ as $\lambda \rightarrow +\infty$, where b_n is the Bernoulli number of the second kind given by

$$\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} b_n t^n.$$

For more number-theoretical properties of the degenerate Bernoulli numbers, the readers may refer to [6, 7] and [8].

The analogues of (1.1) and (1.2) for b_n have been given in [9] and [10]. Moreover, in [11], Zhang and Yang obtained a generalization of (1.2) for $\beta_n(\lambda)$:

$$\begin{aligned} & \sum_{\substack{j_1+j_2+\dots+j_N=n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} \beta_{j_1}(\lambda) \beta_{j_2}(\lambda) \cdots \beta_{j_N}(\lambda) \\ &= N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^{N-1-k} \sigma_{N-1, k}(\lambda, n) \cdot \frac{\beta_{n-k}(\lambda)}{n-k}, \end{aligned} \tag{1.4}$$

where

$$\sigma_{N-1, k}(\lambda, n) = \sum_{1 \leq i_k < i_{k-1} < \dots < i_2 < i_1 \leq N-1} \prod_{j=1}^k ((\lambda - 1)i_j - (n - j)\lambda).$$

The Euler numbers E_n are another important kind of numbers, which are defined by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

The Euler numbers have many similar properties as the Bernoulli numbers. For example, in the same paper, Dilcher also proved that

$$\begin{aligned} & \sum_{\substack{j_1+j_2+\dots+j_N=n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{2n}{2j_1, 2j_2, \dots, 2j_N} E_{2j_1} E_{2j_2} \cdots E_{2j_N} \\ &= \frac{2^{2n+N-1}}{(N-1)!} \sum_{j=0}^{[(N-1)/2]} \left(\sum_{k=0}^{2j} \binom{N-1-k}{2j-k} s(N, N-k) \cdot \frac{N^{2j-k}}{2^{2j-k}} \right) E_{2n+N-1-2j} \left(\frac{N}{2} \right), \end{aligned} \tag{1.5}$$

where $2n > N$ and the Euler polynomials $E_n(x)$ are given by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

In [12], Wang obtained an explicit expression for sums of products of l Bernoulli polynomials and $n - l$ Euler polynomials.

In [3], Carlitz also defined the degenerate Euler numbers $\varepsilon_n(\lambda)$ by

$$\frac{2}{(1 + \lambda t)^{1/\lambda} + (1 + \lambda t)^{-1/\lambda}} = \sum_{n=0}^{\infty} \varepsilon_n(\lambda) \frac{t^n}{n!}.$$

Motivated by (1.4), we shall establish a generalization of (1.5) for $\varepsilon_n(\lambda)$ in this note. We define a class of generalized Stirling-like polynomials of the first kind as follows:

$$\tau_{m,k}(\lambda, x, n) = \sum_{1 \leq i_k < i_{k-1} < \dots < i_2 < i_1 \leq m} \prod_{j=1}^k (x + (\lambda - 1)i_j - (n + m + 1 - j)\lambda).$$

In particular, here we set $\tau_{m,0}(\lambda, x, n) = 1$.

Theorem 1.1 For $1 \leq N \leq n$, we have

$$\begin{aligned} & \sum_{\substack{j_1 + j_2 + \dots + j_N = n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} \varepsilon_{j_1}(\lambda) \varepsilon_{j_2}(\lambda) \dots \varepsilon_{j_N}(\lambda) \\ &= \frac{2^{n+N-1}}{(N-1)!} \sum_{k=0}^{N-1} (-1)^k \tau_{N-1,k} \left(\frac{\lambda}{2}, \frac{N}{2}, n \right) \varepsilon_{n+N-1-k} \left(\frac{\lambda}{2}, \frac{N}{2} \right), \end{aligned} \tag{1.6}$$

where the degenerate Euler polynomials $\varepsilon_n(\lambda, x)$ are given by

$$\frac{2(1 + \lambda t)^{x/\lambda}}{(1 + \lambda t)^{1/\lambda} + 1} = \sum_{n=0}^{\infty} \varepsilon_n(\lambda, x) \frac{t^n}{n!}.$$

In fact, we shall prove a polynomial extension of (1.6) in the next section. In the third section, we also establish a generalization of (1.4) for the degenerate Bernoulli polynomial $\beta_n(\lambda, x)$ (see, e.g., [3]).

2 Degenerate Euler numbers and polynomials

Note that $\varepsilon_n(\lambda) = 2^n \varepsilon_n(\lambda/2, 1/2)$. So (1.6) is evidently a consequence of the following theorem.

Theorem 2.1 Let $y = x_1 + x_2 + \dots + x_N$. Then for $1 \leq N \leq n$, we have

$$\begin{aligned} & \sum_{\substack{j_1 + j_2 + \dots + j_N = n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} \varepsilon_{j_1}(\lambda, x_1) \varepsilon_{j_2}(\lambda, x_2) \dots \varepsilon_{j_N}(\lambda, x_N) \\ &= \frac{2^{N-1}}{(N-1)!} \sum_{k=0}^{N-1} (-1)^k \tau_{N-1,k}(\lambda, y, n) \varepsilon_{n+N-1-k}(\lambda, y). \end{aligned} \tag{2.1}$$

The degenerate Euler polynomials of order m are defined by

$$\left(\frac{2}{(1 + \lambda t)^{1/\lambda} + 1}\right)^m (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \varepsilon_n^{(m)}(\lambda, x) \frac{t^n}{n!}. \tag{2.2}$$

Clearly, $\varepsilon_n^{(1)}(\lambda, x) = \varepsilon_n(\lambda, x)$.

Lemma 2.1 For $1 \leq m \leq n$, we have

$$\varepsilon_n^{(m)}(\lambda, x) = \frac{2^{m-1}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \tau_{m-1,k}(\lambda, x, n) \varepsilon_{n+m-1-k}(\lambda, x). \tag{2.3}$$

Proof Observe that

$$\begin{aligned} & \frac{2^{m+1}(1 + \lambda t)^{x/\lambda}}{((1 + \lambda t)^{1/\lambda} + 1)^{m+1}} \\ &= \frac{2(1 + \lambda t)}{m} \cdot \frac{d}{dt} \left(\frac{2^m(1 + \lambda t)^{x/\lambda}}{((1 + \lambda t)^{1/\lambda} + 1)^m} \right) \\ & \quad - \frac{2(x - m)}{m} \cdot \frac{2^m(1 + \lambda t)^{x/\lambda}}{((1 + \lambda t)^{1/\lambda} + 1)^m}. \end{aligned}$$

Comparing the coefficients of t^n in both sides of the above equation, we have

$$\varepsilon_n^{(m+1)}(\lambda, x) = \frac{2}{m} (\varepsilon_{n+1}^{(m)}(\lambda, x) - (x + (\lambda - 1)m - (n + m)\lambda) \varepsilon_n^{(m)}(\lambda, x)). \tag{2.4}$$

Below we use induction on m to show (2.3). It is easy to see that (2.3) holds for $m = 1$. Now let $m > 1$ and assume that (2.3) holds for the smaller values of m . Then, by the induction hypothesis, we have

$$\begin{aligned} & \frac{2}{m} \cdot \varepsilon_{n+1}^{(m)}(\lambda, x) - \frac{2}{m} \cdot (x + (\lambda - 1)m - (n + m)\lambda) \varepsilon_n^{(m)}(\lambda, x) \\ &= \frac{2^m}{m!} \sum_{k=0}^{m-1} (-1)^k \tau_{m-1,k}(\lambda, x, n + 1) \varepsilon_{n+m-k}(\lambda, x) \\ & \quad - \frac{2^m(x + (\lambda - 1)m - (n + m)\lambda)}{m!} \sum_{k=1}^m (-1)^{k-1} \tau_{m-1,k-1}(\lambda, x, n) \varepsilon_{n+m-k}(\lambda, x). \end{aligned}$$

It is easy to verify that

$$\tau_{m,k}(\lambda, x, n) = \tau_{m-1,k}(\lambda, x, n + 1) + (x + (\lambda - 1)m - (n + m)\lambda) \tau_{m-1,k-1}(\lambda, x, n)$$

for $1 \leq k \leq m - 1$, and

$$\tau_{m,m}(\lambda, x, n) = (x + (\lambda - 1)m - (n + m)\lambda) \tau_{m-1,m-1}(\lambda, x, n).$$

So by (2.4), we get

$$\begin{aligned} \varepsilon_n^{(m+1)}(\lambda, x) &= \frac{2^m}{m!} \left(\varepsilon_{n+m}(\lambda, x) + (-1)^m \tau_{m,m}(\lambda, x, n) \varepsilon_n(\lambda, x) + \sum_{k=1}^{m-1} (-1)^k \tau_{m,k}(\lambda, x, n) \varepsilon_{n+m-k}(\lambda, x) \right) \\ &= \frac{2^m}{m!} \sum_{k=0}^m (-1)^k \tau_{m,k}(\lambda, x, n) \varepsilon_{n+m-k}(\lambda, x). \end{aligned}$$

This concludes our proof. □

Let us turn to the proof of Theorem 2.1. Clearly,

$$\begin{aligned} &\frac{2(1 + \lambda t)^{x_1/\lambda}}{(1 + \lambda t)^{1/\lambda} + 1} \cdot \frac{2(1 + \lambda t)^{x_2/\lambda}}{(1 + \lambda t)^{1/\lambda} + 1} \cdots \frac{2(1 + \lambda t)^{x_N/\lambda}}{(1 + \lambda t)^{1/\lambda} + 1} \\ &= \frac{2^N (1 + \lambda t)^{(x_1+x_2+\dots+x_N)/\lambda}}{((1 + \lambda t)^{1/\lambda} + 1)^N}. \end{aligned}$$

Hence we have

$$\sum_{\substack{j_1+j_2+\dots+j_N=n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} \varepsilon_{j_1}(\lambda, x_1) \varepsilon_{j_2}(\lambda, x_2) \cdots \varepsilon_{j_N}(\lambda, x_N) = \varepsilon_n^{(N)}(\lambda, y).$$

Thus (2.1) immediately follows from (2.3).

3 Degenerate Bernoulli numbers and polynomials

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Clearly, $B_n(0) = B_n$. In [1], Dilcher proved that

$$\begin{aligned} &\sum_{\substack{j_1+j_2+\dots+j_N=n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} B_{j_1}(x_1) B_{j_2}(x_2) \cdots B_{j_N}(x_N) \\ &= N \binom{n}{N} \sum_{j=0}^{N-1} (-1)^{N-1-j} \left(\sum_{k=0}^j \binom{n-1-k}{j-k} s(N, N-k) y^{j-k} \right) \cdot \frac{B_{n-j}(y)}{n-j}, \end{aligned} \tag{3.1}$$

where $y = x_1 + x_2 + \dots + x_N$.

The degenerate Bernoulli polynomials $\beta_n(\lambda, x)$ are defined by

$$\frac{t(1 + \lambda t)^{x/\lambda}}{(1 + \lambda t)^{1/\lambda} - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}. \tag{3.2}$$

In this section, we shall give a generalization of (3.1) for $\beta_n(\lambda, x)$.

Define $\sigma_{m,k}(\lambda, x, n)$ by

$$\sigma_{m,k}(\lambda, x, n) = \sum_{1 \leq i_k < i_{k-1} < \dots < i_2 < i_1 \leq m} \prod_{j=1}^k (x + (\lambda - 1)i_j - (n - j)\lambda).$$

In particular, we set $\sigma_{m,0}(\lambda, x, n) = 1$.

Theorem 3.1 *Let $y = x_1 + x_2 + \dots + x_N$. Then for $1 \leq N \leq n$, we have*

$$\begin{aligned} & \sum_{\substack{j_1+j_2+\dots+j_N=n \\ j_1, j_2, \dots, j_N \geq 0}} \binom{n}{j_1, j_2, \dots, j_N} \beta_{j_1}(\lambda, x_1) \beta_{j_2}(\lambda, x_2) \dots \beta_{j_N}(\lambda, x_N) \\ &= N \binom{n}{N} \sum_{k=0}^{N-1} (-1)^{N-1-k} \sigma_{N-1,k}(\lambda, y, n) \cdot \frac{\beta_{n-k}(\lambda, y)}{n-k}. \end{aligned} \tag{3.3}$$

Proof Similarly as in the proof of Theorem 2.1, it suffices to show that

$$\beta_n^{(m)}(\lambda, x) = m \binom{n}{m} \sum_{k=0}^{m-1} (-1)^{m-1-k} \sigma_{m-1,k}(\lambda, x, n) \cdot \frac{\beta_{n-k}(\lambda, x)}{n-k} \tag{3.4}$$

for $1 \leq m \leq n$, where the degenerate Bernoulli polynomials of order m are defined by

$$\left(\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} \right)^m (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \beta_n^{(m)}(\lambda, x) \frac{t^n}{n!}. \tag{3.5}$$

We shall use induction on m . Clearly (3.4) holds for $m = 1$. Let $m \geq 1$ and assume that (3.4) holds for m . Note that

$$\begin{aligned} \frac{t^{m+1}(1 + \lambda t)^{x/\lambda}}{((1 + \lambda t)^{1/\lambda} - 1)^{m+1}} &= \left((1 + \lambda t) + (x - m) \cdot \frac{t}{m} \right) \cdot \frac{t^m(1 + \lambda t)^{x/\lambda}}{((1 + \lambda t)^{1/\lambda} - 1)^m} \\ &\quad - (1 + \lambda t) \cdot \frac{t}{m} \cdot \frac{d}{dt} \left(\frac{t^m(1 + \lambda t)^{x/\lambda}}{((1 + \lambda t)^{1/\lambda} - 1)^m} \right). \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the above equation, we get

$$\beta_n^{(m+1)}(\lambda, x) = \frac{m-n}{m} \cdot \beta_n^{(m)}(\lambda, x) + \frac{n}{m} \cdot (x + (\lambda - 1)m - (n - 1)\lambda) \beta_{n-1}^{(m)}(\lambda, x). \tag{3.6}$$

Applying (3.4) for $\beta_n^{(m)}(\lambda, x)$ and $\beta_{n-1}^{(m)}(\lambda, x)$, we have

$$\begin{aligned} & \frac{m-n}{m} \cdot \beta_n^{(m)}(\lambda, x) + \frac{n}{m} \cdot (x + (\lambda - 1)m - (n - 1)\lambda) \cdot \frac{\beta_{n-1}^{(m)}(\lambda, x)}{n-1} \\ &= (m+1) \binom{n}{m+1} \left(- \sum_{k=0}^{m-1} (-1)^{m-1-k} \sigma_{m-1,k}(\lambda, x, n) \cdot \frac{\beta_{n-k}(\lambda, x)}{n-k} \right. \\ &\quad \left. + (x + (\lambda - 1)m - (n - 1)\lambda) \sum_{k=1}^m (-1)^{m-k} \sigma_{m-1,k-1}(\lambda, x, n-1) \cdot \frac{\beta_{n-k}(\lambda, x)}{n-k} \right). \end{aligned}$$

It is not difficult to check that

$$\sigma_{m,k}(\lambda, x, n) = \sigma_{m-1,k}(\lambda, x, n) + (x + (\lambda - 1)m - (n - 1)\lambda)\sigma_{m-1,k-1}(\lambda, x, n - 1)$$

for $1 \leq k \leq m - 1$, and

$$\sigma_{m,m}(\lambda, x, n) = (x + (\lambda - 1)m - (n - 1)\lambda)\sigma_{m-1,m-1}(\lambda, x, n - 1).$$

It follows from (3.6) that

$$\begin{aligned} \beta_n^{(m+1)}(\lambda, x) &= (m+1) \binom{n}{m+1} \left((-1)^m \frac{\beta_n(\lambda, x)}{n} + \sigma_{m,m}(\lambda, x, n) \frac{\beta_{n-m}(\lambda, x)}{n-m} \right) \\ &\quad + (m+1) \binom{n}{m+1} \sum_{k=1}^{m-1} (-1)^{m-k} \sigma_{m,k}(\lambda, x, n) \frac{\beta_{n-k}(\lambda, x)}{n-k} \\ &= (m+1) \binom{n}{m+1} \sum_{k=0}^m (-1)^{m-k} \sigma_{m,k}(\lambda, x, n) \cdot \frac{\beta_{n-k}(\lambda, x)}{n-k}. \end{aligned}$$

All proofs thus are done. □

Competing interests

The authors did not provide this information.

Authors' contributions

The authors did not provide this information.

Author details

¹Department of Basic Courses, JinLing Institute of Technology, Nanjing, 211169, People's Republic of China. ²Department of Mathematics, Nanjing University, Nanjing, 210093, People's Republic of China.

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