Watanabe *Journal of Inequalities and Applications* 2013, **2013**:137
http://www.journalofinequalitiesandapplications.com/content/2013/1/137

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

On a relation between Schur, Hardy-Littlewood-Pólya and Karamata's theorem and an inequality of some products of $x^p - 1$ derived from the Furuta inequality

Keiichi Watanabe*

*Correspondence:
wtnbk@math.sc.niigata-u.ac.jp
Department of Mathematics, Faculty of Science, Niigata University, Niigata, 950-2181, Japan

Abstract

We show a functional inequality of some products of $x^p - 1$ as an application of an operator inequality. Furthermore, we will show it can be deduced from a classical theorem on majorization and convex functions.

MSC: Primary 26D07; secondary 26A09; 26A51; 39B62; 47A63

Keywords: inequalities; fractional powers; convex functions; majorization; matrix inequalities; Furuta inequality

1 Introduction

It is easy to see the inequalities

$$595(x^6 - 1)(x^8 - 1)^2(x^9 - 1) \leq 1728(x^2 - 1)(x^5 - 1)(x^7 - 1)(x^{17} - 1)$$

or

$$48(x^2 - 1)(x^3 - 1)(x^5 - 1)(x^7 - 1)(x^{11} - 1) \leq 385(x - 1)^2(x^4 - 1)^2(x^{18} - 1)$$

for arbitrary $1 < x$ if they are provided as the matter to be proved. However, if we would like to estimate functions of the form

$$\prod(x^{p_j} - 1)$$

by simpler ones, how can we guess what forms and coefficients are possible?

Example The following inequality does not hold on an interval contained in $1 < x$.

$$225(x^2 - 1)^2(x^8 - 1)^2 \leq 256(x - 1)(x^5 - 1)^2(x^9 - 1).$$

In Section 2, we prove a certain functional inequality as mentioned above, although the efficiency and possible applications to other branches of mathematics are still to be clarified.

In Section 3, we show that the functional inequality derived in Section 2 can be easily deduced from Schur, Hardy-Littlewood-Pólya and Karamata's theorem on majorization and convex functions. Although the proof presented in Section 2 looks like a detour, one should note that it naturally arises as a byproduct of the Furuta inequality, which is an epochmaking extension of the celebrated Löwner-Heinz inequality [1, 2]. It seems worthy to compare various ways to derive fundamental functional inequalities, for it might contribute to clarify relations between their background theories and to suggest further developments.

2 An inequality of some products of $x^p - 1$

The proof of the following theorem is based on an operator inequality by Furuta [3] and an argument related to the best possibility of that by Tanahashi [4]. The main feature of the argument is applying an order-preserving operator inequality to matrices which contain variables as their entries. It might be a new method to obtain functional inequalities systematically.

Theorem 2.1 [5] *Let* $0 \le p$, $1 \le q$ *and* $0 \le r$ *with* $p + r \le (1 + r)q$. *If* $0 < x$, *then*

$$x^{\frac{1+r-\frac{p+r}{q}}{2}}\left(x^p - 1\right)\left(x^{\frac{p+r}{q}} - 1\right) \le \frac{p}{q}\left(x^{p+r} - 1\right)(x - 1).$$

Proposition 2.2 *Let* $1 \le p$, $0 \le r$. *Then, for arbitrary* $0 < x$,

$$(p + r)\left(x^p - 1\right)\left(x^{1+r} - 1\right) \le p(1 + r)\left(x^{p+r} - 1\right)(x - 1). \tag{1}$$

Proof Put $q = \frac{p+r}{1+r}$. Since $1 \le p$, we have $1 \le q$, and hence Proposition 2.2 immediately follows from Theorem 2.1. □

Theorem 2.3 *Let* $0 < p_2 \le p_1$, $0 < q_2 \le q_1$, $p_1 + p_2 = q_1 + q_2$ *and* $p_1 \le q_1$. *Then, for arbitrary* $0 < x$,

$$q_1 q_2\left(x^{p_1} - 1\right)\left(x^{p_2} - 1\right) \le p_1 p_2\left(x^{q_1} - 1\right)\left(x^{q_2} - 1\right). \tag{2}$$

Proof For a moment, we add $1 \le p_1, p_2, q_1$ and $q_2 = 1$ to the assumption. Apply Proposition 2.2 with $p = p_2$, $r = p_1 - 1$, then the inequality (1) implies

$$q_1\left(x^{p_1} - 1\right)\left(x^{p_2} - 1\right) \le p_1 p_2\left(x^{q_1} - 1\right)(x - 1).$$

In general, note that $q_2 \le p_2$. Dividing by q_2, we have

$$1 \le \frac{p_2}{q_2} \le \frac{p_1}{q_2}, \qquad 1 \le \frac{q_1}{q_2}, \qquad \frac{p_1}{q_2} + \frac{p_2}{q_2} = \frac{q_1}{q_2} + 1 \quad \text{and} \quad \frac{p_1}{q_2} \le \frac{q_1}{q_2}.$$

By the first part of the proof,

$$\frac{q_1}{q_2}\left(x^{\frac{p_1}{q_2}} - 1\right)\left(x^{\frac{p_2}{q_2}} - 1\right) \le \frac{p_1}{q_2} \cdot \frac{p_2}{q_2}\left(x^{\frac{q_1}{q_2}} - 1\right)(x - 1)$$

for arbitrary $0 < x$. By substituting x^{q_2} to x in the above inequality, it is immediate to see the inequality (2). □

Definition 2.4 For a finite sequence $p_1, \ldots, p_n$ of real numbers, we denote its decreasing rearrangement by $p_{[1]} \geq \cdots \geq p_{[n]}$.

For two vectors $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$, p is said to be majorized by q and denoted by $(p_1, \ldots, p_n) \prec (q_1, \ldots, q_n)$ if the following inequalities are satisfied:

$$\sum_{i=1}^{k} p_{[i]} \leq \sum_{i=1}^{k} q_{[i]}, \quad k = 1, \ldots, n-1,$$
$$\sum_{i=1}^{n} p_{[i]} = \sum_{i=1}^{n} q_{[i]}.$$

Theorem 2.5 *Let n be a natural number. Suppose* $0 < p_j, q_j$, $j = 1, \ldots, n$ *and* $(p_1, \ldots, p_n) \prec (q_1, \ldots, q_n)$. *Then, for arbitrary* $1 < x$,

$$\prod_{j=1}^{n} q_j\left(x^{p_j} - 1\right) \leq \prod_{j=1}^{n} p_j\left(x^{q_j} - 1\right). \tag{3}$$

If n is even, the inequality (3) *holds for arbitrary* $0 < x < 1$. *If n is odd, the reverse inequality of* (3) *holds for arbitrary* $0 < x < 1$.

Proof The case $n = 2$ is exactly Theorem 2.3. Suppose that the case n is valid. We may assume $0 < p_{n+1} \leq p_n \leq \cdots \leq p_1$, $0 < q_{n+1} \leq q_n \leq \cdots \leq q_1$ and

$$\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \quad (k = 1, \ldots, n) \quad \text{and} \quad \sum_{i=1}^{n+1} p_i = \sum_{i=1}^{n+1} q_i.$$

There exists a number k such that $1 \leq k \leq n$ and

$$q_{n+1} \leq \cdots \leq q_{k+1} \leq p_{n+1} \leq q_k \leq \cdots \leq q_1.$$

Take a real number q' which is determined by $q_k + q_{k+1} = p_{n+1} + q'$. Then

$$q_{k+1} \leq q' = q_k + q_{k+1} - p_{n+1} \leq q_k.$$

By the case $n = 2$,

$$q_k q_{k+1}\left(x^{p_{n+1}} - 1\right)\left(x^{q'} - 1\right) \leq p_{n+1} q'\left(x^{q_k} - 1\right)\left(x^{q_{k+1}} - 1\right). \tag{4}$$

Since

$$p_1 + \cdots + p_{n+1} = q_1 + \cdots + q_{n+1} = p_{n+1} + q' + \sum_{j \neq k, k+1} q_j,$$

we have

$$p_1 + \cdots + p_n = q' + \sum_{j \neq k, k+1} q_j \tag{5}$$

and

$$q_{n+1} \le \cdots \le q_{k+1} \le q' \le q_k \le \cdots \le q_1.$$

Note that

$$\begin{aligned} &p_1 + \cdots + p_{n-1} \le q_1 + \cdots + q_{n-1}, \\ &\vdots \\ &p_1 \le q_1 \end{aligned}$$

by the assumption of the induction.

If $k = n$, then the n-tuples $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_{n-1}, q'\}$ satisfy the assumption of the case n, so we may assume $k \neq n$ by using the inequality (4).

Equality (5) and $q_{n+1} \le p_n$ yield

$$p_1 + \cdots + p_{n-1} \le q' + \sum_{j \neq k, k+1, n+1} q_j.$$

If $k = n - 1$, then the n-tuples $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_{n-2}, q', q_{n+1}\}$ satisfy the assumption of the case n, so we may assume $k \neq n, n - 1$. For $k \le n - 2$, we have

$$\begin{aligned} p_1 + \cdots + p_{n-1} &\le q_1 + \cdots + q_{n-1} \\ &= p_{n+1} + q' + \sum_{j \le n-1, j \neq k, k+1} q_j \le p_{n-1} + q' + \sum_{j \le n-1, j \neq k, k+1} q_j, \end{aligned}$$

and hence

$$p_1 + \cdots + p_{n-2} \le q' + \sum_{j \le n-1, j \neq k, k+1} q_j.$$

Similarly, we have

$$\begin{aligned} &p_1 + \cdots + p_{n-3} \le q' + \sum_{j \le n-2, j \neq k, k+1} q_j, \\ &\vdots \\ &p_1 + \cdots + p_k \le q' + \sum_{j=1}^{k-1} q_j, \\ &p_1 + \cdots + p_{k-1} \le q_1 + \cdots + q_{k-1}, \\ &\vdots \\ &p_1 + p_2 \le q_1 + q_2, \\ &p_1 \le q_1. \end{aligned}$$

Therefore, n-tuples $\{p_1,\dots,p_n\}$, $\{q_1,\dots,q_{k-1},q',q_{k+2},\dots,q_{n+1}\}$ satisfy the assumption of the case n, and so we can obtain

$$q'\prod_{j\neq k,k+1} q_j \prod_{j=1}^{n}\left(x^{p_j}-1\right) \le \prod_{j=1}^{n} p_j\left(x^{q'}-1\right)\prod_{j\neq k,k+1}\left(x^{q_j}-1\right) \tag{6}$$

for arbitrary $1<x$.

From (4) and (6), it is immediate to see that

$$\prod_{j=1}^{n+1} q_j \prod_{j=1}^{n+1}\left(x^{p_j}-1\right) \le \prod_{j=1}^{n+1} p_j \prod_{j=1}^{n+1}\left(x^{q_j}-1\right)$$

for $1<x$.

The last assertion of the theorem can be easily seen by substituting $\frac{1}{x}$ for $0<x<1$ and multiplying $x^{p_1+\cdots+p_n}$ to both sides.

This completes the proof. □

Remark 2.6 Each following example of the case $n=5$ does not satisfy one of the conditions for parameters in the assumption of Theorem 2.5, and the inequality does not hold for all $1<x$.

(i) $p_1 > q_1$

$$4\cdot 6\cdot 8\left(x^2-1\right)^2\left(x^3-1\right)^2\left(x^{10}-1\right) \le 2^2\cdot 3^2\cdot 10(x-1)^2\left(x^4-1\right)\left(x^6-1\right)\left(x^8-1\right).$$

(ii) $p_1+p_2 > q_1+q_2$

$$6\cdot 8\cdot 12\left(x^2-1\right)^3\left(x^{11}-1\right)^2 \le 2^3\cdot 11^2(x-1)^2\left(x^6-1\right)\left(x^8-1\right)\left(x^{12}-1\right).$$

(iii) $p_1+p_2+p_3 > q_1+q_2+q_3$

$$5^2\cdot 10^2\left(x^2-1\right)^2\left(x^9-1\right)^3 \le 2^2\cdot 9^3(x-1)\left(x^5-1\right)^2\left(x^{10}-1\right)^2.$$

(iv) $p_1+p_2+p_3+p_4 > q_1+q_2+q_3+q_4$

$$3^4\cdot 9(x-1)\left(x^5-1\right)^4 \le 5^4\left(x^3-1\right)^4\left(x^9-1\right).$$

Remark 2.7 There exists an example of the case $n=3$ such that $p_1>q_1$, but the inequality holds for $1<x$.

$$5\cdot 6\left(x^2-1\right)\left(x^3-1\right)\left(x^7-1\right) \le 2\cdot 3\cdot 7(x-1)\left(x^5-1\right)\left(x^6-1\right).$$

Indeed,

$$\begin{aligned}&2\cdot 3\cdot 7(x-1)\left(x^5-1\right)\left(x^6-1\right) - 5\cdot 6\left(x^2-1\right)\left(x^3-1\right)\left(x^7-1\right)\\ &\quad= 6(x-1)^7(x+1)\left(x^2+x+1\right)\left(2x^2+3x+2\right).\end{aligned}$$

3 A proof by Schur, Hardy-Littlewood-Pólya and Karamata's theorem

Theorem 2.5 is a special case of a more general theorem on majorization and convex functions.

Theorem 3.1 (C.1. Proposition in [6], Theorem 108 in [7], Karamata [8]) *Let n be a natural number and p_j, q_j be real numbers from an interval (α,β). If $(p_1,\ldots,p_n) \prec (q_1,\ldots,q_n)$, then*

$$\sum_{j=1}^{n} f(p_j) \leq \sum_{j=1}^{n} f(q_j)$$

for every real-valued convex function f on (α,β).

Proposition 3.2 *Let $1 < x$ be a fixed real number. Then*

$$f(t) = \log\left(\frac{x^t - 1}{t}\right)$$

is convex on the interval $(0,\infty)$.

Although it is definitely elementary to prove this proposition, we will give it for the sake of completeness.

Proof One can calculate the derivatives of f with respect to t,

$$f'(t) = \frac{x^t(\log x)t - x^t + 1}{t(x^t - 1)},$$
$$f''(t) = \frac{-x^t(\log x)^2 t^2 + x^{2t} - 2x^t + 1}{t^2(x^t - 1)^2}.$$

The signature of f'' is the same as g, where

$$g(t) = -x^t(\log x)^2 t^2 + x^{2t} - 2x^t + 1.$$

It is easy to see

$$g'(t) = x^t(\log x)\bigl(-t^2(\log x)^2 - 2t\log x + 2x^t - 2\bigr).$$

The signature of g' is the same as g_1, where

$$g_1(t) = -t^2(\log x)^2 - 2t\log x + 2x^t - 2.$$

It is also easy to see

$$g_1'(t) = 2(\log x)\bigl(-t\log x - 1 + x^t\bigr).$$

The signature of g_1' is the same as g_2, where

$$g_2(t) = -t\log x - 1 + x^t.$$

Now we have

$$g_2'(t) = -\log x + x^t \log x = (x^t - 1)\log x > 0 \quad (0 < t).$$

Therefore, g_2 is increasing on $0 < t$ and $g_2(0) = 0$ so that $0 < g_2(t)$ $(0 < t)$, and hence $0 < g_1'(t)$ $(0 < t)$.

Again, therefore, g_1 is increasing on $0 < t$ and $g_1(0) = 0$ so that $0 < g_1(t)$ $(0 < t)$, and hence $0 < g'(t)$ $(0 < t)$.

Once again, therefore, g is increasing on $0 < t$ and $g(0) = 0$ so that $0 < g(t)$ $(0 < t)$, and hence $0 < f''(t)$ $(0 < t)$, namely, f is convex on the interval $(0, \infty)$. This completes the proof of Proposition 3.2. □

The completion of the proof of Theorem 2.5 by using Schur, Hardy-Littlewood-Pólya and Karamata's theorem.

For arbitrary $1 < x$, $f(t) = \log(\frac{x^t - 1}{t})$ is a convex function on the interval $(0, \infty)$, so we can apply Theorem 3.1 to obtain

$$\sum_{j=1}^{n} \log\left(\frac{x^{p_j} - 1}{p_j}\right) \le \sum_{j=1}^{n} \log\left(\frac{x^{q_j} - 1}{q_j}\right),$$

and hence we have

$$\prod_{j=1}^{n} q_j\left(x^{p_j} - 1\right) \le \prod_{j=1}^{n} p_j\left(x^{q_j} - 1\right).$$

The rest is identical to the proof of Theorem 2.5. This completes the proof.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
The author is grateful to the referee, for the careful reading of the paper and for the helpful suggestions and comments. The author was supported in part by Grants-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

Received: 18 November 2012 Accepted: 13 March 2013 Published: 29 March 2013

References

1. Löwner, K: Über monotone Matrixfunktionen. Math. Z. **38**, 177-216 (1934)
2. Heinz, E: Beiträge zur Störungstheorie der Spektralzerlegung. Math. Ann. **123**, 415-438 (1951)
3. Furuta, T: $A \ge B \ge 0$ assures $(B^r A^p B^r)^{1/q} \ge B^{(p+2r)/q}$ for $r \ge 0$, $p \ge 0$, $q \ge 1$ with $(1 + 2r)q \ge p + 2r$. Proc. Am. Math. Soc. **101**(1), 85-88 (1987)
4. Tanahashi, K: Best possibility of the Furuta inequality. Proc. Am. Math. Soc. **124**, 141-146 (1996)
5. Watanabe, K: An application of matrix inequalities to certain functional inequalities involving fractional powers. J. Inequal. Appl. **2012**, Article ID 221 (2012). doi:10.1186/1029-242X-2012-221
6. Marshall, AW, Olkin, I, Arnold, BC: Inequalities: Theory of Majorization and Its Applications, 2nd edn. Springer, Berlin (2011)
7. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1952)
8. Karamata, J: Sur une inegalite relative aux fonctions convexes. Publ. Math. Univ. (Belgr.) **1**, 145-148 (1932)

doi:10.1186/1029-242X-2013-137
Cite this article as: Watanabe: On a relation between Schur, Hardy-Littlewood-Pólya and Karamata's theorem and an inequality of some products of $x^p - 1$ derived from the Furuta inequality. *Journal of Inequalities and Applications* 2013 2013:137.