Gu and Cai *Journal of Inequalities and Applications* 2013, **2013**:439
http://www.journalofinequalitiesandapplications.com/content/2013/1/439

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

A sharp inequality for multilinear singular integral operators with non-smooth kernels

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Abstract
In this paper, we establish a sharp inequality for some multilinear singular integral operators with non-smooth kernels. As an application, we obtain the weighted L^p-norm inequality and $L \log L$-type inequality for the multilinear operators.
MSC: 42B20; 42B25

Keywords: multi-linear operator; singular integral operator with non-smooth kernel; sharp inequality; BMO; A_p-weight

1 Definitions and results

As the development of singular integral operators and their commutators, multilinear singular integral operators have been well studied (see [1–6]). In this paper, we study some multilinear operator associated to the singular integral operators with non-smooth kernels as follows.

Definition 1 A family of operators D_t, $t > 0$, is said to be 'approximations to the identity' if, for every $t > 0$, D_t can be represented by the kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x, y)f(y)\,dy$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies

$$\left|a_t(x, y)\right| \leq h_t(x, y) = Ct^{-n/2}s\left(|x - y|^2/t\right),$$

where s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s\left(r^2\right) = 0$$

for some $\epsilon > 0$.

Definition 2 A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{R^n} K(x, y)f(y)\,dy$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an 'approximation to the identity' $\{B_t, t > 0\}$ such that TB_t has an associated kernel $k_t(x,y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} \left|K(x,y) - k_t(x,y)\right| dx \le c_2 \quad \text{for all } y \in R^n.$$

(2) There exists an 'approximation to the identity' $\{A_t, t > 0\}$ such that $A_t T$ has an associated kernel $K_t(x,y)$ which satisfies

$$\left|K_t(x,y)\right| \le c_4 t^{-n/2} \quad \text{if } |x-y| \le c_3 t^{1/2},$$

and

$$\left|K(x,y) - K_t(x,y)\right| \le c_4 t^{\delta/2} |x-y|^{-n-\delta} \quad \text{if } |x-y| \ge c_3 t^{1/2}$$

for some $c_3, c_4 > 0$, $\delta > 0$.

Let m_j be positive integers ($j = 1, \dots, l$), $m_1 + \cdots + m_l = m$, and let b_j be functions on R^n ($j = 1, \dots, l$). Set, for $1 \le j \le m$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x-y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} K(x,y) f(y)\, dy.$$

Note that when $m = 0$, T^b is just the multilinear commutator of T and b_j (see [7]). However, when $m > 0$, T_b is a non-trivial generalization of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–4]). Hu and Yang (see [8]) proved a variant sharp estimate for the multilinear singular integral operators. In [7], Pérez and Trujillo-Gonzalez proved a sharp estimate for the multilinear commutator when $b_j \in Osc_{\exp L^{r_j}}(R^n)$ and noted that $Osc_{\exp L^{r_j}} \subset BMO$. The main purpose of this paper is to prove a sharp function inequality for the multilinear singular integral operator with non-smooth kernel when $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As an application, we obtain an L^p ($p > 1$) norm inequality and an $L \log L$-type inequality for the multilinear operators. In [9–12], the boundedness of a singular integral operator with non-smooth kernel is obtained. In [13], the boundedness of the commutator associated to the singular integral operator with non-smooth kernel is obtained. Our works are motivated by these papers.

First, let us introduce some notations. Throughout this paper, Q denotes a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left|f(y) - f_Q\right| dy,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)\,dx$. It is well known that (see [14, 15])

$$f^{\#}(x) \approx \sup_{Q\ni x} \inf_{c\in \mathbf{C}} \frac{1}{|Q|}\int_Q \left|f(y)-c\right|\,dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$. Let M be a Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q \left|f(y)\right|\,dy.$$

For $k \in N$, we denote by M^k the operator M iterated k times, *i.e.*, $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M\bigl(M^{k-1}(f)\bigr)(x) \quad \text{when } k \geq 2.$$

The sharp maximal function $M_A(f)$ associated with the 'approximations to the identity' $\{A_t, t > 0\}$ is defined by

$$M_A^{\#}(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q \left|f(y) - A_{t_Q}(f)(y)\right|\,dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $0 < r < \infty$, we denote $M_A^{\#}(f)_r$ by

$$M_A^{\#}(f)_r = \bigl[M_A^{\#}\bigl(|f|^r\bigr)\bigr]^{1/r}.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ. For a function f, we denote the Φ-average by

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{Q\ni x} \|f\|_{\Phi,Q}.$$

The Young functions used in this paper are $\Phi(t) = t(1 + \log t)^r$ and $\tilde{\Phi}(t) = \exp(t^{1/r})$, the corresponding average and maximal functions are denoted by $\|\cdot\|_{L(\log L)^r,Q}$, $M_{L(\log L)^r}$ and $\|\cdot\|_{\exp L^{1/r},Q}$, $M_{\exp L^{1/r}}$. Following [11, 12, 16], we know the generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q \left|f(y)g(y)\right|\,dy \leq \|f\|_{\Phi,Q}\|g\|_{\tilde{\Phi},Q}$$

and the following inequality, for $r, r_j \geq 1$, $j = 1,\dots,l$ with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$\|f\|_{L(\log L)^{1/r},Q} \leq M_{L(\log L)^{1/r}}(f) \leq CM_{L(\log L)^l}(f) \leq CM^{l+1}(f),$$

$$\|b - b_Q\|_{\exp L^r,Q} \leq C\|b\|_{BMO},$$

$$|b_{2^{k+1}Q} - b_{2Q}| \leq Ck\|b\|_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$ (see [14]).

We shall prove the following theorems.

Theorem 1 *If T is a singular integral operator with non-smooth kernel as given in Definition* 2, *let $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $0 < r < 1$ and $\tilde{x} \in R^n$,*

$$M_A^\#\big(T^b(f)\big)_r(\tilde{x}) \le C \prod_{j=1}^{l} \bigg(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \bigg) M^{l+1}(f)(\tilde{x}).$$

Theorem 2 *If T is a singular integral operator with non-smooth kernel as given in Definition* 2, *let $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then T^b is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_p$, that is,*

$$\|T^b(f)\|_{L^p(w)} \le C \prod_{j=1}^{l} \bigg(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \bigg) \|f\|_{L^p(w)}.$$

Theorem 3 *If T is a singular integral operator with non-smooth kernel as given in Definition* 2, *let $w \in A_1$, $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,*

$$w\big(\big\{x \in R^n : \big|T^b(f)(x)\big| > \lambda\big\}\big) \le C \int_{R^n} \frac{|f(x)|}{\lambda} \bigg[1 + \log^+\bigg(\frac{|f(x)|}{\lambda}\bigg)\bigg]^l w(x)\,dx.$$

2 Proof of the theorem

To prove the theorems, we need the following lemma.

Lemma 1 (see [1]) *Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$\big|R_m(b; x, y)\big| \le C|x - y|^m \sum_{|\alpha|=m} \bigg(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} \big|D^\alpha b(z)\big|^q\,dz \bigg)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2 ([14, p.485]) *Let $0 < p < q < \infty$ and for any function $f \ge 0$, we define that for $1/r = 1/p - 1/q$,*

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda \big|\big\{x \in R^n : f(x) > \lambda\big\}\big|^{1/q}, \qquad N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},$$

where the sup *is taken for all measurable sets E with $0 < |E| < \infty$. Then*

$$\|f\|_{WL^q} \le N_{p,q}(f) \le \big(q/(q-p)\big)^{1/p} \|f\|_{WL^q}.$$

Lemma 3 (see [17]) *Let $r_j \ge 1$ for $j = 1, \ldots, l$, we denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then*

$$\frac{1}{|Q|} \int_Q \big|f_1(x) \cdots f_l(x) g(x)\big|\,dx \le \|f\|_{\exp L^{r_1}, Q} \cdots \|f\|_{\exp L^{r_l}, Q} \|g\|_{L(\log L)^{1/r}, Q}.$$

Lemma 4 ([9, 10]) *Let T be a singular integral operator with non-smooth kernel as given in Definition 2. Then T is bounded on $L^p(R^n)$ for every $1 < p < \infty$ and bounded from $L^1(R^n)$ to $WL^1(R^n)$.*

Lemma 5 (see [9, 12]) *For any $\gamma > 0$, there exists a constant $C > 0$ independent of γ such that*

$$\left|\left\{x \in R^n : M(f)(x) > D\lambda, M_A^\#(f)(x) \le \gamma\lambda\right\}\right| \le C\gamma\left|\left\{x \in R^n : M(f)(x) > \lambda\right\}\right|$$

for $\lambda > 0$, where D is a fixed constant which only depends on n. Thus

$$\|M(f)\|_{L^p} \le C\|M_A^\#(f)\|_{L^p}$$

for every $f \in L^p(R^n)$, $1 < p < \infty$.

Lemma 6 *Let $\{A_t, t > 0\}$ be an 'approximation to the identity' and $b \in BMO(R^n)$. Then, for every $f \in L^p(R^n)$, $p > 1$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|}\int_Q \left|A_{t_Q}\big((b - b_Q)f\big)(x)\right| dx \le C\|b\|_{BMO} M^2(f)(\tilde{x}),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Proof We write, for any cube Q with $\tilde{x} \in Q$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \left|A_{t_Q}\big((b - b_Q)f\big)(x)\right| dx &\le \frac{1}{|Q|}\int_Q\int_{R^n} h_{t_Q}(x,y)\left|\big(b(y) - b_Q\big)f(y)\right| dy\,dx \\ &\le \frac{1}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x,y)\left|\big(b(y) - b_Q\big)f(y)\right| dy\,dx \\ &\quad + \sum_{k=1}^{\infty}\frac{1}{|Q|}\int_Q\int_{2^{k+1}Q\setminus 2^kQ} h_{t_Q}(x,y)\left|\big(b(y) - b_Q\big)f(y)\right| dy\,dx \\ &= I_1 + I_2.\end{aligned}$$

We have, by the generalized Hölder inequality,

$$\begin{aligned}I_1 &\le C\frac{1}{|Q||2Q|}\int_Q\int_{2Q}\left|\big(b(y) - b_Q\big)f(y)\right| dy\,dx \\ &\le C\|b - b_Q\|_{\exp L, 2Q}\|f\|_{L(\log L), 2Q} \\ &\le C\|b\|_{BMO} M^2(f)(\tilde{x}).\end{aligned}$$

For I_2, notice for $x \in Q$ and $y \in 2^{k+1}Q\setminus 2^kQ$, then $|x - y| \ge 2^{k-1}t_Q$ and $h_{t_Q}(x,y) \le C\frac{s(2^{2(k-1)})}{|Q|}$, then

$$\begin{aligned}I_2 &\le C\sum_{k=1}^{\infty} s\big(2^{2(k-1)}\big)\frac{1}{|Q|^2}\int_Q\int_{2^{k+1}Q}\left|\big(b(y) - b_Q\big)f(y)\right| dy\,dx \\ &\le C\sum_{k=1}^{\infty} 2^{kn} s\big(2^{2(k-1)}\big)\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\big(b(y) - b_Q\big)f(y)\right| dy\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=1}^{\infty}2^{kn}s\big(2^{2(k-1)}\big)\|b-b_Q\|_{\exp L,2^{k+1}Q}\|f\|_{L(\log L),2^{k+1}Q}\\
&\le C\sum_{k=1}^{\infty}2^{(k-1)n}s\big(2^{2(k-1)}\big)\|b\|_{BMO}M^2(f)(\tilde{x})\\
&\le C\|b\|_{BMO}M^2(f)(\tilde{x}),
\end{aligned}$$

where the last inequality follows from

$$\sum_{k=1}^{\infty}2^{(k-1)n}s\big(2^{2(k-1)}\big)\le C\sum_{k=1}^{\infty}2^{-(k-1)\varepsilon}<\infty$$

for some $\epsilon > 0$. This completes the proof. □

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q\big|\big|T^b(f)(x)\big|^r-\big|A_{t_Q}T^b(f)(x)\big|^r\big|\,dx\right)^{1/r}\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\big\|D^{\alpha_j}b_j\big\|_{BMO}\right)M^{l+1}(f)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(b_j;x,y) = R_{m_j}(\tilde{b}_j;x,y)$ and $D^\alpha\tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f = f\chi_{\tilde{Q}} + f\chi_{R^n\setminus\tilde{Q}} = f_1 + f_2$,

$$\begin{aligned}
T^b(f)(x) &= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f(y)\,dy = \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f_1(y)\,dy\\
&\quad-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m}K(x,y)f_1(y)\,dy\\
&\quad-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)f_1(y)\,dy\\
&\quad+\sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)f_1(y)\,dy\\
&\quad+\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)f_2(y)\,dy\\
&= T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_1\right)-T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\\
&\quad-T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\\
&\quad+T\left(\sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\\
&\quad+T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)
\end{aligned}$$

and

$$\begin{aligned}
A_{t_Q}T^b(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K_t(x,y) f_1(y)\,dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K_t(x,y) f_1(y)\,dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_t(x,y) f_1(y)\,dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K_t(x,y) f_1(y)\,dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K_t(x,y) f_2(y)\,dy \\
&= A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right) \\
&\quad - A_{t_Q}T\left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right) \\
&\quad - A_{t_Q}T\left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
&\quad + A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right) \\
&\quad + A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_2\right),
\end{aligned}$$

then

$$\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q \big| |T^b(f)(x)|^r - |A_{t_Q}T^b(f)(x)|^r \big|\,dx\right]^{1/r} \\
&\quad \le \left[\frac{1}{|Q|}\int_Q |T^b(f)(x) - A_{t_Q}T^b(f)(x)|^r\,dx\right]^{1/r} \\
&\quad \le \left[\frac{C}{|Q|}\int_Q \left| T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad + \left[\frac{C}{|Q|}\int_Q \left| T\left(\sum_{|\alpha_1|=m_1} \frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m} f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad + \left[\frac{C}{|Q|}\int_Q \left| T\left(\sum_{|\alpha_2|=m_2} \frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad + \left[\frac{C}{|Q|}\int_Q \left| T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_Q \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m} f_1\right)\right|^r dx\right]^{1/r} \\
&\qquad + \left[\frac{C}{|Q|}\int_Q \left| A_{t_Q}T\left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m} f_1\right)\right|^r dx\right]^{1/r}
\end{aligned}$$

$$+\left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r}$$
$$+\left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r}$$
$$+\left[\frac{C}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}f_1\right)\right|^r dx\right]^{1/r}$$
$$+\left[\frac{C}{|Q|}\int_Q\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}f_2\right)\right|^r dx\right]^{1/r}$$
$$:= I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.$$

Now, let us estimate I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8 and I_9, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_m(\tilde{b}_j;x,y)\le C|x-y|^m\sum_{|\alpha_j|=m}\|D^{\alpha_j}b_j\|_{BMO},$$

by Lemma 2 and the weak type (1,1) of T (Lemma 4), we obtain

$$\begin{aligned}
I_1&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(f_1)(x)|^r dx\right)^{1/r}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)|Q|^{-1}\frac{\|T(f_1)\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)|Q|^{-1}\|T(f_1)\|_{WL^1}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)|\tilde{Q}|^{-1}\|f_1\|_{L^1}\\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)M(f)(\tilde{x}).
\end{aligned}$$

For I_2, we get, by Lemma 2 and the generalized Hölder inequality,

$$\begin{aligned}
I_2&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}\left|T\left(D^{\alpha_1}\tilde{b}_1f_1\right)(x)\right|^r dx\right)^{1/r}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\frac{\|T(D^{\alpha_1}\tilde{b}_1f_1)\chi_Q\|_{L^r}}{|Q|^{1/r-1}}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}\left\|T\left(D^{\alpha_1}\tilde{b}_1f_1\right)\right\|_{WL^1}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\sum_{|\alpha_1|=m_1}|\tilde{Q}|^{-1}\left\|D^{\alpha_1}\tilde{b}_1f_1\right\|_{L^1}
\end{aligned}$$

$$\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} b_1 - (D^{\alpha} b_1)_{\tilde{Q}}\|_{\exp L, \tilde{Q}} \|f\|_{L(\log L), \tilde{Q}}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) M^2(f)(\tilde{x}).$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M^2(f)(\tilde{x}).$$

Similarly, for I_4, taking $r, r_1, r_2 \ge 1$ such that $1/r = 1/r_1 + 1/r_2$, we obtain, by Lemma 3 and the generalized Hölder inequality,

$$I_4 \le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} \left| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)(x)\right|^r dx \right)^{1/r}$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} \frac{\|T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)\chi_Q\|_{L^r}}{|Q|^{1/r-1}}$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} \left\| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)\right\|_{WL^1}$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1} \left\| D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1 \right\|_{L^1}$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^{2} \left\| D^{\alpha_j} b_j - (D^{\alpha_j} b_j)_{\tilde{Q}} \right\|_{\exp L^{r_j}, \tilde{Q}} \cdot \|f\|_{L(\log L)^{1/r}, \tilde{Q}}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M^3(f)(\tilde{x}).$$

For I_5, I_6, I_7 and I_8, by Lemma 6, we get

$$\begin{aligned} I_5 + I_6 + I_7 + I_8 \le{} & C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q \left| A_{t_Q} T(f_1)(x) \right| dx \\ & + C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q \left| A_{t_Q} T\left(D^{\alpha_1} \tilde{b}_1 f_1\right)(x) \right| dx \\ & + C \sum_{|\alpha_1|=m_1} \|D^{\alpha_1} b_1\|_{BMO} \sum_{|\alpha_2|=m_2} \frac{1}{|Q|} \int_Q \left| A_{t_Q} T\left(D^{\alpha_2} \tilde{b}_2 f_1\right)(x) \right| dx \\ & + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q \left| A_{t_Q} T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1\right)(x) \right| dx \\ \le{} & C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M^3(f)(\tilde{x}). \end{aligned}$$

For I_9, we write

$$\begin{aligned}
&(T - A_{t_Q} T)\left(\frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, \cdot)}{|x - \cdot|^m} f_2\right) \\
&= \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{b}_j; x, y)}{|x - y|^m} \big(K(x,y) - K_t(x,y)\big) f_2(y)\, dy \\
&= \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y)}{|x - y|^m} \big(K(x,y) - K_t(x,y)\big) f_2(y)\, dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{D^{\alpha_1}\tilde{b}_1(y)(x-y)^{\alpha_1} R_{m_2}(\tilde{b}_2; x, y)}{|x - y|^m} \big(K(x,y) - K_t(x,y)\big) f_2(y)\, dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_2} R_{m_1}(\tilde{b}_1; x, y)}{|x - y|^m} \big(K(x,y) - K_t(x,y)\big) f_2(y)\, dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_1+\alpha_2}}{|x - y|^m} \big(K(x,y) - K_t(x,y)\big) f_2(y)\, dy \\
&= I_9^{(1)} + I_9^{(2)} + I_9^{(3)} + I_9^{(4)}.
\end{aligned}$$

By Lemma 1 and the following inequality (see [15])

$$|b_{Q_1} - b_{Q_2}| \le C \log\big(|Q_2|/|Q_1|\big) \|b\|_{BMO} \quad \text{for } Q_1 \subset Q_2,$$

we know that for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}
\big|R_m(\tilde{b}; x, y)\big| &\le C|x - y|^m \sum_{|\alpha|=m} \left(\|D^\alpha b\|_{BMO} + \big|(D^\alpha b)_{\tilde{Q}(x,y)} - (D^\alpha b)_{\tilde{Q}}\big|\right) \\
&\le Ck|x - y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO}.
\end{aligned}$$

Note that $|x - y| \ge d = t^{1/2}$ and $|x - y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$. By the conditions on K and K_t, we obtain

$$\begin{aligned}
\big|I_9^{(1)}\big| &= \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{\prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j; x, y)|}{|x - y|^m} \big|K(x,y) - K_t(x,y)\big| \big|f(y)\big|\, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}\right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \frac{d^\delta}{|x_0 - y|^{n+\delta}} \big|f(y)\big|\, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}\right) \sum_{k=1}^{\infty} k^2 2^{-\delta k} \frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \big|f(y)\big|\, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}\right) M(f)(\tilde{x}).
\end{aligned}$$

For $I_9^{(2)}$, we get, by the generalized Hölder inequality,

$$\begin{aligned}
|I_9^{(2)}| &\le C\Big(\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\Big)\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{kd^\delta}{|x_0-y|^{n+\delta}}|D^{\alpha_1}\tilde{b}_1(y)||f(y)|\,dy \\
&\le C\Big(\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}b_2\|_{BMO}\Big) \\
&\quad\times\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k2^{-\delta k}\|D^{\alpha_1}b_1-(D^{\alpha_1}b_1)_{\tilde{Q}}\|_{\exp L,2^k\tilde{Q}}\|f\|_{L(\log L),2^k\tilde{Q}} \\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{BMO}\Big)M^2(f)(\tilde{x}).
\end{aligned}$$

Similarly,

$$|I_9^{(3)}| \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{BMO}\Big)M^2(f)(\tilde{x}).$$

For $I_9^{(4)}$, taking $r, r_1, r_2 \ge 1$ such that $1/r = 1/r_1 + 1/r_2$, by Lemma 3 and the generalized Hölder inequality, we get

$$\begin{aligned}
|I_9^{(4)}| &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{d^\delta}{|x_0-y|^{n+\delta}}|D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f(y)|\,dy \\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}\prod_{j=1}^{2}\|D^{\alpha_j}b_j-(D^{\alpha_j}b_j)_{\tilde{Q}}\|_{\exp L^{r_j},2^k\tilde{Q}}\|f\|_{L(\log L)^{1/r},2^k\tilde{Q}} \\
&\le C\prod_{j=1}^{2}\Big(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{BMO}\Big)M^3(f)(\tilde{x}).
\end{aligned}$$

Thus

$$|I_5| \le C\prod_{j=1}^{2}\Big(\sum_{|\alpha|=m_j}\|D^\alpha b_j\|_{BMO}\Big)M^3(f)(\tilde{x}).$$

This completes the proof of Theorem 1. □

By Theorem 1 and the $L^p(w)$-boundedness of M^{l+1}, we may obtain the conclusions of Theorem 2. By Theorem 1 and [16, 17], we may obtain the conclusions of Theorem 3.

3 Applications

In this section we shall apply Theorems 1, 2 and 3 of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [9]). Given $0 \le \theta < \pi$, define

$$S_\theta = \{z \in C : |\arg(z)| \le \theta\} \cup \{0\}$$

and its interior by S_θ^0. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and if for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta| \| (\eta I - L)^{-1} \| \le C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty(S_\mu^0) = \{f : S_\theta^0 \to C : f \text{ is holomorphic and } \|f\|_{L^\infty} < \infty\},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Set

$$\Psi(S_\mu^0) = \left\{ g \in H_\infty(S_\mu^0) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \le c \frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta)\, d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \ge 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-to-one and has a dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = [h(L)]^{-1} (fh)(L),$$

where $h(z) = z(1 + z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ if

$$\|g(L)\| \le N \|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [9], we get the following.

Theorem 4 *Assume the following conditions are satisfied*:

(i) *The holomorphic semigroup e^{-zL}, $0 \le |\arg(z)| < \pi/2 - \theta$ is represented by the kernels $a_z(x, y)$ which satisfy, for all $\nu > \theta$, an upper bound*

$$|a_z(x, y)| \le c_\nu h_{|z|}(x, y)$$

for $x, y \in R^n$, and $0 \le |\arg(z)| < \pi/2 - \theta$, where $h_t(x, y) = Ct^{-n/2} s(|x - y|^2/t)$ and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon} s(r^2) = 0.$$

(ii) *The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S_\mu^0)$, the operator $g(L)$ satisfies*

$$\|g(L)(f)\|_{L^2} \le c_\nu \|g\|_{L^\infty} \|f\|_{L^2}.$$

Then, for $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$, the multilinear operator $g(L)^b$ associated to $g(L)$ and b_j satisfies:

(a) *For $0 < r < 1$ and $\tilde{x} \in R^n$,*

$$M_A^\#\big(g(L)^b(f)\big)_r(\tilde{x}) \le C \prod_{j=1}^{l} \bigg(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \bigg) M^{l+1}(f)(\tilde{x});$$

(b) *$g(L)^b$ is bounded on $L^p(w)$ for any $1 < p < \infty$ and $w \in A_p$, that is,*

$$\|g(L)^b(f)\|_{L^p(w)} \le C \prod_{j=1}^{l} \bigg(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \bigg) \|f\|_{L^p(w)};$$

(c) *There exists a constant $C > 0$ such that for all $\lambda > 0$ and $w \in A_1$,*

$$w\big(\big\{x \in R^n : \big|g(L)^b(f)(x)\big| > \lambda\big\}\big) \le C \int_{R^n} \frac{|f(x)|}{\lambda} \left[1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right]^l w(x)\,dx.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

Acknowledgements
Project supported by Hunan Provincial Natural Science Foundation of China (12JJ6003) and Scientific Research Fund of Hunan Provincial Education Departments (12K017).

Received: 17 March 2013 Accepted: 12 August 2013 Published: 16 September 2013

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doi:10.1186/1029-242X-2013-439
Cite this article as: Gu and Cai: **A sharp inequality for multilinear singular integral operators with non-smooth kernels.** *Journal of Inequalities and Applications* 2013 **2013**:439.