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# Some properties of relative efficiency of estimators in a two linear regression equations system with identical parameter vectors

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Arts and Sciences, Chongqing,  
402160, China**Abstract**

Two normal linear models with some of the parameters identical are discussed in this article. We introduce four relative efficiencies to define the efficiency of estimator in two linear regression equations system with identical parameter vectors, also we give the lower and upper bounds of the four relative efficiencies.

**Keywords:** best linear unbiased estimator; common parameter; relative efficiency

**1 Introduction**

Consider a system (H) formed by two linear models:

$$y_1 = X_1\beta + Z_1\beta_1 + \varepsilon_1, \quad (1)$$

$$y_2 = X_2\beta + Z_2\beta_2 + \varepsilon_2, \quad (2)$$

where for  $i = 1, 2$ ,  $y_i$  is  $n_i \times 1$  vector of observations,  $X_i$  and  $Z_i$  are  $n_i \times p$  and  $n_i \times t_i$  full rank matrices satisfying  $\text{rank}(X_i, Z_i) = \text{rank}(X_i) + \text{rank}(Z_i)$  with  $\text{rank}(\cdot)$  denoting the rank of a matrix,  $\beta$  and  $\beta_i$  are  $p \times 1$  and  $t_i \times 1$  unknown parameters,  $\varepsilon_i$  is  $n_i \times 1$  random vector supposed to follow a multivariate normal distribution mean 0 and variance covariance matrix  $\sigma_i I$ ,  $\sigma_i$  being a known parameter,  $\varepsilon_1$  and  $\varepsilon_2$  are independent.

Define  $Q_i = I - Z_i(Z_i'Z_i)^{-1}Z_i'$ ,  $T_i = (Z_i'Z_i)^{-1}Z_i'X_i$  and  $r = \frac{\sigma_1}{\sigma_2}$ . Then by Liu [1] we have the following:

- (1) In the single equation (1), the best linear unbiased estimators (BLUE) of  $\beta$  and  $\beta_1$  are given respectively by

$$\hat{\beta} = (X_1'Q_1X_1)^{-1}X_1'Q_1y_1, \quad (3)$$

$$\hat{\beta}_1 = (Z_1'Z_1)^{-1}Z_1'y_1 - T_1\hat{\beta}. \quad (4)$$

- (2) In the single equation (2), the best linear unbiased estimators (BLUE) of  $\beta$  and  $\beta_2$  are given respectively by

$$\tilde{\beta} = (X_2'Q_2X_2)^{-1}X_2'Q_2y_2, \quad (5)$$

$$\tilde{\beta}_2 = (Z_2'Z_2)^{-1}Z_2'y_2 - T_2\tilde{\beta}. \quad (6)$$

(3) For the system (H), the BLUE of  $\beta$ ,  $\beta_1$  and  $\beta_2$  are given respectively by

$$\beta^*(r) = (X_1'Q_1X_1 + rX_2'Q_2X_2)^{-1}(X_1'Q_1y_1 + rX_2'Q_2y_2), \tag{7}$$

$$\beta_1^* = (Z_1'Z_1)^{-1}Z_1'y_1 - T_1\beta^*(r), \tag{8}$$

$$\beta_2^* = (Z_2'Z_2)^{-1}Z_2'y_2 - T_2\beta^*(r). \tag{9}$$

In this article, we only discuss the estimation of the parameter  $\beta$ . Liu [1] gave the comparison between the estimators  $\hat{\beta}$ ,  $\tilde{\beta}$  and  $\beta^*(r)$  in the mean squared error criterion when  $\sigma_i$  are known. He also gave an estimator when  $\sigma_i$  are unknown and discussed the statistical properties of the estimators  $\hat{\beta}$ ,  $\tilde{\beta}$  and  $\beta^*(r)$ . Ma and Wang [2] also studied the estimators  $\hat{\beta}$ ,  $\tilde{\beta}$  and  $\beta^*(r)$  in the mean squared error criterion.

It is easy to compute that

$$\text{Cov}(\hat{\beta}) = \sigma_1(X_1'Q_1X_1)^{-1}, \tag{10}$$

$$\text{Cov}(\tilde{\beta}) = \sigma_2(X_2'Q_2X_2)^{-1}, \tag{11}$$

$$\text{Cov}(\beta^*(r)) = \sigma_1(X_1'Q_1X_1 + rX_2'Q_2X_2)^{-1}. \tag{12}$$

From Equations (10)-(12), we can see that

$$\text{Cov}(\beta^*(r)) \leq \text{Cov}(\hat{\beta}), \quad \text{Cov}(\beta^*(r)) \leq \text{Cov}(\tilde{\beta}). \tag{13}$$

In practice,  $\sigma_i$  may be unknown, in this case we can use  $\hat{\beta}$  or  $\tilde{\beta}$  to replace  $\beta^*(r)$ . However, this will lead to loss, we introduce the relative efficiency to define the loss. Relative efficiency has been studied by many researchers such as Yang [3], Wang and Ip [4], Liu *et al.* [5, 6], Yang and Wang [7], Wang and Yang [8, 9] and Yang and Wu [10].

In this article, we introduce four relative efficiencies in system (H), and we also give the lower and upper bounds of the four relative efficiencies.

The rest of the article is organized as follows. In Section 2, we propose the new relative efficiency. Sections 3 and 4 give the lower and upper bounds of the relative efficiencies proposed in Section 2. Some concluding remarks are given in Section 5.

## 2 New relative efficiency

In order to define the loss when we use  $\hat{\beta}$  or  $\tilde{\beta}$  to replace  $\beta^*(r)$ , we introduce four relative efficiencies as follows:

$$e_1(\beta^*(r)|\hat{\beta}) = \frac{|\text{Cov}(\beta^*(r))|}{|\text{Cov}(\hat{\beta})|}, \tag{14}$$

$$e_2(\beta^*(r)|\tilde{\beta}) = \frac{|\text{Cov}(\beta^*(r))|}{|\text{Cov}(\tilde{\beta})|}, \tag{15}$$

$$e_3(\beta^*(r)|\hat{\beta}) = \frac{\text{tr}(\text{Cov}(\beta^*(r)))}{\text{tr}(\text{Cov}(\hat{\beta}))}, \tag{16}$$

$$e_4(\beta^*(r)|\tilde{\beta}) = \frac{\text{tr}(\text{Cov}(\beta^*(r)))}{\text{tr}(\text{Cov}(\tilde{\beta}))}, \tag{17}$$

where  $|A|$  and  $\text{tr}(A)$  denote the determinant and trace of matrix  $A$ , respectively. By Equation (13), we have  $0 < e_i(\cdot|\cdot) \leq 1$ ,  $i = 1, 2, 3, 4$ . In the next section we will give the lower and upper bounds of  $e_1(\beta^*(r)|\hat{\beta})$  and  $e_2(\beta^*(r)|\tilde{\beta})$ .

### 3 The lower and upper bounds of $e_1(\beta^*(r)|\hat{\beta})$ and $e_2(\beta^*(r)|\tilde{\beta})$

In this section we give the lower and upper bounds of  $e_1(\beta^*(r)|\hat{\beta})$  and  $e_2(\beta^*(r)|\tilde{\beta})$ . Firstly, we give some lemmas and notations which are needed in the following discussion. Let  $A$  be an  $n \times n$  nonnegative definite matrix,  $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$  stands for the ordered eigenvalues of matrix  $A$ .

**Lemma 3.1** [11] *Let  $A$  be an  $n \times n$  nonnegative definite matrix, and let  $B$  be an  $n \times n$  nonnegative definite matrix, then we have*

$$\lambda_n(A)\lambda_i(B) \leq \lambda_i(AB) \leq \lambda_1(A)\lambda_i(B), \quad i = 1, 2, \dots, n. \tag{18}$$

**Lemma 3.2** [12] *Let  $\Delta_1 = \text{diag}(\tau_1, \tau_2, \dots, \tau_p)$ ,  $\tau_1 \geq \tau_2 \geq \dots \geq \tau_p > 0$ ,  $\Delta_2 = \text{diag}(\mu_1, \mu_2, \dots, \mu_p)$ ,  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p \geq 0$  and  $A$  be an  $p \times p$  orthogonal matrix, then we have*

$$\sum_{i=1}^p \tau_i \mu_{p+1-i} \leq \text{tr}(\Delta_1 A' \Delta_2 A) \leq \sum_{i=1}^p \tau_i \mu_i. \tag{19}$$

Now we will give the lower and upper bounds of  $e_1(\beta^*(r)|\hat{\beta})$ .

**Theorem 3.1** *Let  $\beta^*(r)$  and  $\hat{\beta}$  be given in Equations (7) and (3), let  $e_1(\beta^*(r)|\hat{\beta})$  be defined in Equation (14), then we have*

$$\frac{1}{\prod_{i=1}^p (1 + r\theta_p^{-1}\eta_i)} \leq e_1(\beta^*(r)|\hat{\beta}) \leq \frac{1}{\prod_{i=1}^p (1 + r\theta_1^{-1}\eta_i)}, \tag{20}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1'Q_1X_1$ ,  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2'Q_2X_2$ .

*Proof* By the definition of  $e_1(\beta^*(r)|\hat{\beta})$ , we have

$$\begin{aligned} e_1(\beta^*(r)|\hat{\beta}) &= \frac{|\text{Cov}(\beta^*(r))|}{|\text{Cov}(\hat{\beta})|} \\ &= \frac{|\sigma_1(X_1'Q_1X_1 + rX_2'Q_2X_2)^{-1}|}{|\sigma_1(X_1'Q_1X_1)^{-1}|} \\ &= \frac{|X_1'Q_1X_1|}{|X_1'Q_1X_1 + rX_2'Q_2X_2|}. \end{aligned} \tag{21}$$

It is easy to see that  $X_1'Q_1X_1 > 0$  and  $X_2'Q_2X_2 > 0$ . Define

$$A = (X_1'Q_1X_1)^{-1/2} (X_2'Q_2X_2) (X_1'Q_1X_1)^{-1/2},$$

then  $A > 0$ , there exists an orthogonal matrix  $N$  such that

$$NAN' = \text{diag}(\zeta_1, \dots, \zeta_p) \triangleq \Delta, \tag{22}$$

where  $\zeta_1 \geq \dots \geq \zeta_p$  is the eigenvalues of  $A$ . Now we define  $M = N(X'_1 Q_1 X_1)^{-1/2}$ , then we have

$$M(X'_1 Q_1 X_1)M' = NN' = I_p, \tag{23}$$

$$\begin{aligned} M(X'_2 Q_2 X_2)M' &= N(X'_1 Q_1 X_1)^{-1/2} (X'_2 Q_2 X_2) (X'_1 Q_1 X_1)^{-1/2} N' \\ &= NAN' = \Delta. \end{aligned} \tag{24}$$

Thus

$$X'_1 Q_1 X_1 = M^{-1} M'^{-1}, \tag{25}$$

$$X'_2 Q_2 X_2 = M^{-1} \Delta M'^{-1}. \tag{26}$$

Then we put Equations (25) and (26) into Equation (21), and we have

$$\begin{aligned} e_1(\beta^*(r)|\hat{\beta}) &= \frac{|X'_1 Q_1 X_1|}{|X'_1 Q_1 X_1 + rX'_2 Q_2 X_2|} \\ &= \frac{|M^{-1} M'^{-1}|}{|M^{-1} M'^{-1} + rM^{-1} \Delta M'^{-1}|} \\ &= \frac{|M^{-1}| |M'^{-1}|}{|M^{-1}| |I_p + r\Delta| |M'^{-1}|} = \frac{1}{|I_p + r\Delta|}. \end{aligned} \tag{27}$$

Since  $A = (X'_1 Q_1 X_1)^{-1/2} (X'_2 Q_2 X_2) (X'_1 Q_1 X_1)^{-1/2}$  has the same eigenvalues of  $(X'_2 Q_2 X_2) \times (X'_1 Q_1 X_1)^{-1}$ , we have  $\lambda_i(A) = \lambda_i((X'_2 Q_2 X_2)(X'_1 Q_1 X_1)^{-1})$ ,  $i = 1, 2, \dots, p$ . Then by Lemma 3.1 we have

$$\begin{aligned} \lambda_p((X'_1 Q_1 X_1)^{-1}) \lambda_i(X'_2 Q_2 X_2) &\leq \lambda_i((X'_2 Q_2 X_2)(X'_1 Q_1 X_1)^{-1}) \\ &\leq \lambda_1((X'_1 Q_1 X_1)^{-1}) \lambda_i(X'_2 Q_2 X_2). \end{aligned} \tag{28}$$

On the other hand,

$$\lambda_p((X'_1 Q_1 X_1)^{-1}) = \lambda_1^{-1}(X'_1 Q_1 X_1) = \theta_1^{-1}, \tag{29}$$

$$\lambda_1((X'_1 Q_1 X_1)^{-1}) = \lambda_p^{-1}(X'_1 Q_1 X_1) = \theta_p^{-1}, \tag{30}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X'_1 Q_1 X_1$ . By Equations (28)-(30), we obtain

$$\theta_1^{-1} \eta_i \leq \lambda_i((X'_2 Q_2 X_2)(X'_1 Q_1 X_1)^{-1}) \leq \theta_p^{-1} \eta_i, \quad i = 1, \dots, p, \tag{31}$$

where  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X'_2 Q_2 X_2$ . Thus by Equations (27) and (31), we have

$$\frac{1}{\prod_{i=1}^p (1 + r\theta_p^{-1} \eta_i)} \leq e_1(\beta^*(r)|\hat{\beta}) \leq \frac{1}{\prod_{i=1}^p (1 + r\theta_1^{-1} \eta_i)}. \tag{32}$$

□

**Corollary 3.1** *Let  $\beta^*(r)$  and  $\hat{\beta}$  be given in Equations (7) and (3), let  $e_1(\beta^*(r)|\hat{\beta})$  be defined in Equation (14),  $X_1'Q_1X_1$  and  $X_2'Q_2X_2$  communicate, then we have*

$$\frac{\theta_p^p}{(\theta_1 + r\eta_1)^p} \leq e_1(\beta^*(r)|\hat{\beta}) \leq \frac{\theta_1^p}{(\theta_p + r\eta_p)^p}, \tag{33}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1'Q_1X_1$ ,  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2'Q_2X_2$ .

*Proof* Since  $X_1'Q_1X_1$  and  $X_2'Q_2X_2$  communicate, there exists an orthogonal matrix  $G$  such that

$$G'X_1'Q_1X_1G = \text{diag}(\theta_1, \dots, \theta_p) \triangleq \Sigma, \tag{34}$$

$$G'X_2'Q_2X_2G = \text{diag}(\eta_1, \dots, \eta_p) \triangleq \Omega, \tag{35}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1'Q_1X_1$ ,  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2'Q_2X_2$ .

By the definition of  $e_1(\beta^*(r)|\hat{\beta})$ , we have

$$\begin{aligned} e_1(\beta^*(r)|\hat{\beta}) &= \frac{|\text{Cov}(\beta^*(r))|}{|\text{Cov}(\hat{\beta})|} \\ &= \frac{|X_1'Q_1X_1|}{|X_1'Q_1X_1 + rX_2'Q_2X_2|} \\ &= \frac{|G\Sigma G'|}{|G\Sigma G' + rG\Omega G'|} \\ &= \frac{\prod_{i=1}^p \theta_i}{\prod_{i=1}^p (\theta_i + r\eta_i)}. \end{aligned} \tag{36}$$

Thus we have

$$\frac{\theta_p^p}{(\theta_1 + r\eta_1)^p} \leq e_1(\beta^*(r)|\hat{\beta}) \leq \frac{\theta_1^p}{(\theta_p + r\eta_p)^p}. \tag{37}$$

□

Using the same way, we can give the lower and upper bounds of  $e_2(\beta^*(r)|\tilde{\beta})$ .

**Theorem 3.2** *Let  $\beta^*(r)$  and  $\tilde{\beta}$  be given in Equations (7) and (5), let  $e_2(\beta^*(r)|\tilde{\beta})$  be defined in Equation (15), then we have*

$$\frac{1}{\prod_{i=1}^p (r + \eta_p^{-1}\theta_i)} \leq e_2(\beta^*(r)|\tilde{\beta}) \leq \frac{1}{\prod_{i=1}^p (r + \eta_1^{-1}\theta_i)}, \tag{38}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1'Q_1X_1$ ,  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2'Q_2X_2$ .

**Corollary 3.2** *Let  $\beta^*(r)$  and  $\tilde{\beta}$  be given in Equations (7) and (5), let  $e_2(\beta^*(r)|\tilde{\beta})$  be defined in Equation (15),  $X_1'Q_1X_1$  and  $X_2'Q_2X_2$  communicate, then we have*

$$\frac{\eta_p^p}{(\theta_1 + r\eta_1)^p} \leq e_2(\beta^*(r)|\tilde{\beta}) \leq \frac{\eta_1^p}{(\theta_p + r\eta_p)^p}. \tag{39}$$

#### 4 The lower and upper bounds of $e_3(\beta^*(r)|\hat{\beta})$ and $e_4(\beta^*(r)|\tilde{\beta})$

In this section we give the lower and upper bounds of  $e_3(\beta^*(r)|\hat{\beta})$  and  $e_4(\beta^*(r)|\tilde{\beta})$ . Firstly we give the lower and upper bounds of  $e_3(\beta^*(r)|\hat{\beta})$ .

**Theorem 4.1** *Let  $\beta^*(r)$  and  $\hat{\beta}$  be given in Equations (7) and (3), let  $e_3(\beta^*(r)|\hat{\beta})$  be defined in Equation (16), then we have*

$$\frac{\sum_{i=1}^p \theta_{p+1-i}^{-1} (1+r\zeta_i)^{-1}}{\sum_{i=1}^p \theta_i^{-1}} \leq e_3(\beta^*(r)|\hat{\beta}) \leq \frac{\sum_{i=1}^p \theta_{p+1-i}^{-1} (1+r\zeta_{p+1-i})^{-1}}{\sum_{i=1}^p \theta_i^{-1}}, \tag{40}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1'Q_1X_1$ ,  $\zeta_1 \geq \dots \geq \zeta_p$  is the ordered eigenvalues of  $(X_1'Q_1X_1)^{-1/2}(X_2'Q_2X_2)(X_1'Q_1X_1)^{-1/2}$ .

*Proof* Since  $X_1'Q_1X_1 > 0$ , there exists an orthogonal matrix  $K_1$  such that

$$X_1'Q_1X_1 = K_1'\Sigma K_1, \quad \Sigma = \text{diag}(\theta_1, \dots, \theta_p), \tag{41}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1'Q_1X_1$ . Similar to Theorem 3.1, we define

$$A = (X_1'Q_1X_1)^{-1/2} (X_2'Q_2X_2) (X_1'Q_1X_1)^{-1/2}.$$

Since  $A > 0$ , there exists an orthogonal matrix  $K_2$  such that

$$A = K_2'\Delta K_2, \quad \Delta = \text{diag}(\zeta_1, \dots, \zeta_p), \tag{42}$$

where  $\zeta_1 \geq \dots \geq \zeta_p$  is the order eigenvalues of  $A$ .

We can easily compute that

$$\text{tr}(\text{Cov}(\hat{\beta})) = \sigma_1 \text{tr}((X_1'Q_1X_1)^{-1}) = \sigma_1 \sum_{i=1}^p \theta_i^{-1} \tag{43}$$

and

$$\begin{aligned} \text{tr}(\text{Cov}(\beta^*(r))) &= \sigma_1 \text{tr}((X_1'Q_1X_1 + rX_2'Q_2X_2)^{-1}) \\ &= \sigma_1 \text{tr}((X_1'Q_1X_1)^{-1/2} (I_p + rA)^{-1} (X_1'Q_1X_1)^{-1/2}) \\ &= \sigma_1 \text{tr}((I_p + rA)^{-1} (X_1'Q_1X_1)^{-1}) \\ &= \sigma_1 \text{tr}((I_p + r\Delta)^{-1} K_2 K_1' \Sigma^{-1} K_1 K_2') \\ &= \sigma_1 \text{tr}((I_p + r\Delta)^{-1} K' \Sigma^{-1} K), \end{aligned} \tag{44}$$

where  $K = K_1 K_2'$  is an orthogonal matrix. Thus we have

$$\begin{aligned} e_3(\beta^*(r)|\hat{\beta}) &= \frac{\text{tr}(\text{Cov}(\beta^*(r)))}{\text{tr}(\text{Cov}(\hat{\beta}))} \\ &= \frac{\text{tr}((I_p + r\Delta)^{-1} K' \Sigma^{-1} K)}{\sum_{i=1}^p \theta_i^{-1}}. \end{aligned} \tag{45}$$

Using Lemma 3.2, we have

$$\begin{aligned} \sum_{i=1}^p \theta_{p+1-i}^{-1} (1 + r\zeta_i)^{-1} &\leq \text{tr}((I_p + r\Delta)^{-1} K' \Sigma^{-1} K) \\ &\leq \sum_{i=1}^p \theta_{p+1-i}^{-1} (1 + r\zeta_{p+1-i})^{-1}. \end{aligned} \tag{46}$$

Thus

$$\frac{\sum_{i=1}^p \theta_{p+1-i}^{-1} (1 + r\zeta_i)^{-1}}{\sum_{i=1}^p \theta_i^{-1}} \leq e_3(\beta^*(r)|\hat{\beta}) \leq \frac{\sum_{i=1}^p \theta_{p+1-i}^{-1} (1 + r\zeta_{p+1-i})^{-1}}{\sum_{i=1}^p \theta_i^{-1}}. \tag{47}$$

□

**Corollary 4.1** *Let  $\beta^*(r)$  and  $\hat{\beta}$  be given in Equations (7) and (3), let  $e_3(\beta^*(r)|\hat{\beta})$  be defined in Equation (16),  $X_1' Q_1 X_1$  and  $X_2' Q_2 X_2$  communicate, then we have*

$$\frac{\theta_p}{\theta_1 + r\eta_1} \leq e_3(\beta^*(r)|\hat{\beta}) \leq \frac{\theta_1}{\theta_p + r\eta_p}, \tag{48}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1' Q_1 X_1$ ,  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2' Q_2 X_2$ .

*Proof* Since  $X_1' Q_1 X_1$  and  $X_2' Q_2 X_2$  communicate, there exists an orthogonal matrix  $G$  such that

$$G' X_1' Q_1 X_1 G = \text{diag}(\theta_1, \dots, \theta_p) = \Sigma, \tag{49}$$

$$G' X_2' Q_2 X_2 G = \text{diag}(\eta_1, \dots, \eta_p) = \Omega, \tag{50}$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1' Q_1 X_1$ ,  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2' Q_2 X_2$ .

By the definition of  $e_3(\beta^*(r)|\hat{\beta})$ , we have

$$\begin{aligned} e_3(\beta^*(r)|\hat{\beta}) &= \frac{\text{tr}(\text{Cov}(\beta^*(r)))}{\text{tr}(\text{Cov}(\hat{\beta}))} \\ &= \frac{\sum_{i=1}^p (\theta_i + r\eta_i)^{-1}}{\sum_{i=1}^p \theta_i^{-1}}. \end{aligned} \tag{51}$$

Thus we have

$$\frac{\theta_p}{\theta_1 + r\eta_1} \leq e_3(\beta^*(r)|\hat{\beta}) \leq \frac{\theta_1}{\theta_p + r\eta_p}. \tag{52}$$

□

Then we can give the lower and upper bounds of  $e_4(\beta^*(r)|\tilde{\beta})$ .

**Theorem 4.2** *Let  $\beta^*(r)$  and  $\tilde{\beta}$  be given in Equations (7) and (5), let  $e_4(\beta^*(r)|\tilde{\beta})$  be defined in Equation (17), then we have*

$$\frac{\sum_{i=1}^p \eta_{p+1-i}^{-1} (r + \iota_i)^{-1}}{\sum_{i=1}^p \eta_i^{-1}} \leq e_4(\beta^*(r)|\tilde{\beta}) \leq \frac{\sum_{i=1}^p \eta_{p+1-i}^{-1} (r + \iota_{p+1-i})^{-1}}{\sum_{i=1}^p \eta_i^{-1}}, \tag{53}$$

where  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2'Q_2X_2$ ,  $\iota_1 \geq \dots \geq \iota_p$  is the ordered eigenvalues of  $(X_2'Q_2X_2)^{-1/2}(X_1'Q_1X_1)(X_2'Q_2X_2)^{-1/2}$ .

**Corollary 4.2** Let  $\beta^*(r)$  and  $\tilde{\beta}$  be given in Equations (7) and (5), let  $e_4(\beta^*(r)|\tilde{\beta})$  be defined in Equation (17),  $X_1'Q_1X_1$  and  $X_2'Q_2X_2$  communicate, then we have

$$\frac{\eta_p}{\theta_1 + r\eta_1} \leq e_4(\beta^*(r)|\tilde{\beta}) \leq \frac{\eta_1}{\theta_p + r\eta_p}, \quad (54)$$

where  $\theta_1 \geq \dots \geq \theta_p$  is the ordered eigenvalues of  $X_1'Q_1X_1$ ,  $\eta_1 \geq \dots \geq \eta_p$  is the ordered eigenvalues of  $X_2'Q_2X_2$ .

## 5 Concluding remarks

In this article, we have introduced four relative efficiencies in two linear regression equations system with identical parameter vectors, and we have also given the lower and upper bounds for the four relative efficiencies.

### Competing interests

The author declares that they have no competing interests.

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### References

1. Liu, AY: Estimation of the parameters in two linear models with only some of the parameter vectors identical. *Stat. Probab. Lett.* **29**, 369-375 (1996)
2. Ma, TF, Wang, SG: Estimation of the parameters in a two linear regression equations system with identical parameter vectors. *Stat. Probab. Lett.* **79**, 1135-1140 (2009)
3. Yang, H: Extensions of the Kantorovich inequality and the error ratio efficiency of the mean square. *Math. Appl.* **4**, 85-90 (1988)
4. Wang, SG, Ip, WC: A matrix version of the Wielandt inequality and its applications to statistics. *Linear Algebra Appl.* **296**, 171-181 (1999)
5. Liu, SZ: Efficiency comparisons between the OLSE and the BLUE in a singular linear model. *J. Stat. Plan. Inference* **84**, 191-200 (2000)
6. Liu, SZ, Lu, CY, Puntanen, S: Matrix trace Wielandt inequalities with statistical applications. *J. Stat. Plan. Inference* **139**, 2254-2260 (2009)
7. Yang, H, Wang, LT: An alternative form of the Watson efficiency. *J. Stat. Plan. Inference* **139**, 2767-2774 (2009)
8. Wang, LT, Yang, H: Several matrix Euclidean norm inequalities involving Kantorovich inequality. *J. Inequal. Appl.* **2009**, Article ID 291984 (2009)
9. Wang, LT, Yang, H: Matrix Euclidean norm Wielandt inequalities and their applications to statistics. *Stat. Pap.* **53**, 521-530 (2012)
10. Yang, H, Wu, JB: Some matrix norm Kantorovich inequalities and their applications. *Commun. Stat., Theory Methods* **22**, 4078-4085 (2011)
11. Wang, SG, Jia, ZZ: *Matrix Inequality*. Anhui Education Press, Hefei (1994)
12. Yuan, JC: Relative efficiencies of mixed estimator with respect to LS estimator in linear regression model. *J. China Univ. Sci. Technol.* **30**, 285-291 (2000)

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