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Local saturation of a positive linear convolution operator

Hee Sun Jung^{1*} and Ryozi Sakai²

*Correspondence:

hsun90@skku.edu

¹Department of Mathematics
Education, Sungkyunkwan
University, Seoul, 110-745, Republic
of Korea

Full list of author information is
available at the end of the article

Abstract

Let $\{H_n(t)\}$ be a sequence of non-negative, even, and continuous functions on \mathbb{R} . In this paper, we consider a convolution operator $J_n(f; x) = \int_0^\infty f(t)H_n(t-x) dt$, $f \in L_p(\mathbb{R}^+)$, and then investigate the local saturation of $J_n(f; x)$.

MSC: 44A35

Keywords: convolution operator; local saturation

1 Introduction and theorems

Through this paper we let $\{H_n(t)\}$, $n = 1, 2, \dots$, be a sequence of non-negative, even, and continuous functions on $\mathbb{R} := (-\infty, \infty)$, and there exist $M, N > 0$, and $T > 0$ such that $\{H_n(t)\}$ satisfied uniformly

$$\sup_{|t| \geq T} H_n(t) \leq M, \quad n \geq N, \quad (1.1)$$

$$\int_{-\infty}^{\infty} H_n(t) dt = 1, \quad n = 1, 2, \dots, \quad (1.2)$$

$$\int_{-\infty}^{\infty} t^2 H_n(t) dt = \mu_n \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (1.3)$$

and there exist two positive constants α, β with $2\beta \geq \alpha > 3$ such that uniformly for n ,

$$\int_{-\infty}^{\infty} |t|^\alpha H_n(t) dt \leq C\mu_n^\beta. \quad (1.4)$$

As an example for $H_n(t)$ we give the following:

Let

$$H_n(t) = \frac{n}{\sqrt{\pi}} e^{-(nt)^2}.$$

Then it satisfies (1.2),

$$\mu_n = \frac{1}{n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \int_{-\infty}^{\infty} t^4 H_n(t) dt = \frac{3}{4} \frac{1}{n^4}.$$

Let us denote $\mathbb{R}^+ := [0, \infty)$. In what follows we assume that $H_n(t)$ satisfies the conditions (1.1)-(1.4). Using H_n , we define the convolution operators for $f \in L_p(\mathbb{R}^+)$,

$$J_n(f; x) := \int_0^\infty f(t)H_n(t-x) dt, \quad n = 1, 2, \dots \tag{1.5}$$

Swetits and Wood [1] studied the operators on a finite interval $[0, r]$;

$$K_n(f; x) := \int_0^r f(t)H_n^*(t-x) dt, \quad f \in L_p([0, r]), n = 1, 2, \dots,$$

where H_n^* is defined on $[-r, r]$ as H_n , and then they gave a local saturation theorem. Furthermore, there is a rich bibliography concerning the convergence of positive linear operators on $[0, \infty)$ (e.g. see [2-4] and the references cited therein).

In this paper we extend [1] to the infinite interval \mathbb{R}^+ . Then we use a similar methods as [1]. For $1 < p < \infty$ and $c \geq 0$, we define $L_p^2([c, \infty))$ as the space of those functions f such that $f \in L_p(\mathbb{R}^+)$ and f' is a locally absolutely continuous function on $[c, \infty)$, with $f' \in L_p([c, \infty))$ and $f'' \in L_p([c, \infty))$. Let $C_c^2(\mathbb{R}^+)$ be the space of continuous, compactly supported and continuously second differentiable functions on \mathbb{R}^+ . Furthermore, the total variation of a real-valued function f defined on an interval $[a, b] \subset \mathbb{R}$ is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{np-1} |f(x_{i+1}) - f(x_i)|.$$

Here the supremum is taken over the set \mathcal{P} of all partitions $P = \{x_0, x_1, \dots, x_{np}\}$ of the interval considered. If $F : [b, \infty) \rightarrow \mathbb{R}$ and $x \in [b, \infty)$, we define

$$T_F(x) = \sup \left\{ \sum_1^n |F(x_j) - F(x_{j-1})| : b = x_0 < x_1 < \dots < x_n = x, n \in \mathbb{N} \right\}.$$

If $T_F(\infty) := \lim_{x \rightarrow \infty} T_F(x)$ is finite, we say that F is of bounded variation on $[b, \infty)$, and $T_F(\infty)$ is called the total variation of F on $[b, \infty)$. We define $BV[b, \infty)$ to be the set of all functions on $[b, \infty)$ whose total variation on $[b, \infty)$ is finite.

Then we first give the pointwise convergence theorem.

Theorem 1.1 (cf. [5]) *Let $f \in C_c^2(\mathbb{R}^+)$, and let $x \in [a, \infty)$, $a > 0$. Then we have a pointwise convergence as follows:*

$$\lim_{n \rightarrow \infty} \frac{1}{\mu_n} (J_n(f; x) - f(x)) = \frac{1}{2} f''(x). \tag{1.6}$$

Equation (1.6) holds uniformly on $[a, \infty)$.

Then the following is a direct convergence theorem.

Theorem 1.2 *Let $0 < b < a < \infty$.*

(i) *If $1 < p < \infty$, then we have for $f \in L_p^2([b, \infty))$*

$$\|J_n(f; x) - f(x)\|_{L_p([a, \infty))} = O(\mu_n) \quad \text{as } n \rightarrow \infty. \tag{1.7}$$

(ii) If $p = 1$, then we have for $f' \in \text{BV}([b, \infty))$ with $f \in L_1([0, \infty))$

$$\|J_n(f; x) - f(x)\|_{L_1([a, \infty))} = O(\mu_n) \quad \text{as } n \rightarrow \infty. \tag{1.8}$$

(iii) Let $1 \leq p < \infty$ and f be linear on $[b, \infty)$. Then we have

$$\|J_n(f; x) - f(x)\|_{L_p([a, \infty))} = o(\mu_n) \quad \text{as } n \rightarrow \infty. \tag{1.9}$$

Finally, we give an inverse theorem as follows.

Theorem 1.3 Let $0 < b < a < \infty$. Let $f \in L_p(\mathbb{R}^+)$ and $f' \in L_p(\mathbb{R}^+)$.

- (i) For $1 < p < \infty$, the condition (1.7) implies $f \in L_p^2([a, \infty))$.
- (ii) For $p = 1$, the condition (1.8) implies $f' \in \text{BV}([a, \infty))$.
- (iii) For $1 \leq p < \infty$, the condition (1.9) implies that f is linear on $[a, \infty)$.

This paper is organized as follows. In Section 2, we will give some fundamental lemmas in order to prove the main results and we will prove the results in Section 3.

2 Fundamental lemmas

Throughout this paper C, C_1, C_2, \dots denote positive constants independent of n, x, t or function $f(x)$. The same symbol does not necessarily denote the same constant in different occurrences.

To prove the theorems we need some lemmas.

Lemma 2.1 Let $\delta > 0$ and $\ell = 0, 1, 2$. Then for $2\beta \geq \alpha > 3$ defined in (1.4)

$$\int_{|u| \geq \delta} |u|^\ell H_n(u) \, du \leq C \frac{\mu_n^\beta}{\delta^{\alpha-\ell}}. \tag{2.1}$$

Proof Let $\ell = 0, 1, 2$. Then we have from (1.4)

$$\begin{aligned} \int_{|u| \geq \delta} |u|^\ell H_n(u) \, du &= \delta^\ell \int_{|u| \geq \delta} \left(\frac{|u|}{\delta}\right)^\ell H_n(u) \, du \\ &\leq \delta^\ell \int_{|u| \geq \delta} \left(\frac{|u|}{\delta}\right)^\alpha H_n(u) \, du \leq \delta^{\ell-\alpha} \int_{|u| \geq \delta} |u|^\alpha H_n(u) \, du \\ &= \delta^{\ell-\alpha} \int_{-\infty}^{\infty} |u|^\alpha H_n(u) \, du \leq C \frac{\mu_n^\beta}{\delta^{\alpha-\ell}}. \quad \square \end{aligned}$$

Lemma 2.2 Let $e_0(x) := 1, e_1(x) := x$ and let δ be a positive constant. Then for $n = 1, 2, \dots$, we have

$$|J_n((\cdot - x); x)| \leq C \frac{\mu_n^\beta}{\delta^{\alpha-1}}, \quad 0 < \delta \leq x \tag{2.2}$$

and

$$|J_n((\cdot - x)^2; x)| \leq \mu_n, \quad x \geq 0. \tag{2.3}$$

Moreover, we have for $0 < \delta \leq x$

$$|J_n(e_0; x) - e_0| \leq C \frac{\mu_n^\beta}{\delta^\alpha} \quad \text{and} \quad |J_n(e_1; x) - e_1| \leq C \frac{\mu_n^\beta}{\delta^{\alpha-1}}. \tag{2.4}$$

Here, α and β are defined in (1.4).

Proof For $0 < \delta \leq x$, we have by (2.1)

$$\begin{aligned} |J_n((\cdot - x); x)| &= \left| \int_0^\infty (t - x)H_n(t - x) dt \right| = \left| \int_{-x}^\infty yH_n(y) dy \right| \\ &= \left| \int_x^\infty yH_n(y) dy \right| \quad \because H_n(t) \text{ is even} \\ &\leq \int_{y \geq \delta} yH_n(y) dy \leq C \frac{\mu_n^\beta}{\delta^{\alpha-1}}. \end{aligned}$$

For (2.3), we have from (1.3)

$$J_n((\cdot - x)^2; x) = \int_0^\infty (t - x)^2 H_n(t - x) dt = \int_{-x}^\infty y^2 H_n(y) dy \leq \mu_n.$$

Since we know for $0 < \delta \leq x$

$$J_n(e_0; x) = \int_0^\infty H_n(t - x) dt = \int_{-x}^\infty H_n(y) dy = 1 - \int_{-\infty}^{-x} H_n(y) dy = 1 - \int_x^\infty H_n(y) dy,$$

and from (2.1)

$$\left| \int_x^\infty H_n(y) dy \right| \leq C \frac{\mu_n^\beta}{\delta^\alpha}, \tag{2.5}$$

we have for $0 < \delta \leq x$,

$$|J_n(e_0; x) - e_0| \leq C \frac{\mu_n^\beta}{\delta^\alpha}.$$

Next, we give an estimate for e_1 . Since

$$\begin{aligned} J_n(e_1; x) &= \int_0^\infty tH_n(t - x) dt = \int_{-x}^\infty (x + y)H_n(y) dy \\ &= x \left(1 - \int_{-\infty}^{-x} H_n(y) dy \right) + \int_x^\infty yH_n(y) dy \\ &= x - x \int_x^\infty H_n(y) dy + \int_x^\infty yH_n(y) dy \end{aligned}$$

and by (2.1), for $0 < \delta \leq x$,

$$\left| x \int_x^\infty H_n(y) dy \right| + \left| \int_x^\infty yH_n(y) dy \right| \leq 2 \int_x^\infty yH_n(y) dy \leq C \frac{\mu_n^\beta}{\delta^{\alpha-1}},$$

we have for $0 < \delta \leq x$,

$$|J_n(e_1; x) - e_1| \leq C \frac{\mu_n^\beta}{\delta^{\alpha-1}}. \quad \square$$

Let $b > 0$ and then we define

$$\chi([0, b]; t) := \begin{cases} 1, & t \in [0, b], \\ 0, & t \notin [0, b]. \end{cases}$$

Lemma 2.3 *Let $0 < b < a$, $\delta := a - b$ and let $1 \leq p < \infty$. If $f \in L_p(\mathbb{R}^+)$, then*

$$\|J_n(\chi([0, b])f)\|_{L_p[a, \infty)} \leq C \frac{\mu_n^\beta}{\delta^\alpha} \|f\|_{L_p(\mathbb{R}^+)}.$$

Proof Let $p = 1$. Then we have

$$\begin{aligned} \|J_n(\chi([0, b])f)\|_{L_1[a, \infty)} &= \int_a^\infty \left| \int_0^\infty \chi([0, b]; t) f(t) H_n(t-x) dt \right| dx \\ &\leq \int_a^\infty \int_0^b |f(t)| H_n(t-x) dt dx \\ &\leq \int_0^b |f(t)| \int_a^\infty H_n(t-x) dx dt \\ &\leq \left(\sup_{t \in [0, b]} \int_a^\infty H_n(t-x) dx \right) \|f\|_{L_1(\mathbb{R}^+)}. \end{aligned}$$

For $t \in [0, b]$ and $x \geq a$, we see $|t-x| \geq a-b =: \delta > 0$. From (1.4), we have the following:

$$\begin{aligned} \sup_{t \in [0, b]} \int_a^\infty H_n(t-x) dx &\leq \sup_{t \in [0, b]} \int_a^\infty \left(\frac{|t-x|}{\delta} \right)^\alpha H_n(t-x) dx \\ &\leq \frac{1}{\delta^\alpha} \int_{-\infty}^\infty |t|^\alpha H_n(t) dt \leq C \frac{\mu_n^\beta}{\delta^\alpha}. \end{aligned} \quad (2.6)$$

Hence we have

$$\|J_n(\chi([0, b])f)\|_{L_1[a, \infty)} \leq C \frac{\mu_n^\beta}{\delta^\alpha} \|f\|_{L_1(\mathbb{R}^+)}.$$

Let $1 < p < \infty$, $0 < b < a \leq x$ and $p+q = pq$. By Hölder's inequality,

$$\begin{aligned} |J_n(\chi([0, b])f, x)| &= \left| \int_0^\infty \chi([0, b]; t) f(t) H_n(t-x) dt \right| \\ &\leq C \left(\int_0^\infty \chi([0, b]; t) H_n(t-x) dt \right)^{1/q} \\ &\quad \times \left(\int_0^\infty \chi([0, b]; t) |f(t)|^p H_n(t-x) dt \right)^{1/p}. \end{aligned}$$

Here, from (2.1) we have

$$\begin{aligned} \int_0^\infty \chi([0, b]; t) H_n(t-x) dt &= \int_0^b H_n(t-x) dt = \int_{x-b}^x H_n(y) dy \\ &\leq \int_\delta^\infty H_n(y) dy \leq C \frac{\mu_n^\beta}{\delta^\alpha}. \end{aligned} \tag{2.7}$$

Hence,

$$\begin{aligned} &\|J_n(\chi([0, b])f)\|_{L_p[a, \infty)} \\ &\leq C \left(\frac{\mu_n^\beta}{\delta^\alpha}\right)^{1/q} \left(\int_a^\infty \int_0^\infty \chi([0, b]; t) |f(t)|^p H_n(t-x) dt dx\right)^{1/p}. \end{aligned} \tag{2.8}$$

Also, we see by (2.6)

$$\begin{aligned} &\int_a^\infty \int_0^\infty \chi([0, b]; t) |f(t)|^p H_n(t-x) dt dx \\ &\leq \int_0^\infty \int_a^\infty \chi([0, b]; t) |f(t)|^p H_n(t-x) dx dt \\ &\leq \sup_{t \in [0, b]} \int_a^\infty H_n(t-x) dx \int_0^\infty |f(t)|^p dt \leq C \|f\|_{L_p(\mathbb{R}^+)}^p \frac{\mu_n^\beta}{\delta^\alpha}. \end{aligned}$$

Thus, by (2.8) we conclude. □

3 Proof of theorems

Proof of Theorem 1.1 For $x \in [a, \infty)$ and $t \geq 0$, we set

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(\eta)}{2}(t-x)^2, \quad x \leq \eta \leq t.$$

Then we see

$$\begin{aligned} &J_n(f; x) - f(x) - \frac{1}{2}f''(x)\mu_n \\ &= f(x)(J_n(e_0; x) - e_0) + f'(x) \int_0^\infty (t-x)H_n(t-x) dt \\ &\quad + \frac{1}{2} \left[\int_0^\infty f''(\eta)(t-x)^2 H_n(t-x) dt - f''(x)\mu_n \right], \quad x \leq \eta \leq t \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

For J_1 , we have from (2.4)

$$|J_1| = |f(x)| O\left(\frac{\mu_n^\beta}{a^\alpha}\right),$$

and for J_2 , we have from (2.2)

$$|J_2| = |f'(x)J_n((\cdot - x); x)| = |f'(x)| O\left(\frac{\mu_n^\beta}{a^{\alpha-1}}\right).$$

Now, we will estimate J_3 . We have

$$\begin{aligned} & \int_0^\infty f''(\eta)(t-x)^2 H_n(t-x) dt - f''(x)\mu_n \\ &= \int_0^\infty (f''(\eta) - f''(x))(t-x)^2 H_n(t-x) dt - f''(x) \int_{-\infty}^0 (t-x)^2 H_n(t-x) dt. \end{aligned}$$

For the second term, we have by (2.1)

$$\left| f''(x) \int_{-\infty}^0 (t-x)^2 H_n(t-x) dt \right| \leq |f''(x)| \int_a^\infty u^2 H_n(u) du \leq C |f''(x)| \frac{\mu_n^\beta}{a^{\alpha-2}}.$$

For a given $\varepsilon > 0$, there exists a positive constant $\delta_1 > 0$ (depending only on ε) such that for $|x - \eta| < \delta_1$

$$|f''(x) - f''(\eta)| < \varepsilon. \tag{3.1}$$

For the first term, we have using (1.3)

$$\begin{aligned} & \left| \int_{|x-t| \leq \delta_1} (f''(\eta) - f''(x))(t-x)^2 H_n(t-x) dt \right| \\ & \leq \varepsilon \int_{|u| \leq \delta_1} u^2 H_n(u) du \leq \varepsilon \mu_n, \end{aligned}$$

and by (2.1)

$$\begin{aligned} & \left| \int_{|x-t| \geq \delta_1} (f''(\eta) - f''(x))(t-x)^2 H_n(t-x) dt \right| \\ & \leq 2 \|f''\|_{L_\infty(\mathbb{R})} \int_{|u| \geq \delta_1} u^2 H_n(u) du \leq \|f''\|_{L_\infty(\mathbb{R})} O\left(\frac{\mu_n^\beta}{\delta_1^{\alpha-2}}\right). \end{aligned}$$

Then we have for some positive constants C_1 , C_2 , and C_3

$$\begin{aligned} \left| J_n(f; x) - f(x) - \frac{1}{2} f''(x) \mu_n \right| & \leq C_1 |f(x)| \frac{\mu_n^\beta}{a^\alpha} + C_2 |f'(x)| \frac{\mu_n^\beta}{a^{\alpha-1}} \\ & \quad + \varepsilon \mu_n + C_3 \|f''\|_{L_\infty(\mathbb{R})} \frac{\mu_n^\beta}{\delta_1^{\alpha-2}}. \end{aligned}$$

Therefore, we have for an arbitrary $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\mu_n} \left| J_n(f; x) - f(x) - \frac{1}{2} f''(x) \mu_n \right| \leq \varepsilon.$$

Thus, (1.6) is proved. Moreover, noting (3.1), we see that (1.6) holds uniformly on $[a, \infty)$. \square

Proof of Theorem 1.2 Let $\delta := a - b > 0$. (i) We start under the condition $1 \leq p < \infty$ for the convenience of considering (ii). On the way of the proof we switch over to the assumption

with $1 < p < \infty$. Let $1 < p < \infty$ and let $\chi := \chi([0, b])$ and let $\chi_1(t) = 1 - \chi(t)$. Since

$$f(t) = (\chi f)(t) + (\chi_1 f)(t), \quad f(x) = (\chi_1 f)(x), \quad x \in [a, \infty),$$

we have

$$\|J_n(f) - f\|_{L_p([a, \infty))} \leq \|J_n(\chi f)\|_{L_p([a, \infty))} + \|J_n(\chi_1 f) - \chi_1 f\|_{L_p([a, \infty))} =: K_1 + K_2. \tag{3.2}$$

By Lemma 2.3,

$$K_1 = \|J_n(\chi f)\|_{L_p([a, \infty))} \leq C \|f\|_{L_p(\mathbb{R}^+)} \frac{\mu_n^\beta}{\delta^\alpha}. \tag{3.3}$$

We estimate K_2 . Let $f \in L_p^2([a, \infty))$. Then

$$\begin{aligned} J &:= \|J_n((\chi_1 f)(t) - (\chi_1 f)(x), x)\|_{L_p([a, \infty))} \\ &= \left(\int_a^\infty \left| \int_0^\infty (\chi_1(t)f(t) - f(x))H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &\leq \left(\int_a^\infty \left| \left(\int_0^b + \int_b^\infty \right) (\chi_1(t)f(t) - f(x))H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &\leq \left(\int_a^\infty |f(x)|^p \left| \int_0^b H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &\quad + \left(\int_a^\infty \left| \int_b^\infty (f(t) - f(x))H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &=: I_1 + I_2. \end{aligned} \tag{3.4}$$

Here, for the first term, using

$$\int_0^b H_n(t-x) dt \leq C \frac{\mu_n^\beta}{\delta^\alpha},$$

which is shown in (2.7), we have

$$I_1 = \left(\int_a^\infty |f(x)|^p \left| \int_0^b H_n(t-x) dt \right|^p dx \right)^{1/p} \leq C \|f\|_{L_p([a, \infty))} \frac{\mu_n^\beta}{\delta^\alpha}. \tag{3.5}$$

From this we suppose $1 < p < \infty$. By

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (t-u)f''(u) du,$$

we have

$$\begin{aligned} I_2 &= \left(\int_a^\infty \left| \int_b^\infty (f(t) - f(x))H_n(t-x) dt \right|^p dx \right)^{1/p} \\ &\leq \left(\int_a^\infty \left| \int_b^\infty f'(x)(t-x)H_n(t-x) dt \right|^p dx \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
 & + \left(\int_a^\infty \left| \int_b^\infty \int_x^t (t-u)f''(u) du H_n(t-x) dt \right|^p dx \right)^{1/p} \\
 & =: I_{2,1} + I_{2,2}.
 \end{aligned}$$

Here we note that $yH_n(y)$ is an odd function. Then we have by (2.1)

$$\begin{aligned}
 \left| \int_b^\infty (t-x)H_n(t-x) dt \right| & = \left| \int_{b-x}^\infty yH_n(y) dy \right| = \int_{x-b}^\infty yH_n(y) dy \\
 & \leq \int_{y \geq \delta} yH_n(y) dy \leq C \frac{\mu_n^\beta}{\delta^{\alpha-1}}
 \end{aligned}$$

and the first term $I_{2,1}$ is estimated as

$$\begin{aligned}
 I_{2,1} & = \left(\int_a^\infty \left(|f'(x)| \left| \int_b^\infty (t-x)H_n(t-x) dt \right| \right)^p dx \right)^{1/p} \\
 & \leq C \frac{\mu_n^\beta}{\delta^{\alpha-1}} \left(\int_a^\infty |f'(x)|^p dx \right)^{1/p} \leq C \|f'\|_{L_p([a,\infty))} \frac{\mu_n^\beta}{\delta^{\alpha-1}}.
 \end{aligned} \tag{3.6}$$

Now we set

$$\theta(f'', x) := \sup_{b \leq t, t \neq x} \frac{1}{t-x} \int_x^t |f''(u)| du, \quad a \leq x,$$

and denote the Hardy-Littlewood majorant of f'' at x . Since $f'' \in L_p([b, \infty))$ and $1 < p < \infty$, we have

$$\|\theta(f'', x)\|_{L_p([a,\infty))} \leq A_p \|f''\|_{L_p([b,\infty))}, \tag{3.7}$$

where $A_p > 0$ depend only on p [6, Theorem 1, p.201]. Since

$$\left| \int_x^t (t-u)f''(u) du \right| \leq \pm \int_x^t |t-u| |f''(u)| du \leq (t-x)^2 \frac{1}{t-x} \int_x^t |f''(u)| du,$$

where $\pm \int_x^t du \geq 0$, we have by (3.7) and (1.3)

$$\begin{aligned}
 I_{2,2} & \leq \left(\int_a^\infty \left| \int_b^\infty \left((t-x)^2 \frac{1}{t-x} \int_x^t |f''(u)| du \right) H_n(t-x) dt \right|^p dx \right)^{1/p} \\
 & \leq \left(\int_a^\infty |\theta(f''; x)|^p \left| \int_b^\infty (t-x)^2 H_n(t-x) dt \right|^p dx \right)^{1/p} \\
 & \leq \int_{-\infty}^\infty u^2 H_n(u) du \left(\int_a^\infty |\theta(f''; x)|^p dx \right)^{1/p} \\
 & \leq A_p \mu_n \|f''\|_{L_p([b,\infty))}.
 \end{aligned} \tag{3.8}$$

Hence, from (3.6) and (3.8), we have for some positive constant $C > 0$

$$I_2 \leq I_{2,1} + I_{2,2} \leq C (\|f'\|_{L_p([a,\infty))} + \|f''\|_{L_p([b,\infty))}) \mu_n. \tag{3.9}$$

So, from (3.5) and (3.9),

$$J \leq I_1 + I_2 \leq C(\|f\|_{L_p(\mathbb{R}^+)} + \|f'\|_{L_p([a,\infty))} + \|f''\|_{L_p([b,\infty))})\mu_n. \tag{3.10}$$

Now, we see by (3.10) and (2.4)

$$\begin{aligned} K_2 &= \|J_n(\chi_1 f) - (\chi_1 f)\|_{L_p([a,\infty))} \\ &\leq \|J_n((\chi_1 f)(t) - (\chi_1 f)(x), x)\|_{L_p([a,\infty))} + \|(\chi_1 f)(x)(J_n(e_0, x) - e_0)\|_{L_p([a,\infty))} \\ &= J + \|(\chi_1 f)(x)(J_n(e_0, x) - e_0)\|_{L_p([a,\infty))} \\ &\leq C(\|f\|_{L_p(\mathbb{R}^+)} + \|f'\|_{L_p([a,\infty))} + \|f''\|_{L_p([b,\infty))})\mu_n \\ &\quad + \|J_n(e_0, x) - e_0\|_{L_\infty([a,\infty))} \|\chi_1 f\|_{L_p([a,\infty))} \\ &\leq C_1(\|f\|_{L_p(\mathbb{R}^+)} + \|f'\|_{L_p([a,\infty))} + \|f''\|_{L_p([b,\infty))})\mu_n = O(\mu_n). \end{aligned}$$

Consequently, with (3.2) and (3.3) we conclude (i).

(ii) Let $p = 1$ and $f' \in \text{BV}[b, \infty)$. Then we have for $x, t \in [b, \infty)$,

$$f(t) - f(x) = f'(x)(t - x) + \int_x^t (t - u) df'(u). \tag{3.11}$$

On the proof of (i) we recall the part which we assumed as $p = 1$. From (3.3) and (3.5) we may only estimate

$$I'_2 := \int_a^\infty \left| \int_b^\infty (f(t) - f(x)) H_n(t - x) dt dx \right| \tag{3.12}$$

(see (3.4)). By (3.11) we see

$$\begin{aligned} I'_2 &\leq \left| \int_a^\infty \int_b^\infty f'(x)(t - x) H_n(t - x) dt dx \right| \\ &\quad + \left| \int_a^\infty \int_b^\infty \int_x^t (t - u) df'(u) H_n(t - x) dt dx \right| \\ &=: I'_{2,1} + I'_{2,2}. \end{aligned} \tag{3.13}$$

Now, we see that

$$\begin{aligned} I'_{2,1} &= \left| \int_a^\infty \int_b^\infty f'(x)(t - x) H_n(t - x) dt dx \right| \\ &= \left| \int_a^\infty f'(x) \int_{x-b}^\infty u H_n(u) du dx \right| \\ &= \left| \int_{a-b}^\infty u H_n(u) \int_a^{u+b} f'(x) dx du \right| \\ &\leq \sup_{a \leq x < \infty} |f'(x)| \left| \int_\delta^\infty (u - \delta) u H_n(u) du \right| \\ &\leq \sup_{[b,\infty)} |f'(x)| O(\mu_n) \end{aligned}$$

by (1.3). We estimate $I'_{2,2}$. We fix an arbitrary $\eta > 0$. Then we have by means of the substitution $u = y + x$ with a new variable y

$$\begin{aligned} I'_{2,2} &= \left| \int_a^\infty \int_b^\infty \int_x^t (t-u) df'(u) H_n(t-x) dt dx \right| \\ &= \left| \int_a^\infty \int_b^\infty \int_0^{t-x} (t-x-y) df'(x+y) H_n(t-x) dt dx \right| \\ &\leq 2 \int_a^\infty \int_b^\infty \int_0^{|t-x|} |t-x| |df'(x+y)| H_n(t-x) dt dx \quad \because |t-x-y| \leq 2|t-x| \\ &\leq 2 \int_a^\infty \sum_{j=0}^\infty \int_{j\eta \leq |t-x| \leq (j+1)\eta} \int_0^{(j+1)\eta} |df'(x+y)| |t-x| H_n(t-x) dt dx \\ &\leq 2 \sum_{j=0}^\infty \int_{j\eta \leq |u| \leq (j+1)\eta} |u| H_n(u) du \int_a^\infty \int_0^{(j+1)\eta} |df'(x+y)| dx. \end{aligned}$$

Then

$$\begin{aligned} &\int_a^\infty \int_0^{(j+1)\eta} |df'(x+y)| dx \\ &= \int_a^\infty \int_x^{x+(j+1)\eta} |df'(v)| dx \\ &= \int_{a \leq v \leq a+(j+1)\eta} \int_{a \leq x \leq v} dx |df'(v)| + \int_{a+(j+1)\eta \leq v < \infty} \int_{v-(j+1)\eta \leq x \leq v} dx |df'(v)| \\ &:= A_1 + A_2. \end{aligned}$$

Since we can see

$$A_1 = \int_{a \leq v \leq a+(j+1)\eta} \int_{a \leq x \leq v} dx |df'(v)| = \int_{a \leq v \leq a+(j+1)\eta} (v-a) |df'(v)| \leq (j+1)\eta \text{BV}(f'; \mathbb{R}^+)$$

and

$$A_2 = \int_{a+(j+1)\eta \leq v < \infty} \int_{v-(j+1)\eta \leq x \leq v} dx |df'(v)| \leq (j+1)\eta \text{BV}(f'; \mathbb{R}^+),$$

we have

$$\int_a^\infty \int_0^{(j+1)\eta} |df'(x+y)| dx \leq 2(j+1)\eta \text{BV}(f'; \mathbb{R}^+).$$

Now, we estimate $\int_{j\eta \leq |u| \leq (j+1)\eta} |u| H_n(u) du$ for a non-negative integer j . Let $j = 0$. Then from (1.2) and (1.3), we have

$$\int_{0 \leq |u| \leq \eta} |u| H_n(u) du \leq \left(\int_{0 \leq |u| \leq \eta} H_n(u) du \right)^{1/2} \left(\int_{0 \leq |u| \leq \eta} u^2 H_n(u) du \right)^{1/2} = O(\mu_n^{1/2}).$$

Let $j \geq 1$. Then by (2.1) we have

$$\int_{j\eta \leq |u| \leq (j+1)\eta} |u| H_n(u) du \leq C \frac{\mu_n^\beta}{(j\eta)^{\alpha-1}}.$$

Therefore, we have

$$\begin{aligned}
 I'_{2,2} &\leq \int_{0 \leq |u| \leq \eta} |u| H_n(u) du \int_a^\infty \int_0^\eta |df'(x+y)| dx \\
 &\quad + \sum_{j=1}^\infty \int_{j\eta \leq |u| \leq (j+1)\eta} |u| H_n(u) du \int_a^\infty \int_0^{(j+1)\eta} |df'(x+y)| dx \\
 &\leq O(\mu_n^{1/2}) \eta \text{BV}(f'; \mathbb{R}^+) + \sum_{j=1}^\infty \frac{1}{(j\eta)^{\alpha-1}} O(\mu_n^\beta) (j+1) \eta \text{BV}(f'; \mathbb{R}^+) \\
 &= O(\mu_n^{1/2}) \eta \text{BV}(f'; \mathbb{R}^+) + O(\mu_n^\beta) \frac{1}{\eta^{\alpha-2}} \text{BV}(f'; \mathbb{R}^+) \quad (\because \alpha > 3).
 \end{aligned}$$

If we let $\eta = \mu_n^{1/2}$, then we have $I'_{2,2} = O(\mu_n) \text{BV}(f'; \mathbb{R}^+)$, because $\beta - (\alpha - 2)/2 \geq 1$. Consequently, (ii) is proved.

(iii) It follows from (2.4). Consequently, for a linear function f on $[b, \infty)$

$$\|J_n(f; x) - f(x)\|_{L_p([a, \infty))} = O(\mu_n^\beta) = o(\mu_n). \quad \square$$

Proof of Theorem 1.3 (i), (ii) hold as follows: Let $f \in L_p(\mathbb{R}^+)$. First, we choose $\psi \in C^2(\mathbb{R}^+)$ with $\psi'' \in L_q([a, \infty))$ (for any $1 \leq q \leq \infty$) such that

$$\psi(a) = \psi'(a) = \psi''(a) = 0, \quad \lim_{r \rightarrow \infty} \psi^{(i)}(r) = 0, \quad i = 0, 1, 2$$

and

$$\psi(x) = 0 \quad \text{for } x \notin [a, \infty).$$

We use the bilinear functional;

$$A_n(f, \psi) = \frac{1}{\mu_n} \int_0^\infty (J_n(f, x) - f(x)) \psi(x) dx.$$

We will show that for fixed ψ , $A_n(\cdot, \psi)$ is uniformly bounded on $L_p(\mathbb{R}^+)$. We see

$$\int_0^\infty J_n(f, x) \psi(x) dx = \int_0^\infty \psi(x) \int_0^\infty f(t) H_n(t-x) dt dx.$$

For $x, t \in [0, \infty)$ we can write

$$\psi(x) = \psi(t) + \psi'(t)(x-t) + \int_x^t (x-u) \psi''(u) du.$$

Hence,

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty f(t) \psi(x) H_n(t-x) dx dt \\
 &= \int_0^\infty f(t) \psi(t) \int_0^\infty H_n(t-x) dx dt
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\infty f(t)\psi'(t) \int_0^\infty (x-t)H_n(t-x) dx dt \\
 & + \int_0^\infty f(t) \int_0^\infty \int_x^t (x-u)\psi''(u) du H_n(t-x) dx dt \\
 & = I_1'' + I_2'' + I_3''.
 \end{aligned}$$

From (1.2)

$$I_1'' = \int_0^\infty f(t)\psi(t) \int_{-\infty}^t H_n(u) du dt = \int_0^\infty f(t)\psi(t) \left(1 - \int_t^\infty H_n(u) du\right) dt.$$

Since $\psi(t) = 0$ for $t \notin [a, \infty)$ (so, we may take $t \geq a$), by (2.1) we have

$$\begin{aligned}
 & \left| \int_0^\infty f(t)\psi(t) \int_t^\infty H_n(u) du dt \right| \\
 & \leq \left| \int_0^\infty f(t)\psi(t) \int_t^\infty \left(\frac{u}{t}\right)^\alpha H_n(u) du dt \right| \quad \because (u/t)^\alpha \geq 1 \\
 & \leq O(\mu_n^\beta) \int_a^\infty |f(t)\psi(t)| \frac{1}{t^\alpha} dt \quad \text{by (1.4)} \\
 & \leq O(\mu_n^\beta) \sup_{a \leq t < \infty} |\psi(t)| \int_a^\infty |f(t)| \frac{1}{t^\alpha} dt \\
 & \leq O(\mu_n^\beta) \sup_{a \leq t < \infty} |\psi(t)| \|f\|_{L_p([a, \infty))} \|t^{-\alpha}\|_{L_q([a, \infty))},
 \end{aligned}$$

where $1/p + 1/q = 1$. Thus, we have

$$I_1'' = \int_0^\infty f(t)\psi(t) dt + O(\mu_n^\beta) \|f\|_{L_p([a, \infty))}.$$

We estimate I_2'' . We may $a \leq t$, because $\psi(t) = 0$ for $t \notin [a, \infty)$. Noting (1.4) and $\alpha < 3$,

$$\begin{aligned}
 |I_2''| & = \int_a^\infty |f(t)\psi'(t)| \int_t^\infty u H_n(u) du dt \\
 & \leq \int_a^\infty |f(t)\psi'(t)| \int_t^\infty \frac{u^\alpha}{t^{\alpha-1}} H_n(u) du dt \\
 & = O(\mu_n^\beta) \int_a^\infty |f(t)\psi'(t)| \frac{1}{t^{\alpha-1}} dt \\
 & = O(\mu_n^\beta) \|\psi'\|_{L_\infty([a, \infty))} \|f\|_{L_p([a, \infty))} \|t^{1-\alpha}\|_{L_q([a, \infty))} \\
 & = O(\mu_n^\beta) \|f\|_{L_p([a, \infty))}.
 \end{aligned}$$

Finally, we estimate I_3'' . Let $1 < p < \infty$ and $1/p + 1/q = 1$. As the estimation for $I_{2,2}$, we have by (1.3)

$$\begin{aligned}
 |I_3''| & = \left| \int_0^\infty f(t) \int_0^\infty \int_x^t (x-u)\psi''(u) du H_n(t-x) dx dt \right| \\
 & = \left| \int_0^\infty f(t) \int_0^\infty \frac{1}{t-x} \int_x^t \psi''(u) du (t-x)^2 H_n(t-x) dx dt \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \int_a^\infty |f(t)|\theta(\psi'', t) \int_0^\infty (t-x)^2 H_n(t-x) dx dt \\ &= O(\mu_n) \int_a^\infty |f(t)|\theta(\psi'', t) dt \leq O(\mu_n) \|f\|_{L_p([a, \infty))} \|\psi''\|_{L_q([a, \infty))} \\ &= O(\mu_n) \|f\|_{L_p([a, \infty))}. \end{aligned}$$

We estimate I_3'' for $p = 1$. There exists η between x and t such that

$$\begin{aligned} |I_3''| &\leq \left| \int_0^\infty f(t) \int_0^\infty \psi''(\eta)(t-x)^2 H_n(t-x) dx dt \right| \\ &\leq \sup_{a \leq t < \infty} |\psi''(t)| \int_a^\infty |f(t)| \int_0^\infty (t-x)^2 H_n(t-x) dx dt \\ &\leq O(\mu_n) \sup_{a \leq t < \infty} |\psi''(t)| \|f\|_{L_1([a, \infty))} = O(\mu_n) \|f\|_{L_1([a, \infty))}. \end{aligned}$$

Here we used (1.3). Consequently, it follows that $|A_n(f, \psi)|$ is uniformly bounded on $L_p(\mathbb{R}^+)$. Next, from Theorem 1.1 we see that for $f \in C_c^2(\mathbb{R}^+)$,

$$\lim_{n \rightarrow \infty} A_n(f, \psi) = \frac{1}{2} \int_a^\infty f''(x)\psi(x) dx = \frac{1}{2} \int_a^\infty f(x)\psi''(x) dx. \tag{3.14}$$

Since $\{A_n(\cdot, \psi)\}$ is uniformly bounded on $L_p(\mathbb{R}^+)$, and $C_c^2(\mathbb{R}^+)$ is dense in $L_p(\mathbb{R}^+)$, (3.14) yields

$$\lim_{n \rightarrow \infty} A_n(f, \psi) = \frac{1}{2} \int_a^\infty f(x)\psi''(x) dx \tag{3.15}$$

for any $f \in L_p(\mathbb{R}^+)$. Now, for any fixed $f \in L_p(\mathbb{R}^+)$, we consider the sequence of linear functional $\{A_n(f, \cdot)\}$. Since $\|J_n(f) - f\|_{L_p([a, \infty))} = O(\mu_n)$, $n \rightarrow \infty$, there exist $h \in L_p([a, \infty))$ ($p > 1$) and $h \in \text{BV}[a, \infty)$ ($p = 1$) and a subsequence $\{A_{n_j}(f, \cdot)\}$ such that

$$\lim_{j \rightarrow \infty} A_{n_j}(f, \psi) = \begin{cases} \int_0^\infty h(x)\psi(x) dx, & p > 1, \\ \int_0^\infty \psi(x) dh(x), & p = 1. \end{cases} \tag{3.16}$$

From (3.15) and (3.16) we obtain

$$\frac{1}{2} \int_0^\infty f(x)\psi''(x) dx = \begin{cases} \int_0^\infty h(x)\psi(x) dx, & p > 1, \\ \int_0^\infty \psi(x) dh(x), & p = 1. \end{cases} \tag{3.17}$$

A particular solution to (3.17) is

$$\frac{1}{2}f(x) = \begin{cases} \int_\eta^x \int_\eta^\xi h(\mu) d\mu d\xi, & p > 1, \\ \int_\eta^x \int_\eta^\xi dh(\mu) d\xi, & p = 1. \end{cases}$$

The homogeneous problem

$$\int_a^\infty f(x)\psi''(x) dx = 0$$

has the general solution $f(x) = C_1x + C_2$ for $a \leq x < \infty$, since we can take $\psi \in C^2([a, \infty))$ arbitrarily as $\psi'' \in L_q([a, \infty))$, $\psi(a) = \psi'(a) = 0$, $\lim_{r \rightarrow \infty} \psi^{(i)}(r) = 0$, $i = 0, 1$. Hence, if $1 < p < \infty$, $f \in L_p^2([a, \infty))$, and if $p = 1$ then $f' \in \text{BV}([a, \infty))$. Hence, (i) and (ii) hold. We will show (iii). Now, if

$$\|J_n(f) - f\|_{L_p([a, \infty))} = o(\mu_n), \quad n \rightarrow \infty,$$

then

$$\begin{aligned} |A_n(f, \psi)| &\leq \frac{1}{\mu_n} \int_a^\infty |J_n(f, x) - f(x)| |\psi(x)| dx \\ &\leq \left(\sup_{a \leq x < \infty} |\psi(x)| \right) \frac{C_p}{\mu_n} \|J_n(f) - f\|_{L_p([a, \infty))}, \end{aligned}$$

where $C_p > 0$ is independent of n . Hence

$$\lim_{n \rightarrow \infty} A_n(f, \psi) = 0. \tag{3.18}$$

Considering (3.15), (3.16), (3.17), and (3.18), we obtain

$$\int_a^\infty f(x) \psi''(x) dx = 0,$$

and so

$$\int_a^\infty f''(x) \psi(x) dx = 0,$$

consequently, we have $f''(x) = 0$, that is, f is linear on $[a, \infty)$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript and participated in the sequence alignment. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics Education, Sungkyunkwan University, Seoul, 110-745, Republic of Korea. ²Department of Mathematics, Meijo University, Nagoya, 468-8502, Japan.

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