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Convergence theorems for a system of equilibrium problems and fixed point problems of a strongly nonexpansive sequence

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Abstract

The purpose of this paper is to prove a strong convergence theorem of an iterative scheme associated to a strongly nonexpansive sequence for finding a common element of the set of equilibrium problems and the set of fixed point problems of a pair of sequences of nonexpansive mappings where one of them is a strongly nonexpansive sequence. Moreover, in the last section, by using our main result, we obtain a strong convergence theorem of an iterative scheme associated to a strongly nonexpansive sequence for finding a common element of the set of a finite family of equilibrium problems and the set of fixed point problems of a pair of sequences of nonexpansive mappings where one of them is a strongly nonexpansive sequence in a Hilbert space, and we also give some examples to support our main result.

Keywords: nonexpansive mappings; strongly nonexpansive sequence; equilibrium problem; fixed point

1 Introduction

Throughout this paper, we assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. A mapping T of C into itself is called *nonexpansive* if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in H$. The set of fixed points of T is denoted by $F(T)$, *i.e.*, $F(T) = \{x \in H : Tx = x\}$. It is known that $F(T)$ is closed and convex if T is nonexpansive. Let P_C be a metric projection of H onto C, *i.e.*, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

We use "$\rightharpoonup$" and "$\to$" to denote weak and strong convergence, respectively. Let $\{T_n\}$ be a sequence of mappings of C into H. The set of common fixed points of $\{T_n\}$ is denoted by $F(\{T_n\}) = \bigcap_{n=1}^{\infty} F(T_n)$. Recall the main concepts as follows:

(1) A sequence $\{z_n\}$ in C is said to be an *approximate fixed point sequence* of $\{T_n\}$ if $z_n - T_n z_n \to 0$. The set of all *bounded* approximate fixed point sequences of $\{T_n\}$ is denoted by $\widetilde{F}(\{T_n\})$; see [1]. It is clear that if $\{T_n\}$ has a common fixed point, then $\widetilde{F}(\{T_n\})$ is nonempty.

(2) A sequence $\{T_n\}$ is said to be a *strongly nonexpansive sequence* if each T_n is nonexpansive and

$$x_n - y_n - (T_n x_n - T_n y_n) \to 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$.

(3) A sequence $\{T_n\}$ having a common fixed point is said to satisfy the *condition (Z)* if every weak cluster point of $\{x_n\}$ is a common fixed point whenever $\{x_n\} \in \widetilde{F}(\{T_n\})$.

(4) A sequence $\{T_n\}$ of nonexpansive mappings of C into H is said to satisfy the *condition (R)* if

$$\lim_{n\to\infty} \sup_{y\in D} \|T_{n+1}y - T_n y\| = 0$$

for every nonempty bounded subset D of C; see [2].

Example 1.1 Let $\mathbb{R}$ be a set of real numbers. For every $n \in \mathbb{N}$, the mapping $T_n : \mathbb{R} \to \mathbb{R}$ is defined by $T_n x = \frac{1}{n}x$ for all $x \in \mathbb{R}$.

Then $\{T_n\}$ is a nonexpansive sequence, but it is not a strongly nonexpansive sequence.

Example 1.2 For every $n \in \mathbb{N}$, the mapping $T_n : [0,1] \to [0,1]$ is defined by $T_n x = (1 - \frac{1}{n})x$ for all $x \in [0,1]$.

Then $\{T_n\}$ is a strongly nonexpansive sequence.

Solution It is easy to see that T_n is a nonexpansive mapping for all $n \in \mathbb{N}$.

Let $\{x_n\}$ and $\{y_n\}$ be sequences in $[0,1]$ with $\{x_n - y_n\}$ being bounded and $|x_n - y_n| - |T_n x_n - T_n y_n| \to 0$ as $n \to \infty$.

Since $x_n - y_n - (T_n x_n - T_n y_n) = \frac{1}{n}(x_n - y_n)$, for all $n \in \mathbb{N}$, then we have

$$x_n - y_n - (T_n x_n - T_n y_n) \to 0 \quad \text{as } n \to \infty.$$

Then $\{T_n\}$ is a strongly nonexpansive sequence.

Let $G : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for G is to determine its equilibrium points, *i.e.*, the set

$$EP(G) = \{x \in G : G(x,y) \ge 0, \forall y \in C\}.$$

It is a unified model of several problems, namely, variational inequality problem, complementary problem, saddle point problem, optimization problem, fixed point problem, *etc.*; see [3–5]. Several iterative methods have been proposed to solve the equilibrium problem; see, for instance, [6–8]. In 2005, Combettes and Hirstoaga [4] introduced some iterative schemes of finding the best approximation to the initial data when $EP(G)$ is nonempty and proved a strong convergence theorem.

Also in [4], Combettes and Hiratoaga, following [3], defined $S_r : H \to C$ by

$$S_r(x) = \left\{z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\right\}.$$

They proved that under suitable hypotheses S_r is single-valued and firmly nonexpansive with $F(S_r) = EP(G)$.

In 2007, Takahashi and Takahashi [9] proved the following theorem.

Theorem 1.3 *Let C be a nonempty closed convex subset of H. Let G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying*

(A1) $G(x,x) = 0$, $\forall x \in C$;

(A2) *G is monotone, i.e.,* $G(x,y) + G(y,x) \leq 0$, $\forall x,y \in C$;

(A3) $\forall x,y,z \in C$, $\lim_{t\to 0^+} G(tz + (1-t)x, y) \leq G(x,y)$;

(A4) $\forall x \in C$, $y \mapsto G(x,y)$ *is convex and lower semicontinuous*;

and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(G) \neq \emptyset$. Let f be a contraction of H into itself, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,1)$ satisfy

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

and $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(G)$, where $z = P_{F(S)\cap EP(G)}f(z)$.

Very recently, in 2011, Aoyama and Kimura [10] proved a strong convergence theorem of the iterative scheme of $\{x_n\}$ associated to a strongly nonexpansive sequence as follows.

Theorem 1.4 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let $\{S_n\}$ and $\{T_n\}$ be sequences of nonexpansive self-mappings of C. Suppose that $F = F(\{S_n\}) \cap F(\{T_n\})$ is nonempty, both $\{S_n\}$ and $\{T_n\}$ satisfy the conditions (R) and (Z), and $\{S_n\}$ or $\{T_n\}$ is a strongly nonexpansive sequence. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Let $x, u \in C$ and let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and

$$x_{n+1} = \beta_n x_n + (1-\beta_n)S_n\bigl(\alpha_n u + (1-\alpha_n)T_n x_n\bigr)$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_F u$.

For $x_1, u, v \in C$, let $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ be the sequences defined by

$$\begin{cases} F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \\ y_n = \delta_n u_n + (1-\delta_n)v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S_n(\alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n), \quad \forall n \geq 1, \end{cases} \tag{1.1}$$

where $f : C \to C$ is a contractive mapping with $\alpha \in (0, \frac{1}{2})$ and $\{S_n\}$, $\{T_n\}$ are sequences of nonexpansive mappings, one of them is a strongly nonexpansive sequence.

In this paper, inspired and motivated by [10] and [9], we prove that a strong convergence theorem of the iterative scheme $\{x_n\}$ defined by (1.1) converges strongly to $z = P_{\mathbb{F}} f(z)$, where $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{S_n\}) \cap F(\{T_n\})$, under the conditions (R) and (Z) and suitable conditions of $\{r_n\}$, $\{s_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$.

2 Preliminaries

In this section, we need the following lemmas to prove our main result in the next section.

Lemma 2.1 (See [11]) *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if the following inequality holds*:

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2 (See [12]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

(1) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (See [13]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

for all integers $n \geq 0$ and

$$\limsup_{n\to\infty} \bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Lemma 2.4 (See [14]) *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$S(x) = \sum_{n=1}^{\infty} \lambda_n T_n x$$

for all $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

Lemma 2.5 (See [4]) *Let C be a nonempty closed convex subset of a Hilbert space H, and let $G : C \times C \to \mathbb{R}$ satisfy*

(A1) $G(x,x) = 0$, $\forall x \in C$;
(A2) *G is monotone, i.e.*, $G(x,y) + G(y,x) \leq 0$, $\forall x, y \in C$;
(A3) $\forall x, y, z \in C$, $\lim_{t\to 0^+} G(tz + (1-t)x, y) \leq G(x,y)$;
(A4) $\forall x \in C$, $y \mapsto G(x,y)$ *is convex and lower semicontinuous.*

For $x \in H$ *and* $r > 0$, *define a mapping* $S_r : H \to C$ *as follows*:

$$S_r(x) = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Then S_r *is well defined and the following hold*:
(1) S_r *is single-valued*;
(2) S_r *is firmly nonexpansive, i.e.*, $\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle$, $\forall x, y \in H$;
(3) $F(S_r) = EP(G)$;
(4) $EP(G)$ *is closed and convex.*

Lemma 2.6 (See [11]) (*Demiclosedness principle*) *Assume that* T *is a nonexpansive self-mapping of a closed convex subset* C *of a Hilbert space* H. *If* T *has a fixed point, then* $I - T$ *is demiclosed. That is, whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I-T)x_n\}$ *converges strongly to some* y, *it follows that* $(I-T)x = y$. *Here,* I *is the identity mapping of* H.

Lemma 2.7 *Let* H *be a real Hilbert space. Then, for all* $x, y \in H$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.8 (See [10]) *Let* H *be a Hilbert space, let* C *be a nonempty subset of* H, *and let* $\{S_n\}$ *and* $\{T_n\}$ *be the sequences of nonexpansive self-mappings of* C. *Suppose that* $\{S_n\}$ *and* $\{T_n\}$ *satisfy the condition* (R) *and that* $\{T_n y : n \in \mathbb{N}, y \in D\}$ *is bounded for any bounded subset* D *of* C. *Then* $\{S_n T_n\}$ *satisfies the condition* (R).

Lemma 2.9 (See [1]) *Let* H *be a Hilbert space, let* C *be a nonempty subset of* H, *and let* $\{S_n\}$ *and* $\{T_n\}$ *be the sequences of nonexpansive self-mappings of* C. *Suppose that* $\{S_n\}$ *or* $\{T_n\}$ *is a strongly nonexpansive sequence and that* $\widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\})$ *is nonempty. Then* $\widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\}) = \widetilde{F}(\{S_n T_n\})$.

3 Main result

Theorem 3.1 *Let* H *be a Hilbert space, let* C *be a nonempty closed convex subset of* H. *Let* F_1 *and* F_2 *be two bifunctions from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4), *respectively, and let* $\{S_n\}$ *and* $\{T_n\}$ *be sequences of nonexpansive self-mappings of* C *with* $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{S_n\}) \cap F(\{T_n\}) \neq \emptyset$. *Let* $\{T_n\}$ *or* $\{S_n\}$ *be a sequence of strongly nonexpansive mappings, and let* $f : C \to C$ *be a contractive mapping with* $\alpha \in (0, \frac{1}{2})$. *Let* $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ *be sequences generated by* $x_1, u, v \in C$ *and*

$$\begin{cases} F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \\ y_n = \delta_n u_n + (1-\delta_n)v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S_n(\alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\},\{\beta_n\} \in [0,1]$, $\{r_n\},\{s_n\} \in (a,b) \in [0,1]$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=0}^{\infty} |r_{n+1} - r_n|$, $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$;

(iv) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$;

(v) $\{S_n\}$ *and* $\{T_n\}$ *satisfy the conditions R and Z.*

Then the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ *converge strongly to* $z = P_{\mathbb{F}} f(z)$.

Proof Let $v \in \mathbb{F}$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - v\| &= \big\| \beta_n(x_n - v) + (1-\beta_n)\big(S_n\big(\alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n\big) - v\big)\big\| \\
&\leq \beta_n\|x_n - v\| + (1-\beta_n)\big\|\alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n - v\big\| \\
&\leq \beta_n\|x_n - v\| + (1-\beta_n)\big(\alpha_n\big\|f(T_n y_n) - v\big\| + (1-\alpha_n)\|T_n y_n - v\|\big) \\
&\leq \beta_n\|x_n - v\| + (1-\beta_n)\big(\alpha_n\big\|f(T_n y_n) - f(v)\big\| + \alpha_n\big\|f(v) - v\big\| \\
&\quad + (1-\alpha_n)\|T_n y_n - v\|\big) \\
&\leq \beta_n\|x_n - v\| + (1-\beta_n)\big(\alpha_n\alpha\|y_n - v\| + \alpha_n\big\|f(v) - v\big\| \\
&\quad + (1-\alpha_n)\|y_n - v\|\big) \\
&= \beta_n\|x_n - v\| + (1-\beta_n)\big(\alpha_n\big\|f(v) - v\big\| \\
&\quad + \big(1-\alpha_n(1-\alpha)\big)\|y_n - v\|\big).
\end{aligned} \tag{3.2}$$

From Lemma 2.5 and (3.1), we have $EP(F_1) = F(S_{r_n})$, $EP(F_2) = F(S_{s_n})$, $S_{r_n}x_n = u_n$ and $S_{s_n}x_n = v_n$. By $v \in \mathbb{F}$ and the nonexpansiveness of S_{r_n} and S_{s_n}, we have

$$\begin{aligned}
\|y_n - v\| &= \big\|\delta_n(u_n - v) + (1-\delta_n)(v_n - v)\big\| \\
&\leq \delta_n\|u_n - v\| + (1-\delta_n)\|v_n - v\| \\
&= \delta_n\|S_{r_n}x_n - v\| + (1-\delta_n)\|S_{s_n}x_n - v\| \\
&\leq \|x_n - v\|.
\end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\begin{aligned}
\|x_{n+1} - v\| &\leq \beta_n\|x_n - v\| + (1-\beta_n)\big(\alpha_n\big\|f(v) - v\big\| \\
&\quad + \big(1-\alpha_n(1-\alpha)\big)\|y_n - v\|\big) \\
&\leq \beta_n\|x_n - v\| + (1-\beta_n)\big(\alpha_n\big\|f(v) - v\big\| \\
&\quad + \big(1-\alpha_n(1-\alpha)\big)\|x_n - v\|\big) \\
&= \beta_n\|x_n - v\| + (1-\beta_n)\alpha_n\big\|f(v) - v\big\| \\
&\quad + (1-\beta_n)\big(1-\alpha_n(1-\alpha)\big)\|x_n - v\| \\
&= (1-\beta_n)\alpha_n\big\|f(v) - v\big\| + \big(1-\alpha_n(1-\beta_n)(1-\alpha)\big)\|x_n - v\| \\
&\leq \max\left\{\|x_n - v\|, \frac{\|f(v) - v\|}{1-\alpha}\right\}.
\end{aligned}$$

By induction we can conclude that $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{v_n\}$, $\{y_n\}$. Next, we show that $\widetilde{F}(\{S_nA_n\}) = \widetilde{F}(\{S_n\})$ and $\widetilde{F}(\{A_nT_n\}) = \widetilde{F}(\{T_n\})$, where $A_n = \alpha_n f + (1-\alpha_n)I$.

Let $\{z_n\}$ be a bounded sequence in C. From the nonexpansiveness of S_n, we have

$$\|S_nA_nz_n - S_nz_n\| \le \|A_nz_n - z_n\| = \alpha_n\|f(z_n) - z_n\|. \tag{3.4}$$

From (3.4) and $\alpha_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|S_nA_nz_n - S_nz_n\| = 0. \tag{3.5}$$

Let $\{z_n\} \in \widetilde{F}(\{S_nA_n\})$, then we have

$$\|z_n - S_nz_n\| \le \|z_n - S_nA_nz_n\| + \|S_nA_nz_n - S_nz_n\|.$$

From (3.5), we have

$$\lim_{n\to\infty} \|z_n - S_nz_n\| = 0,$$

which implies that $\{z_n\} \in \widetilde{F}(\{S_n\})$. It follows that

$$\widetilde{F}(\{S_nA_n\}) \subseteq \widetilde{F}(\{S_n\}). \tag{3.6}$$

Let $\{z_n\} \in \widetilde{F}(\{S_n\})$, then we have

$$\|z_n - S_nA_nz_n\| \le \|z_n - S_nz_n\| + \|S_nz_n - S_nA_nz_n\|.$$

From (3.5), we have

$$\lim_{n\to\infty} \|z_n - S_nA_nz_n\| = 0,$$

which implies that $\{z_n\} \in \widetilde{F}(\{S_nA_n\})$. It follows that

$$\widetilde{F}(\{S_n\}) \subseteq \widetilde{F}(\{S_nA_n\}). \tag{3.7}$$

From (3.6) and (3.7), we have

$$\widetilde{F}(\{S_n\}) = \widetilde{F}(\{S_nA_n\}). \tag{3.8}$$

Let $\{z_n\}$ be a bounded sequence in C, then we have $\{T_nz_n\}$ is bounded and so is $\{f(T_nz_n)\}$. Since

$$\|A_nT_nz_n - T_nz_n\| = \alpha_n\|f(T_nz_n) - T_nz_n\|$$

and $\alpha_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|A_nT_nz_n - T_nz_n\| = 0. \tag{3.9}$$

Let $\{z_n\} \in \widetilde{F}(\{A_n T_n\})$, then we have

$$\|z_n - T_n z_n\| \leq \|z_n - A_n T_n z_n\| + \|A_n T_n z_n - T_n z_n\|.$$

From (3.9), we have

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0,$$

which implies that

$$\{z_n\} \in \widetilde{F}\bigl(\{T_n\}\bigr).$$

It follows that

$$\widetilde{F}\bigl(\{A_n T_n\}\bigr) \subseteq \widetilde{F}\bigl(\{T_n\}\bigr). \tag{3.10}$$

Let $\{z_n\} \in \widetilde{F}(\{T_n\})$, then we have

$$\|z_n - A_n T_n z_n\| \leq \|z_n - T_n z_n\| + \|T_n z_n - A_n T_n z_n\|.$$

From (3.9), we have

$$\lim_{n\to\infty} \|z_n - A_n T_n z_n\| = 0,$$

which implies that

$$\{z_n\} \in \widetilde{F}\bigl(\{A_n T_n\}\bigr).$$

It follows that

$$\widetilde{F}\bigl(\{T_n\}\bigr) \subseteq \widetilde{F}\bigl(\{A_n T_n\}\bigr). \tag{3.11}$$

From (3.10) and (3.11), we have

$$\widetilde{F}\bigl(\{T_n\}\bigr) = \widetilde{F}\bigl(\{A_n T_n\}\bigr). \tag{3.12}$$

Next, we show that

$$\widetilde{F}\bigl(\{S_n A_n T_n\}\bigr) = \widetilde{F}\bigl(\{S_n\}\bigr) \cap \widetilde{F}\bigl(\{T_n\}\bigr).$$

Since $\mathbb{F}$ is nonempty, from (3.8), (3.12), we have

$$\widetilde{F}\bigl(\{S_n A_n\}\bigr) \cap \widetilde{F}\bigl(\{T_n\}\bigr) = \widetilde{F}\bigl(\{S_n\}\bigr) \cap \widetilde{F}\bigl(\{T_n\}\bigr) \neq \emptyset \tag{3.13}$$

and

$$\widetilde{F}\bigl(\{S_n\}\bigr) \cap \widetilde{F}\bigl(\{A_n T_n\}\bigr) = \widetilde{F}\bigl(\{S_n\}\bigr) \cap \widetilde{F}\bigl(\{T_n\}\bigr) \neq \emptyset. \tag{3.14}$$

Suppose that $\{S_n\}$ is a strongly nonexpansive sequence. From (3.14) and Lemma 2.9, we have

$$\widetilde{F}(\{S_nA_nT_n\}) = \widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{A_nT_n\}) = \widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\}). \tag{3.15}$$

On the other hand, suppose that $\{T_n\}$ is a strongly nonexpansive sequence. From (3.13) and Lemma 2.9, we have

$$\widetilde{F}(\{S_nA_nT_n\}) = \widetilde{F}(\{S_nA_n\}) \cap \widetilde{F}(\{T_n\}) = \widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\}). \tag{3.16}$$

From (3.16) and (3.15), we have $\widetilde{F}(\{S_nA_nT_n\}) = \widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\})$. Next, we show that $\{A_n\}$ and $\{S_nA_nT_n\}$ satisfy the condition (R). It is easy to see that A_n is a nonexpansive mapping for every $n \in \mathbb{N}$ and that $\{A_ny : n \in \mathbb{N}, y \in D\}$ is bounded, where D is a bounded subset of C. Let $y \in D$, then we have

$$\begin{aligned} \|A_{n+1}y - A_ny\| &= \|\alpha_{n+1}f(y) + (1-\alpha_{n+1})y - \alpha_nf(y) - (1-\alpha_n)y\| \\ &\le |\alpha_{n+1} - \alpha_n| \|f(y)\| + |\alpha_{n+1} - \alpha_n| \|y\|. \end{aligned}$$

From the condition (i), we have

$$\lim_{n\to\infty} \sup_{y\in D} \|A_{n+1}y - A_ny\| = 0.$$

It follows that $\{A_n\}$ satisfies the condition (R). From Lemma 2.8, we have that $\{S_nA_n\}$ satisfies the condition (R). From the nonexpansiveness of T_n and $\mathbb{F} \neq \emptyset$, we have $\{T_ny : n \in \mathbb{N}, y \in D\}$ is bounded for any bounded subset D of C. From Lemma 2.8, we have that $\{S_nA_nT_n\}$ satisfies the condition (R).

Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Put

$$x_{n+1} = \beta_nx_n + (1-\beta_n)w_n, \tag{3.18}$$

where $w_n = S_n(\alpha_nf(T_ny_n) + (1-\alpha_n)T_ny_n)$. From the definition of w_n, we have

$$\begin{aligned} \|w_{n+1} - w_n\| &= \|S_{n+1}A_{n+1}T_{n+1}y_{n+1} - S_nA_nT_ny_n\| \\ &\le \|S_{n+1}A_{n+1}T_{n+1}y_{n+1} - S_nA_nT_ny_{n+1}\| + \|S_nA_nT_ny_{n+1} - S_nA_nT_ny_n\| \\ &\le \sup_{y\in D} \|S_{n+1}A_{n+1}T_{n+1}y - S_nA_nT_ny\| + \|y_{n+1} - y_n\|, \end{aligned} \tag{3.19}$$

where D is a bounded subset of C. Besides, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|\delta_{n+1}u_{n+1} + (1-\delta_{n+1})v_{n+1} - \delta_nu_n - (1-\delta_n)v_n\| \\ &= \|\delta_{n+1}u_{n+1} - \delta_{n+1}u_n + \delta_{n+1}u_n - (1-\delta_{n+1})v_n + (1-\delta_{n+1})v_n \end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\delta_{n+1})v_{n+1} - \delta_n u_n - (1-\delta_n)v_n \big\| \\
&= \big\| \delta_{n+1}(u_{n+1}-u_n) + (\delta_{n+1}-\delta_n)u_n + (1-\delta_{n+1})(v_{n+1}-v_n) + (\delta_n - \delta_{n+1})v_n \big\| \\
&\le \delta_{n+1}\|u_{n+1}-u_n\| + |\delta_{n+1}-\delta_n|\|u_n\| + (1-\delta_{n+1})\|v_{n+1}-v_n\| \\
&\quad + |\delta_n - \delta_{n+1}|\|v_n\|. \qquad (3.20)
\end{aligned}
$$

From (3.1) and Lemma 2.5, we have $u_n = S_{r_n}x_n$. This implies that

$$F_1(u_n,u) + \frac{1}{r_n}\langle u-u_n, u_n - x_n\rangle \ge 0 \quad \text{for all } u \in C \qquad (3.21)$$

and

$$F_1(u_{n+1},u) + \frac{1}{r_{n+1}}\langle u-u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0 \quad \text{for all } u \in C. \qquad (3.22)$$

Putting $u = u_{n+1}$ in (3.21) and $u = u_n$ in (3.22), we have

$$F_1(u_n,u_{n+1}) + \frac{1}{r_n}\langle u_{n+1}-u_n, u_n - x_n\rangle \ge 0 \qquad (3.23)$$

and

$$F_1(u_{n+1},u_n) + \frac{1}{r_{n+1}}\langle u_n-u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0. \qquad (3.24)$$

Summing up the last two inequalities and using (A2), we obtain

$$\left\langle u_{n+1}-u_n, \frac{u_n-x_n}{r_n} - \frac{u_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle \ge 0.$$

This implies that

$$\left\langle u_{n+1}-u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle \ge 0.$$

Hence,

$$
\begin{aligned}
\|u_{n+1}-u_n\|^2 &\le \left\langle u_{n+1}-u_n, u_{n+1}-x_n - \frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle \\
&= \left\langle u_{n+1}-u_n, u_{n+1}-x_{n+1} + x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle \\
&= \left\langle u_{n+1}-u_n, x_{n+1}-x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1}-x_{n+1})\right\rangle \\
&\le \|u_{n+1}-u_n\|\left(\|x_{n+1}-x_n\| + \frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|\right) \\
&\le \|u_{n+1}-u_n\|\left(\|x_{n+1}-x_n\| + \frac{1}{a}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|\right).
\end{aligned}
$$

Then we have

$$\|u_{n+1}-u_n\| \le \|x_{n+1}-x_n\| + \frac{1}{a}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|. \qquad (3.25)$$

From (3.1) and Lemma 2.5, we have $v_n = S_{s_n} x_n$. This implies that

$$F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0 \quad \text{for all } v \in C.$$

By using the same method as (3.25), we have

$$\|v_{n+1} - v_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a}|s_{n+1} - s_n|\|v_{n+1} - x_{n+1}\|. \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.20), we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \delta_{n+1}\|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n|\|u_n\| + (1 - \delta_{n+1})\|v_{n+1} - v_n\| \\
&\quad + |\delta_n - \delta_{n+1}|\|v_n\| \\
&\leq \delta_{n+1}\left(\|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|\right) \\
&\quad + (1 - \delta_{n+1})\left(\|x_{n+1} - x_n\| + \frac{1}{a}|s_{n+1} - s_n|\|v_{n+1} - x_{n+1}\|\right) \\
&\quad + 2M|\delta_n - \delta_{n+1}| \\
&\leq \|x_{n+1} - x_n\| + \frac{1}{a}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\quad + \frac{1}{a}|s_{n+1} - s_n|\|v_{n+1} - x_{n+1}\| + 2M|\delta_n - \delta_{n+1}|,
\end{aligned} \tag{3.27}$$

where $M = \sup_{n\in\mathbb{N}}\{\|u_n\|, \|v_n\|\}$. Substituting (3.27) into (3.19), we have

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \sup_{y\in D}\|S_{n+1}A_{n+1}T_{n+1}y - S_nA_nT_ny\| + \|y_{n+1} - y_n\| \\
&\leq \sup_{y\in D}\|S_{n+1}A_{n+1}T_{n+1}y - S_nA_nT_ny\| + \|x_{n+1} - x_n\| \\
&\quad + \frac{1}{a}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\quad + \frac{1}{a}|s_{n+1} - s_n|\|v_{n+1} - x_{n+1}\| + 2M|\delta_n - \delta_{n+1}|.
\end{aligned} \tag{3.28}$$

From (3.28), the conditions (iii), (iv) and $\{S_nA_nT_n\}$ satisfying the condition (R), we have

$$\limsup_{n\to\infty}\bigl(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0. \tag{3.29}$$

From Lemma 2.3 and the definition of x_n, we have

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.30}$$

From the definition of x_n, we have

$$x_{n+1} - x_n = (1 - \beta_n)(w_n - x_n). \tag{3.31}$$

From (3.30), (3.31) and the condition (ii), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Next, we show that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

From the definition of y_n, we have

$$\|y_n - x_n\| \le \delta_n\|u_n - x_n\| + (1-\delta_n)\|v_n - x_n\|. \tag{3.32}$$

Next, we show that

$$\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|v_n - x_n\| = 0.$$

Let $v \in \mathbb{F}$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big\|S_n\big(\alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n\big) - v\big\|^2 \\ &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big\|\alpha_n\big(f(T_n y_n) - v\big) + (1-\alpha_n)(T_n y_n - v)\big\|^2 \\ &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big(\alpha_n\big\|f(T_n y_n) - v\big\|^2 + (1-\alpha_n)\|T_n y_n - v\|^2\big) \\ &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big(\alpha_n\big\|f(T_n y_n) - v\big\|^2 + (1-\alpha_n)\|y_n - v\|^2\big) \\ &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big(\alpha_n\big\|f(T_n y_n) - v\big\|^2 \\ &\quad + (1-\alpha_n)\big(\delta_n\|u_n - v\|^2 + (1-\delta_n)\|v_n - v\|^2\big)\big).\end{aligned} \tag{3.33}$$

From the firm nonexpansiveness of S_{r_n} and $u_n = S_{r_n}x_n$, we have

$$\begin{aligned}\|u_n - v\|^2 &= \|S_{r_n}x_n - S_{r_n}v\|^2 \\ &\le \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}\big(\|u_n - v\|^2 + \|x_n - v\|^2 - \|u_n - x_n\|^2\big).\end{aligned}$$

It implies that

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|u_n - x_n\|^2. \tag{3.34}$$

Since S_{s_n} is a firmly nonexpansive mapping and $v_n = S_{s_n}x_n$, by using the same method as (3.34), we have

$$\|v_n - v\|^2 \le \|x_n - v\|^2 - \|v_n - x_n\|^2. \tag{3.35}$$

Substituting (3.34), (3.35) into (3.33), we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big(\alpha_n\big\|f(T_n y_n) - v\big\|^2 \\ &\quad + (1-\alpha_n)\big(\delta_n\|u_n - v\|^2 + (1-\delta_n)\|v_n - v\|^2\big)\big) \\ &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\big(\alpha_n\big\|f(T_n y_n) - v\big\|^2 \\ &\quad + (1-\alpha_n)\big(\delta_n\big(\|x_n - v\|^2 - \|u_n - x_n\|^2\big)\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\delta_n)\bigl(\|x_n - v\|^2 - \|v_n - x_n\|^2\bigr)\bigr)\bigr) \\
&= \beta_n\|x_n - v\|^2 + (1-\beta_n)\bigl(\alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2 \\
&\quad + (1-\alpha_n)\bigl(\delta_n\|x_n - v\|^2 - \delta_n\|u_n - x_n\|^2 \\
&\quad + (1-\delta_n)\|x_n - v\|^2 - (1-\delta_n)\|v_n - x_n\|^2\bigr)\bigr) \\
&= \beta_n\|x_n - v\|^2 + (1-\beta_n)\bigl(\alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2 \\
&\quad + (1-\alpha_n)\bigl(\|x_n - v\|^2 - \delta_n\|u_n - x_n\|^2 \\
&\quad - (1-\delta_n)\|v_n - x_n\|^2\bigr)\bigr) \\
&= \beta_n\|x_n - v\|^2 + (1-\beta_n)\alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\bigl(\|x_n - v\|^2 - \delta_n\|u_n - x_n\|^2 \\
&\quad - (1-\delta_n)\|v_n - x_n\|^2\bigr) \\
&= \beta_n\|x_n - v\|^2 + (1-\beta_n)\alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2 \\
&\quad + (1-\alpha_n)(1-\beta_n)\|x_n - v\|^2 - \delta_n(1-\alpha_n)(1-\beta_n)\|u_n - x_n\|^2 \\
&\quad - (1-\delta_n)(1-\alpha_n)(1-\beta_n)\|v_n - x_n\|^2 \\
&\le \|x_n - v\|^2 + \alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2 - \delta_n(1-\alpha_n)(1-\beta_n)\|u_n - x_n\|^2 \\
&\quad - (1-\delta_n)(1-\alpha_n)(1-\beta_n)\|v_n - x_n\|^2.
\end{aligned}
\tag{3.36}
$$

From (3.36), we have

$$
\begin{aligned}
\delta_n(1-\alpha_n)(1-\beta_n)\|u_n - x_n\|^2 &\le \|x_n - v\|^2 - \|x_{n+1} - v\|^2 + \alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2 \\
&\quad - (1-\delta_n)(1-\alpha_n)(1-\beta_n)\|v_n - x_n\|^2 \\
&\le \bigl(\|x_n - v\| + \|x_{n+1} - v\|\bigr)\|x_{n+1} - x_n\| + \alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2 \\
&\quad - (1-\delta_n)(1-\alpha_n)(1-\beta_n)\|v_n - x_n\|^2 \\
&\le \bigl(\|x_n - v\| + \|x_{n+1} - v\|\bigr)\|x_{n+1} - x_n\| \\
&\quad + \alpha_n\bigl\|f(T_n y_n) - v\bigr\|^2.
\end{aligned}
$$

From the conditions (i), (ii), (iv) and (3.17), we have

$$
\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.37}
$$

By using the method as (3.37), we have

$$
\lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.38}
$$

From (3.32), (3.37) and (3.38), we have

$$
\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.39}
$$

Next, we show that

$$
\{y_n\} \in \widetilde{F}\bigl(\{S_n\}\bigr) \cap \widetilde{F}\bigl(\{T_n\}\bigr). \tag{3.40}
$$

Since

$$\|S_nA_nT_ny_n - y_n\| \leq \|S_nA_nT_ny_n - x_n\| + \|x_n - y_n\| = \|w_n - x_n\| + \|x_n - y_n\|,$$

from (3.30) and (3.39), we have

$$\lim_{n\to\infty} \|S_nA_nT_ny_n - y_n\| = 0.$$

Since $\{y_n\}$ is bounded, we have

$$\{y_n\} \in \widetilde{F}\big(\{S_nA_nT_n\}\big). \tag{3.41}$$

Since $\widetilde{F}(\{S_nA_nT_n\}) = \widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\})$ and (3.41), we have (3.40).

Next, we show that

$$\lim_{n\to\infty} \|S_nm_n - m_n\| = 0,$$

where $m_n = \alpha_n f(T_ny_n) + (1-\alpha_n)T_ny_n$. From the definition of m_n, we have

$$\begin{aligned}\|S_nm_n - m_n\| &\leq \|S_nm_n - x_n\| + \|m_n - x_n\| \\ &= \|S_nm_n - x_n\| + \big\|\alpha_n\big(f(T_ny_n) - x_n\big) + (1-\alpha_n)(T_ny_n - x_n)\big\| \\ &\leq \|w_n - x_n\| + \alpha_n\big\|f(T_ny_n) - x_n\big\| + (1-\alpha_n)\|T_ny_n - x_n\| \\ &\leq \|w_n - x_n\| + \alpha_n\big\|f(T_ny_n) - x_n\big\| \\ &\quad + \|T_ny_n - y_n\| + \|y_n - x_n\|.\end{aligned}$$

From (3.39), (3.40), (3.30) and the condition (i), we have

$$\lim_{n\to\infty} \|S_nm_n - m_n\| = 0.$$

Next, we show that

$$\limsup_{n\to\infty}\big\langle f(z) - z, m_n - z\big\rangle \leq 0,$$

where $z = P_{\mathbb{F}}f(z)$. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ converging weakly to v, that is, $y_{n_i} \rightharpoonup v$ as $i \to \infty$. From (3.40), $\{S_n\}$ and $\{T_n\}$ satisfying the condition (Z), we have $v \in F(\{S_n\}) \cap F(\{T_n\})$.

Define the mapping $Q: C \to C$ by

$$Q(x) = \delta S_{r_n}x + (1-\delta)S_{s_n}x \quad \text{for all } x \in C,$$

where $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$. From the nonexpansiveness of S_{r_n}, S_{s_n} and Lemma 2.4, we have

$$F(Q) = F(S_{r_n}) \cap F(S_{s_n}) = EP(F_1) \cap EP(F_2).$$

From the definitions of y_n and Q, we have

$$\begin{aligned}\|x_n - Qx_n\| &\le \|x_n - y_n\| + \|y_n - Qx_n\| \\ &\le \|x_n - y_n\| + \|\delta_n u_n + (1-\delta_n)v_n - \delta S_{r_n}x_n - (1-\delta)S_{s_n}x_n\| \\ &\le \|x_n - y_n\| + |\delta_n - \delta|\|u_n\| + |\delta_n - \delta|\|v_n\|. \end{aligned} \tag{3.42}$$

From (3.39), (3.42) and the condition (iv), we have

$$\lim_{n\to\infty}\|x_n - Qx_n\| = 0. \tag{3.43}$$

From (3.39) and $y_{n_i} \rightharpoonup v$ as $i \to \infty$, we have $x_{n_i} \rightharpoonup v$ as $i \to \infty$. By (3.43), $x_{n_i} \rightharpoonup v$ as $i \to \infty$ and Lemma 2.6, we have

$$v \in F(Q) = EP(F_1) \cap EP(F_2).$$

Hence,

$$v \in EP(F_1) \cap EP(F_2) \cap F(\{S_n\}) \cap F(\{T_n\}) = \mathbb{F}. \tag{3.44}$$

By (3.40), (3.44) and the condition (i), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, m_n - z\rangle &= \limsup_{n\to\infty}\big(\alpha_n\langle f(z) - z, f(T_n y_n) - T_n y_n\rangle \\ &\quad + \langle f(z) - z, T_n y_n - z\rangle\big) \\ &= \lim_{i\to\infty}\big(\alpha_{n_i}\langle f(z) - z, f(T_{n_i} y_{n_i}) - T_{n_i} y_{n_i}\rangle \\ &\quad + \langle f(z) - z, T_{n_i} y_{n_i} - z\rangle\big) \\ &= \lim_{i\to\infty}\big(\alpha_{n_i}\langle f(z) - z, f(T_{n_i} y_{n_i}) - T_{n_i} y_{n_i}\rangle \\ &\quad + \langle f(z) - z, T_{n_i} y_{n_i} - y_{n_i}\rangle + \langle f(z) - z, y_{n_i} - z\rangle\big) \\ &= \langle f(z) - z, v - z\rangle \le 0. \end{aligned}$$

Finally, we show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathbb{F}}f(z)$. From the definition of $\{x_n\}$, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\beta_n(x_n - z) + (1-\beta_n)(S_n m_n - z)\|^2 \\ &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|S_n m_n - z\|^2 \\ &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|m_n - z\|^2. \end{aligned} \tag{3.45}$$

Since $m_n = \alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n$, we have

$$\begin{aligned}\|m_n - z\|^2 &= \|\alpha_n(f(T_n y_n) - z) + (1-\alpha_n)(T_n y_n - z)\|^2 \\ &\le (1-\alpha_n)^2\|T_n y_n - z\|^2 + 2\alpha_n\langle f(T_n y_n) - z, m_n - z\rangle \\ &\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle f(T_n y_n) - f(z), m_n - z\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\alpha_n\langle f(z) - z, m_n - z\rangle \\
&\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\alpha\|x_n - z\|\|m_n - z\| \\
&\quad + 2\alpha_n\langle f(z) - z, m_n - z\rangle \\
&\le (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\alpha\big(\|x_n - z\|^2 + \|m_n - z\|^2\big) \\
&\quad + 2\alpha_n\langle f(z) - z, m_n - z\rangle \\
&= (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\alpha\|x_n - z\|^2 + \alpha_n\alpha\|m_n - z\|^2 \\
&\quad + 2\alpha_n\langle f(z) - z, m_n - z\rangle \\
&= \big(1-\alpha_n(1-\alpha)\big)\|x_n - z\|^2 + \alpha_n\alpha\|m_n - z\|^2 \\
&\quad + 2\alpha_n\langle f(z) - z, m_n - z\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|m_n - z\|^2 &\le \frac{1-\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z) - z, m_n - z\rangle \\
&= \frac{1-\alpha_n\alpha + \alpha_n\alpha - \alpha_n(1-\alpha)}{1-\alpha_n\alpha}\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z) - z, m_n - z\rangle \\
&= \left(1 - \frac{\alpha_n(1-2\alpha)}{1-\alpha_n\alpha}\right)\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z) - z, m_n - z\rangle. \qquad (3.46)
\end{aligned}
$$

Substituting (3.46) into (3.45), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\|m_n - z\|^2 \\
&\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\left(\left(1 - \frac{\alpha_n(1-2\alpha)}{1-\alpha_n\alpha}\right)\|x_n - z\|^2\right. \\
&\quad \left. + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z) - z, m_n - z\rangle\right) \\
&\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\left(1 - \frac{\alpha_n(1-2\alpha)}{1-\alpha_n\alpha}\right)\|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n(1-\beta_n)}{1-\alpha_n\alpha}\langle f(z) - z, m_n - z\rangle \\
&= \beta_n\|x_n - z\|^2 + \left((1-\beta_n) - \frac{\alpha_n(1-2\alpha)(1-\beta_n)}{1-\alpha_n\alpha}\right)\|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n(1-\beta_n)}{1-\alpha_n\alpha}\langle f(z) - z, m_n - z\rangle \\
&= \left(1 - \frac{\alpha_n(1-2\alpha)(1-\beta_n)}{1-\alpha_n\alpha}\right)\|x_n - z\|^2 \\
&\quad + \frac{\alpha_n(1-\beta_n)(1-2\alpha)}{1-\alpha_n\alpha}\frac{2\langle f(z) - z, m_n - z\rangle}{(1-2\alpha)}. \qquad (3.47)
\end{aligned}
$$

Applying (3.47), the conditions (i), (ii) and Lemma 2.2, we have $\{x_n\}$ converges strongly to $z = P_{\mathbb{F}}f(z)$. From (3.39), (3.37) and (3.38), it is easy to see that $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ converge strongly to $z = P_{\mathbb{F}}f(z)$. This completes the proof. □

4 Applications

In this section, we give three examples for a strongly nonexpansive sequence and prove a strong convergence theorem associated to the variational inequality problem.

Before we give three examples, we need the following definition and lemmas.

Definition 4.1 Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called an α-inverse strongly monotone mapping if there exists an $\alpha > 0$ such that

$$\langle x - y, ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

A mapping $A : C \to H$ is called α-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, ax - Ay\rangle \geq \alpha \|x - y\|^2$$

for all $x, y \in C$.

A mapping $T : C \to C$ is called a κ-strictly pseudo-contractive mapping if there is $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \big\|(I - T)x - (I - T)y\big\|^2 \tag{4.1}$$

for all $x, y \in C$.

Then (4.1) is equivalent to

$$\big\langle x - y, (I - T)x - (I - T)y\big\rangle \geq \frac{1 - \kappa}{2} \big\|(I - T)x - (I - T)y\big\|^2$$

for all $x, y \in C$.

The set of solutions of the variational inequality problem of the mapping $A : C \to H$ is denoted by $VI(C, A)$, that is,

$$VI(C, A) = \big\{x \in C : \langle y - x, Ax\rangle \geq 0, \forall y \in C\big\}.$$

Let $A, B : C \to H$ be two mappings. In 2013, Kangtunyakarn [15] modified $VI(C, A)$ as follows:

$$VI\big(C, aA + (1 - a)B\big) = \big\{x \in C : \big\langle y - x, \big(aA + (1 - a)B\big)x\big\rangle \geq 0, \forall y \in C \text{ and } a \in (0, 1)\big\}.$$

Remark 4.1 If $T : C \to C$ is a κ-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$, then $(I - T)$ is a $\frac{1-\kappa}{2}$-inverse strongly monotone mapping and $F(T) = VI(C, I - T)$.

Lemma 4.2 (See [16]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \quad \Leftrightarrow \quad u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 4.3 (See [15]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C,A) \cap VI(C,B) \neq \emptyset$. Then*

$$VI\big(C, aA + (1-a)B\big) = VI(C,A) \cap VI(C,B), \quad \forall a \in (0,1). \tag{4.2}$$

Furthermore, if $0 < \gamma < 2\eta$, where $\eta = \min\{\alpha, \beta\}$, we have $I - \gamma(aA + (1-a)B)$ is a nonexpansive mapping.

Example 4.4 Let $T : C \to C$ be a κ-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that

$$0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < 1 - \kappa \quad \text{and} \quad \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0,$$

and let $\{T_n\}$ be a sequence of mappings defined by $T_n = P_C(I - \lambda_n(I - T))$. Then $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z).

Proof Since T is a κ-strictly pseudo-contractive mapping, then $I - T$ is $\frac{1-\kappa}{2}$-inverse strongly monotone. From Example 4.3 in [10], we have $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z). □

Example 4.5 Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively, with $\bar{\gamma} = \min\{\alpha, \beta\}$ and $VI(C,A) \cap VI(C,B) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that

$$0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < 2\bar{\gamma} \quad \text{and} \quad \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0,$$

and let $\{T_n\}$ be a sequence of mappings defined by $T_n = P_C(I - \lambda_n D)$, where $D = aA + (1-a)B$ for all $a \in (0,1)$. Then $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z).

Proof Let $x, y \in C$, then we have

$$\begin{aligned}
\langle x - y, Dx - Dy\rangle &= \big\langle x - y, \big(aA + (1-a)B\big)x - \big(aA + (1-a)B\big)y\big\rangle \\
&\geq a\langle x - y, Ax - Ay\rangle + (1-a)\langle x - y, Bx - By\rangle \\
&\geq a\alpha\|Ax - Ay\|^2 + (1-a)\beta\|Bx - By\|^2 \\
&\geq \bar{\gamma}\big(\big\|aAx + (1-a)Bx - aAy - (1-a)By\big\|^2\big) \\
&\geq \bar{\gamma}\|Dx - Dy\|^2.
\end{aligned}$$

Then D is a $\bar{\gamma}$-inverse strongly monotone mapping. From Example 4.3 in [10], we have that $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z). □

Example 4.6 Let $A : C \to H$ be an α-strongly monotone and L-Lipschitzian mapping with $VI(C,A) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that

$$0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{2\alpha}{L^2} \quad \text{and} \quad \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0,$$

and let $\{T_n\}$ be a sequence of mappings defined by $T_n = P_C(I - \lambda_n A)$. Then $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z).

Proof Let $x, y \in C$, then we have

$$\begin{aligned}\langle x-y, Ax-Ay\rangle &\geq \alpha\|x-y\|^2\\ &\geq \frac{\alpha}{L^2}\|Ax-Ay\|^2.\end{aligned}$$

Then A is an $\frac{\alpha}{L^2}$-inverse strongly monotone mapping. From Example 4.3 in [10], we have that $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z). □

Example 4.7 (See [10]) Let $\{R_n\}$ be a sequence of nonexpansive mappings of C into itself having a common fixed point, and let $\{\mu_n\}$ be a sequence in $[0,1]$. For each $n \in \mathbb{N}$, a W-mapping [17] T_n generated by $R_n, R_{n-1}, \dots, R_1$ and $\mu_n, \mu_{n-1}, \dots, \mu_1$ is defined as follows:

$$\begin{aligned}
&U_{n,n} = \mu_n R_n + (1-\mu_n)I,\\
&U_{n,n-1} = \mu_{n-1}R_{n-1}U_{n,n} + (1-\mu_{n-1})I,\\
&U_{n,n-2} = \mu_{n-2}R_{n-2}U_{n,n-1} + (1-\mu_{n-2})I,\\
&\vdots\\
&U_{n,k} = \mu_k R_k U_{n,k+1} + (1-\mu_k)I,\\
&\vdots\\
&U_{n,2} = \mu_2 R_2 U_{n,3} + (1-\mu_2)I,\\
&T_n = U_{n,1} = \mu_1 R_1 U_{n,2} + (1-\mu_1)I.
\end{aligned}$$

If $0 < \mu_1 \leq 1$ and $0 < \mu_n \leq b$, for all $n \geq 2$ and $0 < b < 1$, then $\{T_n\}$ satisfies the conditions (R) and (Z).

By using our main result and these three examples, we obtain the following results.

Theorem 4.8 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let F_1 and F_2 be two bifunctions from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *respectively. Let $T : C \to C$ be a κ-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that*

$$0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < 1-\kappa \quad \text{and} \quad \lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0,$$

and let $\{T_n\}$ be a sequence of mappings defined by $T_n = P_C(I - \lambda_n(I - T))$. Let $\{R_n\}$ be a sequence of nonexpansive mappings of C into itself having a common fixed point, and let $\{\mu_n\}$ be a sequence in $[0,1]$. For each $n \in \mathbb{N}$, W_n is a W-mapping generated by $R_n, R_{n-1}, \dots, R_1$ and $\mu_n, \mu_{n-1}, \dots, \mu_1$. Assume that $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{R_n\}) \cap F(T) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0, \frac{1}{2})$. Let $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences generated by

$x_1, u, v \in C$ and

$$\begin{cases} F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \\ y_n = \delta_n u_n + (1-\delta_n)v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)W_n(\alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n), \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, $\{r_n\}, \{s_n\} \in (a,b) \in [0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=0}^{\infty} |r_{n+1} - r_n|, \sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$;
(iv) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$.

Then the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ converge strongly to $z = P_{\mathbb{F}} f(z)$.

Proof From Example 4.4, we have $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z). From Lemma 4.2, we have $F(T_n) = F(P_C(I - \lambda_n(I - T))) = VI(C, I - T) = F(T)$ for all $n \in \mathbb{N}$. It implies that $F(\{T_n\}) = F(T)$. From [18], we have $F(\{W_n\}) = F(\{R_n\})$. It follows that $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{W_n\}) \cap F(\{T_n\}) \neq \emptyset$. From Example 4.7, we have $\{W_n\}$ is a nonexpansive sequence satisfying the conditions (R) and (Z). By Theorem 3.1, we can conclude the desired result. □

Theorem 4.9 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let F_1 and F_2 be two bifunctions from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *respectively. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively, with $\bar{\gamma} = \min\{\alpha, \beta\}$ and $VI(C,A) \cap VI(C,B) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that*

$$0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < 2\bar{\gamma} \quad \text{and} \quad \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0,$$

and let $\{T_n\}$ be a sequence of mappings defined by $T_n = P_C(I - \lambda_n D)$, where $D = aA + (1-a)B$ for all $a \in (0,1)$. Let $\{R_n\}$ be a sequence of nonexpansive mappings of C into itself having a common fixed point, and let $\{\mu_n\}$ be a sequence in $[0,1]$. *For each $n \in \mathbb{N}$, W_n is a W-mapping generated by $R_n, R_{n-1}, \ldots, R_1$ and $\mu_n, \mu_{n-1}, \ldots, \mu_1$. Assume that $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{R_n\}) \cap VI(C,A) \cap VI(C,B) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0, \frac{1}{2})$. Let $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences generated by $x_1, u, v \in C$ and*

$$\begin{cases} F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \\ y_n = \delta_n u_n + (1-\delta_n)v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)W_n(\alpha_n f(T_n y_n) + (1-\alpha_n)T_n y_n), \quad \forall n \geq 1, \end{cases} \tag{4.4}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, $\{r_n\}, \{s_n\} \in (a,b) \in [0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\sum_{n=0}^{\infty} |r_{n+1} - r_n|, \sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$;
(iv) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$.

Then the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ *converge strongly to* $z = P_{\mathbb{F}} f(z)$.

Proof From Example 4.5, we have $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z). From Lemmas 4.2 and 4.3, we have $F(T_n) = F(P_C(I - \lambda_n D)) = VI(C,D) = VI(C,A) \cap VI(C,B)$ for all $n \in \mathbb{N}$. It implies that $F(\{T_n\}) = VI(C,A) \cap VI(C,B)$. From [18], we have $F(\{W_n\}) = F(\{R_n\})$. It follows that $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{W_n\}) \cap F(\{T_n\}) \neq \emptyset$. From Example 4.7, we have $\{W_n\}$ is a nonexpansive sequence satisfying the conditions (R) and (Z). By Theorem 3.1, we can conclude the desired result. □

Theorem 4.10 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let F_1 and F_2 be two bifunctions from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *respectively. Let $A : C \to H$ be an α-strongly monotone and L-Lipschitzian mapping with $VI(C,A) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that*

$$0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < \frac{2\alpha}{L^2} \quad \textit{and} \quad \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = 0,$$

and let $\{T_n\}$ be a sequence of mappings defined by $T_n = P_C(I - \lambda_n A)$. Let $\{R_n\}$ be a sequence of nonexpansive mappings of C into itself having a common fixed point, and let $\{\mu_n\}$ be a sequence in $[0,1]$. *For each $n \in \mathbb{N}$, W_n is a W-mapping generated by $R_n, R_{n-1}, \ldots, R_1$ and $\mu_n, \mu_{n-1}, \ldots, \mu_1$. Assume that $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{R_n\}) \cap VI(C,A) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with $\alpha \in (0, \frac{1}{2})$. Let $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences generated by $x_1, u, v \in C$ and*

$$\begin{cases} F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ F_2(v_n, v) + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \\ y_n = \delta_n u_n + (1 - \delta_n) v_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) W_n(\alpha_n f(T_n y_n) + (1 - \alpha_n) T_n y_n), \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, $\{r_n\}, \{s_n\} \in (a,b) \in [0,1]$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=0}^{\infty} |r_{n+1} - r_n|, \sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$;
(iv) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$.

Then the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ *converge strongly to* $z = P_{\mathbb{F}} f(z)$.

Proof From Example 4.6, we have $\{T_n\}$ is a strongly nonexpansive sequence satisfying the conditions (R) and (Z). From Lemma 4.2, we have $F(T_n) = F(P_C(I - \lambda_n A)) = VI(C,A)$ for all $n \in \mathbb{N}$. It implies that $F(\{T_n\}) = VI(C,A)$. From [18], we have $F(\{W_n\}) = F(\{R_n\})$. It follows that $\mathbb{F} = EP(F_1) \cap EP(F_2) \cap F(\{W_n\}) \cap F(\{T_n\}) \neq \emptyset$. From Example 4.7, we have $\{W_n\}$ is a nonexpansive sequence satisfying the conditions (R) and (Z). By Theorem 3.1, we can conclude the desired result. □

Theorem 4.11 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let F_1 be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *and let $\{S_n\}$ and $\{T_n\}$ be sequences of nonexpansive self-mappings of C with $\mathbb{F} = EP(F_1) \cap F(\{S_n\}) \cap F(\{T_n\}) \neq \emptyset$. Let $\{T_n\}$ or $\{S_n\}$ be a sequence of strongly nonexpansive mappings, and let $f : C \to C$ be a contractive mapping with $\alpha \in (0, \frac{1}{2})$. Let $\{x_n\}$, $\{u_n\}$ be sequences generated by $x_1, u \in C$ and*

$$\begin{cases} F_1(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n(\alpha_n f(T_n u_n) + (1 - \alpha_n)T_n u_n), \quad \forall n \geq 1, \end{cases} \tag{4.6}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, $\{r_n\}, \{s_n\} \in (a,b) \in [0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$;
(iv) $\{S_n\}$ *and* $\{T_n\}$ *satisfy the conditions R and Z.*

Then the sequences $\{x_n\}$, $\{u_n\}$ converge strongly to $z = P_{\mathbb{F}} f(z)$.

Proof Put $F_1 \equiv F_2$, $s_n = r_n$ and $u_n = v_n$. From Theorem 3.1, we can conclude the desired conclusion. □

The following result can be obtained from Theorem 3.1. We, therefore, omit the proof.

Theorem 4.12 *Let H be a Hilbert space, let C be a nonempty closed convex subset of H. Let F_i be bifunctions from $C \times C$ into $\mathbb{R}$, for every $i = 1, 2, \ldots, N$, satisfying* (A1)-(A4), *and let $\{S_n\}$ and $\{T_n\}$ be sequences of nonexpansive self-mappings of C with $\mathbb{F} = \bigcap_{i=1}^{N} EP(F_i) \cap F(\{S_n\}) \cap F(\{T_n\}) \neq \emptyset$. Let $\{T_n\}$ or $\{S_n\}$ be a sequence of strongly nonexpansive mappings, and let $f : C \to C$ be a contractive mapping with $\alpha \in (0, \frac{1}{2})$. Let $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences generated by $x_1, u^i \in C$, for every $i \in 1, 2, \ldots, N$, and*

$$\begin{cases} F_i(u_n^i, u^i) + \frac{1}{r_n^i}\langle u - u_n^i, u_n^i - x_n\rangle \geq 0, \\ y_n = \sum_{i=1}^{N} \delta_n^i u_n^i, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n(\alpha_n f(T_n y_n) + (1 - \alpha_n)T_n y_n), \quad \forall n \geq 1, \end{cases} \tag{4.7}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, $\{r_n\}, \{s_n\} \in (a,b) \in [0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=0}^{\infty} |r_{n+1}^i - r_n^i| < \infty$, $\forall i = 1, 2, \ldots, N$;
(iv) $\sum_{i=1}^{N} \delta_n^i = 1$;
(v) $\lim_{n\to\infty} \delta_n^i = \delta^i \in (0,1)$, $\forall i = 1, 2, \ldots, N$;
(vi) $\{S_n\}$ *and* $\{T_n\}$ *satisfy the conditions R and Z.*

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n^i\}$, for every $i = 1, 2, \ldots, N$, converge strongly to $z = P_{\mathbb{F}} f(z)$.

Competing interests
The author declares that they have no competing interests.

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