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A study on degree of approximation by Karamata summability method

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Abstract

Vuĉkoviĉ [Maths. Zeitchr. 89, 192 (1965)] and Kathal [Riv. Math. Univ. Parma, Italy 10, 33-38 (1969)] have studied summability of Fourier series by Karamata (K^λ) summability method. In present paper, for the first time, we study the degree of approximation of function $f \in$ Lip (α,r) and $f \in W(L_r,\xi(t))$ by K^λ-summability means of its Fourier series and conjugate of function $\tilde{f} \in \mathrm{Lip}(\alpha, r)$ and $\tilde{f} \in W(L_r, \xi(t))$ by K^λ-summability means of its conjugate Fourier series and establish four quite new theorems.
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1 Introduction

The method K^λ was first introduced by Karamata [1] and Lotosky [2] reintroduced the special case $\lambda = 1$. Only after the study of Agnew [3], an intensive study of these and similar cases took place. Vuĉkoviĉ [4] applied this method for summability of Fourier series. Kathal [5] extended the result of Vuĉkoviĉ [4]. Working in the same direction, Ojha [6], Tripathi and Lal [7] have studied K^λ-summability of Fourier series under different conditions. The degree of approximation of a function $f \in$ Lip α by Cesàro and Nörlund means of the Fourier series has been studied by Alexits [8], Sahney and Goel [9], Chandra [10], Qureshi [11], Qureshi and Neha [12], Rhoades [13], etc. But nothing seems to have been done so far in the direction of present work. Therefore, in present paper, we establish two new theorems on degree of approximation of function f belonging to Lip (α,r) $(r \geq 1)$ and to weighted class $W(L_r, \xi(t))(r \geq 1)$ by K^λ-means on its Fourier series and two other new theorems on degree of approximation of function $\tilde{f}$, conjugate of a 2π-periodic function f belonging to Lip (α,r) $(r > 1)$ and to weighted class $W(L_r,\xi(t))$ $(r \geq 1)$ by K^λ-means on its conjugate Fourier series.

2 Definitions and notations

Let us define, for $n = 0, 1, 2,...,$ the numbers $\begin{bmatrix} n \\ m \end{bmatrix}$, for $0 \leq m \leq n$, by

$$\prod_{\nu-0}^{n-1}(x+\nu) = \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix} x^m = \frac{\Gamma(x+n)}{\Gamma(x)} \qquad (2.1)$$
$$= x(x+1)(x+2)\ldots(x+n-1).$$

The numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ are known as the absolute value of stirling number of first kind

Let $\{s_n\}$ be the sequence of partial sums of an infinite series $\sum u_n$, and let us write

$$s_n^\lambda = \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m s_m \tag{2.2}$$

to denote the nth K^λ-mean of order $\lambda > 0$. If $s_n^\lambda \to s$ as $n \to \infty$, where s is a fixed finite number, then the sequence $\{s_n\}$ or the series $\sum u_n$ is said to be summable by Karamata method (K^λ) of order $\lambda > 0$ to the sum s, and we can write

$$s_n^\lambda \to s\left(K^\lambda\right) \quad \text{as } n \to \infty. \tag{2.3}$$

Let f be a 2π-periodic function and integrable in the sense of Lebesgue. The Fourier series associated with f at a point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x) \tag{2.4}$$

with nth partial sums $s_n(f;x)$.

The conjugate series of Fourier series (2.4) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{2.5}$$

with nth partial sums $\tilde{s}_n(f;x)$.

Throughout this paper, we will call (2.5) as conjugate Fourier series of function f.

L_∞-norm of a function f: $R \to R$ is defined by

$$\| f\|_\infty = \sup\{|f(x)| : x \in R\} \tag{2.6}$$

L_r-norm is defined by

$$\| f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, \quad r \geq 1. \tag{2.7}$$

The degree of approximation of a function f: $R \to R$ by a trigonometric polynomial t_n of degree n under sup norm $\|\ \|_\infty$ is defined by

(Zygmund [14])

$$\| t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\} \tag{2.8}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{t_n} \| t_n - f\|_r. \tag{2.9}$$

This method of approximation is called trigonometric Fourier approximation. A function $f \in \text{Lip}\,\alpha$ if

$$|f(x+t) - f(x)| = O(|t|^\alpha) \quad \text{for} \quad 0 < \alpha \leq 1 \tag{2.10}$$

and

$f \in$ Lip (α, r) for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\left(|t|^\alpha\right), \quad 0 < \alpha \leq 1 \text{ and } r \geq 1 \tag{2.11}$$

(definition 5.38 of McFadden [15]).

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f \in$ Lip $(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{2.12}$$

and that

$f \in W(L_r, \xi(t))$ if

$$\left(\int_0^{2\pi} \left|\left\{f(x+t) - f(x)\right\} \sin^\beta x\right|^r dx\right)^{\frac{1}{r}} = O(|\xi(t)), \quad \beta \geq 0 \tag{2.13}$$

If $\beta = 0$, our newly defined weighted i.e. $W(L_r, \xi(t))$ reduces to Lip $(\xi(t), r)$, if $\xi(t) = t^\alpha$ then Lip $(\xi(t), r)$ coincides with Lip (α, r) and if $r \to \infty$ then Lip (α, r) reduces to Lip α.

We observe that

$$\text{Lip}\,\alpha \subseteq \text{Lip}(\alpha, r) \subseteq \text{Lip}(\xi(t), r) \subseteq W(L_r, \xi(t)) \quad \text{for } 0 < \alpha \leq 1,\ r \geq 1.$$

We write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_n(t) = \frac{\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m \sin\left(m + \frac{1}{2}\right)t}{\Gamma(\lambda + n)\sin\left(\frac{t}{2}\right)}$$

$$\psi(t) = f(x+t) - f(x-t)$$

$$\tilde{K}_n(t) = \frac{\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m \cos\left(m + \frac{1}{2}\right)t}{\Gamma(\lambda + n)\sin\left(\frac{t}{2}\right)}$$

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\left(\frac{t}{2}\right)dt$$

3 The main results

3.1 Theorem 1

If a function f, 2π-periodic, belonging to Lip (α, r) then its degree of approximation by K^λ-summability means on its Fourier series is given by

$$\|s_n - f\|_r = O\left[\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\left\{\frac{\log(n+1)e}{(n+1)}\right\} + \frac{1}{\Gamma(\lambda + n)}\right], \quad 0 < \alpha \leq 1, n = 0, 1, 2..., \tag{3.1}$$

where s_n is K^λ-mean of Fourier series (2.4).

3.2 Theorem 2

If a function f, 2π-periodic, belonging to $W(L_r, \xi(t))$ then its degree of approximation by K^λ-summability means on its Fourier series is given by

$$\| s_n - f \|_r = O\left[\left\{(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}\left\{\frac{\log(n+1)}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{\Gamma(\lambda+n)}\right\}\right] \quad (3.2)$$

provided that $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ is non - increasing } in\ t, \quad (3.3)$$

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t\, dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right), \quad (3.4)$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left((n+1)^{\delta}\right), \quad (3.5)$$

where δ is an arbitrary positive number such that $s(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, $1 \le r \le \infty$, conditions (3.4) and (3.5) hold uniformly in x, s_n is K^{λ}-mean of Fourier series (2.4).

3.3 Theorem 3

If a function $\tilde{f}$, conjugate to a 2π-periodic function f, belonging to Lip(α,r) then its degree of approximation by K^{λ}-summability means on its conjugate Fourier series is given by

$$\| \tilde{s}_n - \tilde{f} \|_r = O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\left\{\frac{\log(n+1)e}{(n+1)^2}\right\} + \frac{1}{\Gamma(\lambda+n)} + 1\right], \quad (3.6)$$

$$0 < \alpha \le 1, n = 0, 1, 2, \ldots,$$

where $\tilde{s}_n$ is K^{λ}-mean of conjugate Fourier series (2.5) and

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot\left(\frac{t}{2}\right) dt.$$

3.4 Theorem 4

If a function $\tilde{f}$, conjugate to a 2π-periodic function f, belonging to $W(L_r, \xi(t))$ then its degree of approximation by K^{λ}-summability means on its conjugate Fourier series is given by

$$\| \tilde{s}_n - \tilde{f} \|_r = O\left\{(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{2}{(n+1)^2} + \frac{\log(n+1)}{(n+1)^2} + \frac{1}{\Gamma(\lambda+n)}\right] \quad (3.7)$$

provided that $\xi(t)$ satisfies the conditions (3.3)-(3.5) in which δ is an arbitrary positive number such that $s(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1, 1 \le r \le \infty$. Conditions (3.4) and (3.5) hold uniformly in x, $\tilde{s}_n$ is K^{λ}-mean of conjugate Fourier series (2.5) and

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot\left(\frac{t}{2}\right) dt. \quad (3.8)$$

4 Lemmas

For the proof of our theorems, following lemmas are required.

4.1 Lemma 1

(Vuĉkoviĉ [14]). Let $\lambda > 0$ and $0 < t < \frac{\pi}{2}$, then

$$\frac{\operatorname{Im}\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda\cos t+n\right)\sin\left(\frac{t}{2}\right)}=\frac{\left|\sin\left(\lambda\log\left(n+1\right).\sin t\right)\right|}{\sin\left(\frac{t}{2}\right)}+O\left(1\right)\quad\text{as } n\to\infty \text{ uniformly in } t.$$

4.2 Lemma 2

$$K_n\left(t\right)=O\left\{\lambda\log\left(n+1\right)\right\}+O\left(1\right).$$

Proof. For $0 < t < \frac{1}{n+1}$, $1-\cos t < \frac{t^2}{2}$, $\sin nt \le nt$ and $\sin\frac{t}{2} \ge \frac{t}{\pi}$

$$\begin{aligned}
\left|K_n\left(t\right)\right| &\le \left|\frac{1}{\Gamma\left(\lambda+n\right)}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\cdot\lambda^m\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \\
&= O\left[\frac{Im\left\{e^{\frac{it}{2}}\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma\left(\lambda+n\right)\sin\frac{t}{2}}\right]\quad\text{by (2.1)} \\
&= O\left[\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda+n\right)\sin\frac{t}{2}}\right]+O\left[\frac{Re\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda+n\right)}\right] \\
&= O\left[\frac{\Gamma\left(\lambda\cos t+n\right)}{\Gamma\left(\lambda+n\right)}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda\cos t+n\right)\sin\frac{t}{2}}\right]+O\left[\frac{\Gamma\left(\lambda\cos t+n\right)}{\Gamma\left(\lambda+n\right)}\right] \\
&= O\left[n^{-\lambda(1-\cos t)}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda\cos t+n\right)\sin\frac{t}{2}}\right]+O\left[n^{-\lambda(1-\cos t)}\right] \\
&= O\left[e^{-\lambda(1-\cos t)\log n}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda\cos t+n\right)\sin\frac{t}{2}}\right]+O\left[e^{-\lambda(1-\cos t)\log n}\right] \\
&= O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda\cos t+n\right)\sin\frac{t}{2}}\right]+O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\right].
\end{aligned}\tag{4.1}$$

Considering first part of (4.1) and using Lemma 1,

$$\begin{aligned}
K_n\left(t\right) &= O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\frac{\left|\sin\left(\lambda\log\left(n+1\right)\cdot\sin t\right)\right|}{\sin\left(\frac{t}{2}\right)}\right]+O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\right] \\
&\quad+O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\right] \\
&= O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\frac{\left|\sin\left(\lambda\log\left(n+1\right)\cdot\sin t\right)\right|}{\sin\left(\frac{t}{2}\right)}\right]+O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\right] \\
&= O\left\{\lambda\log\left(n+1\right)\right\}\left[\frac{\left|\sin\left(\lambda\log\left(n+1\right)\cdot\sin t\right)\right|}{\sin\left(\frac{t}{2}\right)}\right]+O\left(1\right) \\
&= O\left\{\lambda\log\left(n+1\right)\right\}+O\left(1\right).
\end{aligned}$$

4.3 Lemma 3

$$\tilde{K}_n(t) = O\left[\frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}\right] + O\left\{\lambda\log(n+1)\right\}\left|\sin\left(\lambda\log(n+1)\cdot\sin t\right)\right| + O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\left|\sin\left(t/2\right)\right|\right].$$

Proof. For $0 < t < \frac{1}{n+1}$, $1-\cos t < \frac{t^2}{2}$, $\sin nt \le nt$ and $\sin\frac{t}{2} \ge \frac{t}{\pi}$

$$\begin{aligned}
|K_n(t)| &\le \left|\frac{1}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}\cdot\lambda^m\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \\
&= O\left[\frac{Re\left\{e^{\frac{it}{2}}\frac{\Gamma\left(\lambda e^{it}+n\right)}{\Gamma\left(\lambda e^{it}\right)}\right\}}{\Gamma(\lambda+n)\sin\frac{t}{2}}\right] \quad \text{by (2.1)} \\
&= O\left[\frac{Re\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda+n)\sin\frac{t}{2}}\right] + O\left[\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda+n)}\right] \\
&= O\left[\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)\sin\frac{t}{2}}\right] + O\left[\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)}\right] \\
&= O\left[\frac{n^{-\lambda(1-\cos t)}}{\sin\frac{t}{2}}\right] + O\left[n^{-\lambda(1-\cos t)}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)}\right] \\
&= O\left[\frac{e^{-\lambda(1-\cos t)\log n}}{\sin\frac{t}{2}}\right] + O\left[e^{-\lambda(1-\cos t)\log n}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)}\right] \\
&= O\left[\frac{e^{-\frac{\lambda}{2}t^2\log n}}{\sin\frac{t}{2}}\right] + O\left[e^{-\frac{\lambda}{2}t^2\log n}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)}\right] \\
&= O\left[\frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}\right] + O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\frac{Im\,\Gamma\left(\lambda e^{it}+n\right)}{\Gamma(\lambda\cos t+n)}\right].
\end{aligned}$$

Using Lemma 1,

$$\begin{aligned}
K_n(t) &= O\left[\frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}\right] + O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\left|\sin\left(\lambda\log(n+1)\cdot\sin t\right)\right|\right] \\
&\quad + O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\left|\sin\left(t/2\right)\right|\right] \\
&= O\left[\frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}\right] + O\left\{\lambda\log(n+1)\right\}\left|\sin\left(\lambda\log(n+1)\cdot\sin t\right)\right| \\
&\quad + O\left[e^{-\frac{\lambda}{2}t^2\log(n+1)}\cdot\left|\sin\left(t/2\right)\right|\right].
\end{aligned}$$

□

4.4 Lemma 4

(McFadden [15]), Lemma 5.40) If $f(x)$ belongs to Lip(α,r) on [0,π], then $\phi(t)$ belongs to Lip(α,r) on [0,π].

5 Proof of the theorems

5.1 Proof of Theorem 1

Following Titchmarsh [16] and using Riemann-Lebesgue theorem, the mth partial sum $s_m(x)$ of series (2.4) at $t = x$ is given by

$$s_m(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t) \frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt$$

Therefore,

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\left\{s_m(x)-f(x)\right\} = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\,\frac{\Gamma(\lambda)}{\Gamma(\lambda+n)}\sum_{m+0}^{n}\begin{bmatrix} n \\ m \end{bmatrix} \cdot\lambda^m\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$\begin{aligned} s_m(x)-f(x) &= \frac{\Gamma(\lambda)}{2\pi}\int_0^{\pi}\phi(t)K_n(t)\,dt \\ &= \frac{\Gamma(\lambda)}{2\pi}\left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)\,K_n(t)\,dt \\ &= O(I_{1.1}) + O(I_{1.2}) \quad (\text{say}). \end{aligned} \tag{5.1}$$

Now we consider,

$$I_{1.1} = \int_0^{\frac{1}{n+1}} |\phi(t)\,\|\,K_n(t)\,|dt.$$

Using Lemma 2,

$$I_{1.1} = O\left\{\lambda\log(n+1)\right\}\int_0^{\frac{1}{n+1}}|\phi(t)|dt + O\left[\int_0^{\frac{1}{n+1}}|\phi(t)|dt\right].$$

Using Hölder's inequality and Lemma 4,

$$\begin{aligned} I_{1.1} &= O\left[\left\{\lambda\log(n+1)\right\}+1\right]\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\phi(t)}{t^{\alpha}}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left(t^{\alpha-1}\right)^s dt\right]^{\frac{1}{s}} \\ &= O\left[\left\{\lambda\log(n+1)\right\}+1\right]\left(\frac{1}{n+1}\right)\left[\left\{\frac{t^{\alpha s-s+1}}{\alpha s-s+1}\right\}_0^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\ &= O\left\{\frac{\log(n+1)e}{(n+1)}\right\}\left[\frac{1}{(n+1)^{\alpha s-s+1}}\right]^{\frac{1}{s}} \\ &= O\left[\left\{\frac{\log(n+1)e}{(n+1)}\right\}\frac{1}{(n+1)^{\alpha-1+\frac{1}{s}}}\right] \\ &= O\left[\left\{\frac{\log(n+1)e}{(n+1)}\right\}\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1. \end{aligned} \tag{5.2}$$

Since, for $\frac{1}{n+1} \le t \le \pi,\ \sin\frac{t}{2} \ge \frac{t}{\pi}$

$$\begin{aligned} K_n(t) &= O\left[\frac{1}{\Gamma(\lambda+n)\sin\frac{t}{2}}\right] \\ &= O\left[\frac{1}{\Gamma(\lambda+n)\,t}\right]. \end{aligned} \tag{5.3}$$

Next we consider,

$$|I_{1.2}| \le \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||K_n(t)|dt.$$

Using Hölder's inequality, (5.3) and Lemma 4,

$$\begin{aligned} I_{1.2} &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\phi(t)}{t^{\alpha}}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{\delta+\alpha}}{t}\right)^{s}dt\right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\int_{\frac{1}{n+1}}^{\pi}t^{(\delta+\alpha-1)s}dt\right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\left\{\frac{t^{(\delta+\alpha-1)s+1}}{(\delta+\alpha-1)s+1}\right\}_{\frac{1}{n+1}}^{\pi}\right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\frac{1}{(n+1)^{(\delta+\alpha-1)s+1}}\right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\frac{1}{(n+1)^{(\delta+\alpha-1)+\frac{1}{s}}}\right] \\ &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\frac{1}{(n+1)^{\alpha-1+\frac{1}{s}}}\right] \\ &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right]. \end{aligned} \tag{5.4}$$

Combining (5.1), (5.2) and (5.4),

$$\begin{aligned} S_m - f(x) &= O\left[\left(\frac{\log(n+1)e}{(n+1)}\right)\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)\right] + O\left[\left(\frac{1}{\Gamma(\lambda+n)}\right)\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)\right] \\ &= O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\left\{\frac{\log(n+1)e}{(n+1)} + \frac{1}{\Gamma(\lambda+n)}\right\}\right]. \end{aligned}$$

This completes the proof of Theorem 1.

5.2 Proof of Theorem 2

Following the proof of Theorem 1,

$$S_m(x) - f(x) = \frac{\Gamma(\lambda)}{2\pi}\left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)\,dt \\ = O(I_{2.1}) + O(I_{2.2}) \quad (\text{say}). \tag{5.5}$$

We have

$$|\phi(x+t) - \phi(x)| \le |f(u+x+t) - f(u+x)| + |f(u-x-t) - f(u-x)|.$$

Hence, by Minkowiski's inequality,

$$\left[\int_0^{2\pi}\{|\phi(x+t) - \phi(x)\}\sin^\beta x|^r dx\right]^{\frac{1}{r}} \le \left[\int_0^{2\pi}|\left\{f(u+x+t) - f(u+x)\right\}sin^\beta x|^r dx\right]^{\frac{1}{r}} \\ + \left[\int_0^{2\pi}|\left\{f(u-x-t) - f(u-x)\right\}\sin^\beta x|^r dx\right]^{\frac{1}{r}} \\ = O\{\xi(t)\}.$$

Then $f \in W(L_r, \xi(t)) \Rightarrow \phi \in W(L_r, \xi(t))$.
Now we consider,

$$|I_{2.1}| \le \int_0^{\frac{1}{n+1}} |\phi(t)||K_n(t)|dt.$$

Using Lemma 2,

$$I_{2.1} = \left[O\left\{\lambda \log(n+1)\right\} + O(1)\right]\int_0^{\frac{1}{n+1}} |\phi(t)|dt.$$

Using Hölder's inequality and the fact that $\phi(t) \in W(L_r, \xi(t))$,

$$I_{2.1} = O\left[\left\{\lambda \log(n+1)\right\} + 1\right]\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t|\phi(t)|\sin^\beta(t)}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \\ \cdot \left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t\sin^\beta t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left\{\lambda \log(n+1)\, e\right\}\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t\sin^\beta t}\right\}^s dt\right]^{\frac{1}{s}} \quad \text{by (3.4)}$$

Since $\sin t \geq 2t/\pi$,

$$I_{2.1} = O\left(\frac{\log(n+1)e}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}}.$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$\begin{aligned}
I_{2.1} &= O\left\{\left(\frac{\log(n+1)e}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_\epsilon^{\frac{1}{n+1}}\left\{\frac{1}{t^{(1+\beta)s}}\right\}dt\right]^{\frac{1}{s}} \quad \text{for some } 0 < \epsilon < \frac{1}{n+1}\\
&= O\left\{\left(\frac{\log(n+1)e}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(1+\beta)s+1}}{-(1+\beta)s+1}\right\}_\epsilon^{\frac{1}{n+1}}\right]^{\frac{1}{s}}\\
&= O\left[\left\{\left(\frac{\log(n+1)e}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{(1+\beta)-\frac{1}{s}}\right\}\right]\\
&= O\left[\left\{\left(\frac{\log(n+1)e}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.
\end{aligned} \tag{5.6}$$

Next we consider,

$$|I_{2.2}| \leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||K_n(t)|dt.$$

Using Hölder's inequality, $|\sin t| \leq 1$,$\sin t \geq 2t/\pi$, (5.3), conditions (3.3), (3.5) and second mean value theorem for integrals,

$$\begin{aligned}
I_{2.2} &= O\left[\int_{\frac{1}{n+1}}^{\pi}\frac{1}{\Gamma(\lambda+n)t}|\phi(t)|dt\right]\\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\phi(t)|\sin^\beta(t)}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta}\sin^\beta t t}\right\}^s dt\right]^{\frac{1}{s}}\\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+\beta+1}}\right\}^s dt\right]^{\frac{1}{s}}\\
&= O\left[\left\{\frac{1}{\Gamma(\lambda+n)}\right\}\left\{(n+1)^\delta\right\}\right]\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+\beta+1}}\right\}^s dt\right]^{\frac{1}{s}}.
\end{aligned}$$

Putting $t = \dfrac{1}{y}$

$$
\begin{aligned}
I_{2.2} &= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\right\}\left[\int_{\pi}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1'}}\right\}^{s}\frac{dy}{y^{2}}\right]^{\frac{1}{s}} \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\left\{\frac{dy}{d^{s(\delta-\beta-1)+2}}\right\}dt\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi}\leq\eta\leq n+1 \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1}\left\{\frac{dy}{y^{s(\delta-\beta-1)+2}}\right\}dt\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi}\leq 1\leq n+1 \qquad (5.7) \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(\beta+1-\delta)-1}}{s(\beta+1-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}} \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1+\beta-\delta-\frac{1}{s}}\right\} \\
&= O\left\{\frac{\xi\left(\frac{1}{n+1}\right)}{\Gamma(\lambda+n)}\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\} \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.
\end{aligned}
$$

Now combining (5.5)-(5.7),

$$
\begin{aligned}
\left|s_m(x)-f(x)\right| &= O\left[\left\{\left(\frac{\log(n+1)e}{(n+1)}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \\
&\quad + O\left[\left\{\left(\frac{1}{\Gamma(\lambda+n)}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{\log(n+1)e}{(n+1)}+\frac{1}{\Gamma(\lambda+n)}\right].
\end{aligned}
$$

Now using L_r-norm, we get

$$
\begin{aligned}
\left\|s_m(x)-f(x)\right\| &= \left\{\int_0^{2\pi}\left|s_m(x)-f(x)\right|^{r}dx\right\}^{\frac{1}{r}} \\
&= O\Bigg[\int_0^{2\pi}\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \\
&\qquad \cdot\left\{\frac{\log(n+1)e}{(n+1)}+\frac{1}{\Gamma(\lambda+n)}\right\}dx\Bigg]^{\frac{1}{r}} \\
&= \Bigg[\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \\
&\qquad \cdot\left\{\frac{\log(n+1)e}{(n+1)}+\frac{1}{\Gamma(\lambda+n)}\right\}\Bigg]\left[\left\{\int_0^{2\pi}dx\right\}^{\frac{1}{r}}\right] \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{\log(n+1)e}{(n+1)}+\frac{1}{\Gamma(\lambda+n)}\right].
\end{aligned}
$$

This completes the proof of Theorem 2.

5.3 Proof of Theorem 3

Following Lal [7], the mth partial sum $\tilde{S}_m(x)$ of series (2.5) at $t = x$

$$\tilde{S}_m(x) - \left[-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\left(\frac{t}{2}\right)dt\right] = \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

Therefore,

$$\begin{aligned}
&\frac{\Gamma(\lambda)}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\left\{\tilde{S}(x) - \left(-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\left(\frac{t}{2}\right)dt\right)\right\} \\
&= \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\Gamma(\lambda)}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt, \\
\tilde{S}_m(x) - \tilde{f}(x) &= \frac{\Gamma(\lambda)}{2\pi}\int_0^\pi \psi(t)\tilde{K}_n(t)\,dt \\
&= \frac{\Gamma(\lambda)}{2\pi}\left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]|\psi(t)||\tilde{K}_n(t)|dt \\
&= O(I_{3.1}) + O(I_{3.2}).
\end{aligned} \tag{5.8}$$

We consider,

$$|I_{3.1}| = \int_0^{\frac{1}{n+1}} |\psi(t)|\left|\tilde{K}_n(t)\right|dt.$$

Using Lemma 3,

$$\begin{aligned}
I_{3.1} &= O\left[\int_0^{\frac{1}{n+1}}\frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}|\psi(t)|dt\right] \\
&\quad + O\left\{\lambda\log(n+1)\right\}\int_0^{\frac{1}{n+1}}|\sin\left(\lambda\log(n+1).\sin t\right)||\psi(t)|dt \\
&\quad + O\left[\int_0^{\frac{1}{n+1}} e^{-\frac{\lambda}{2}t^2\log(n+1)}|\sin\left(t/2\right)||\psi(t)|dt\right] \\
&= I_{3.1.1} + I_{3.1.2} + I_{3.1.3} \quad \text{(say)}.
\end{aligned} \tag{5.9}$$

Now consider,

$$\begin{aligned}
I_{3.1.1} &= O\left(\int_0^{\frac{1}{n+1}}\frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}|\psi(t)|dt\right) \\
&= O\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\psi(t)}{t^\alpha}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{t^{\alpha-2}e^{-\frac{\lambda}{2}t^2\log(n+1)}\right\}^s dt\right]^{\frac{1}{s}}.
\end{aligned}$$

Using second mean value theorem for integrals,

$$
\begin{aligned}
I_{3.1.1} &= O\left\{\frac{e^{-\frac{\lambda}{2}\frac{\log(n+1)}{(n+1)^2}}}{(n+1)}\right\}\left[\int_{\in}^{\frac{1}{n+1}}\left(t^{\alpha-2}\right)^s dt\right]^{\frac{1}{s}} \quad \text{for } 0 <\in< \frac{1}{n+1} \\
&= O\left\{\frac{e^{-\frac{\lambda}{2}\frac{\log(n+1)}{(n+1)^2}}}{(n+1)}\right\}\left[\left\{\frac{t^{s\alpha-2s+1}}{s\alpha-2s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha s-2s+1}}\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha-2+\frac{1}{s}}}\right] \\
&= O\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha-1-\frac{1}{r}}}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1 \\
&= O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right].
\end{aligned}
\tag{5.10}
$$

Now we consider,

$$I_{3.1.2} = O\left\{\lambda\log(n+1)\right\}\int_0^{\frac{1}{n+1}}|\sin\left(\lambda\log(n+1)\sin t\right)||\psi(t)|dt.$$

Since, for $0 < t < \frac{1}{n+1}$, $\sin nt \le nt$,

$$I_{3.1.2} = O\left\{\lambda\log(n+1)\right\}\int_0^{\frac{1}{n+1}} t|\psi(t)|dt.$$

Using Hölder's inequality and Lemma 4,

$$
\begin{aligned}
I_{3.1.2} &= O\left\{\lambda\log(n+1)\right\}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\psi(t)}{t^\alpha}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left(t^\alpha\right)^s dt\right]^{\frac{1}{s}} \\
&= O\left\{\lambda\log(n+1)\right\}\left(\frac{1}{n+1}\right)\left[\left\{\frac{t^{\alpha s+1}}{\alpha s+1}\right\}_0^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\
&= O\left\{\lambda\log(n+1)\right\}\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha s+1}}\right]^{\frac{1}{s}} \\
&= O\left\{\lambda\log(n+1)\right\}\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha+\frac{1}{s}}}\right] \\
&= O\left\{\log(n+1)\right\}\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha+1-\left(1-\frac{1}{s}\right)}}\right] \\
&= O\left\{\log(n+1)\right\}\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha+1-\frac{1}{r}}}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1 \\
&= O\left\{\frac{\log(n+1)}{(n+1)^2}\right\}\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right].
\end{aligned}
\tag{5.11}
$$

Next we consider,

$$
\begin{aligned}
I_{3.1.3} &= O\int_0^{\frac{1}{n+1}} e^{-\frac{\lambda}{2}t^2\log(n+1)}|\sin(t/2)||\psi(t)|dt \\
&= O\left\{\int_0^{\frac{1}{n+1}} t|\psi(t)|dt\right\} \\
&= O\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\psi(t)}{t^\alpha}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}(t^\alpha)^s dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\left\{\frac{t^{\alpha s+1}}{\alpha s+1}\right\}_0^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha s+1}}\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha+\frac{1}{s}}}\right] \\
&= O\left(\frac{1}{n+1}\right)\left[\frac{1}{(n+1)^{\alpha+1-\left(1-\frac{1}{s}\right)}}\right] \\
&= O\left\{\frac{1}{(n+1)^2}\right\}\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.
\end{aligned} \tag{5.12}
$$

Combining (5.9)-(5.12),

$$
\begin{aligned}
I_{3.1} &= O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right]+O\left\{\frac{\log(n+1)}{(n+1)^2}\right\}\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right] \\
&\quad +O\left\{\frac{1}{(n+1)^2}\right\}\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right] \\
&= O\left[\frac{\log(n+1)\,e}{(n+1)^2(n+1)^{\alpha-\frac{1}{r}}}\right]+O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right].
\end{aligned} \tag{5.13}
$$

Since, for $\frac{1}{n+1} < t < \pi$, $|\sin\left(\frac{t}{2}\right)| \geq \frac{t}{\pi}$

$$
\begin{aligned}
\tilde{K}_n(t) &= O\left[\frac{1}{\Gamma(\lambda+n)\sin\left(\frac{t}{2}\right)}\right] \\
&= O\left[\frac{1}{\Gamma(\lambda+n)\,t}\right].
\end{aligned} \tag{5.14}
$$

Next we consider,

$$I_{3.2} \leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)||\tilde{K}_n(t)|dt.$$

Using Hölder's inequality, (5.14) and Lemma 4,

$$\begin{aligned}
I_{3.2} &= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\psi(t)}{t^{\alpha}}\right\}^{r} dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{\delta+\alpha}}{t}\right)^{s} dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\int_{\frac{1}{n+1}}^{\pi} t^{(\delta+\alpha-1)s} dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\left\{\frac{t^{(\delta+\alpha-1)s+1}}{(\delta+\alpha-1)s+1}\right\}_{\frac{1}{n+1}}^{\pi}\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\frac{1}{(n+1)^{(\delta+\alpha-1)s+1}}\right]^{\frac{1}{s}} \qquad (5.15)\\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\frac{1}{(n+1)^{-\delta}}\left[\frac{1}{(n+1)^{(\delta+\alpha-1)+\frac{1}{s}}}\right] \\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\frac{1}{(n+1)^{\alpha-1+\frac{1}{s}}}\right] \\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.
\end{aligned}$$

Collecting (5.8), (5.13) and (5.15),

$$\begin{aligned}
\tilde{S}_m - \tilde{f}(x) &= O\left[\frac{\log(n+1)e}{(n+1)^2(n+1)^{\alpha-\frac{1}{r}}}\right] + \left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right] \\
&\quad + O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\} \\
&= O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\left\{\frac{\log(n+1)e}{(n+1)} + \frac{1}{\Gamma(\lambda+n)} + 1\right\}\right].
\end{aligned}$$

This completes the proof of Theorem 3.

5.4 Proof of Theorem 4

Following the calculations of Theorem 3,

$$\tilde{S}_m(x) - \tilde{f}(x) = \frac{\Gamma(\lambda)}{2\pi}\left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\,\tilde{K}_n(t)\,dt \qquad (5.16)$$
$$= O(I_{4.1}) + O(I_{4.2}).$$

Now,

$$I_{4.1} = O\left[\int_0^{\frac{1}{n+1}} |\psi(t)||K_n(t)|dt\right].$$

Using Lemma 3,

$$I_{4.1} = O\left(\int_0^{\frac{1}{n+1}} \frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}|\psi(t)|dt\right)$$
$$+ O\left\{\lambda\log(n+1)\right\}\int_0^{\frac{1}{n+1}} |\sin\left(\lambda\log(n+1)\sin t\right)||\psi(t)|dt \qquad (5.17)$$
$$+ O\left[\int_0^{\frac{1}{n+1}} e^{-\frac{\lambda}{2}t^2\log(n+1)}|\sin\left(t/2\right)||\psi(t)|dt\right]$$
$$= I_{4.1.1} + I_{4.1.2} + I_{4.1.3} \quad (\text{say}).$$

Using Minkowiski's inequality, we have a fact that $f \in W(L_r, \xi(t)) \Rightarrow \psi \in W(L_r, \xi(t))$. Now we consider,

$$I_{4.1.1} = O\left(\int_0^{\frac{1}{n+1}} \frac{e^{-\frac{\lambda}{2}t^2\log(n+1)}}{t}|\psi(t)|dt\right)$$
$$= O\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\psi(t)\sin^{\beta}(t)}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\,e^{-\frac{\lambda}{2}t^2\log(n+1)}}{\sin^{\beta}(t)}\right\}^s dt\right]^{\frac{1}{s}}$$
$$= O\left\{e^{-\frac{\lambda}{2}\frac{\log(n+1)}{(n+1)^2}}\right\}O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{\sin^{\beta}(t)}\right)^s dt\right]^{\frac{1}{s}} \quad \text{by (3.4)}$$
$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{\sin^{\beta}(t)}\right)^s dt\right]^{\frac{1}{s}}$$
$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t^{\beta}}\right)^s dt\right]^{\frac{1}{s}} \quad \text{since } \sin t \geq \frac{2t}{\pi}.$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$\begin{aligned}
I_{4.1.1} &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}}\left\{\frac{1}{t^{\beta s}}\right\}dt\right]^{\frac{1}{s}} \quad \text{for some } 0<\in<\frac{1}{n+1} \\
&= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-\beta s+1}}{-\beta s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\
&= O\left[\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta-1+\left(1-\frac{1}{s}\right)}\right\}\right] \\
&= O\left[\left\{\frac{1}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \quad \text{since } \frac{1}{\mathrm{r}}+\frac{1}{s}=1.
\end{aligned} \tag{5.18}$$

Now,

$$I_{4.1.2} = O\left\{\lambda\log(n+1)\right\}\int_0^{\frac{1}{n+1}}\left|\sin\left(\lambda\log(n+1)\sin t\right)\right|\left|\psi(t)\right|dt.$$

Since for $0<t<\dfrac{1}{n+1}$, $\sin\ nt \le nt$,

$$I_{4.1.2} = O\left\{\lambda\log(n+1)\right\}\int_0^{\frac{1}{n+1}} t\left|\psi(t)\right|dt.$$

Hölder's inequality and the fact that $\psi(t) \in W(L_r, \xi(t))$,

$$\begin{aligned}
I_{4.1.2} &= O\left\{\lambda\log(n+1)\right\}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t|\psi(t)|\sin^{\beta}(t)}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left\{\log(n+1)\right\}O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}} \quad \text{by (3.4)} \\
&= O\left(\frac{\log(n+1)}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t^{\beta}}\right\}^s dt\right]^{\frac{1}{s}} \quad \text{since } \sin t \ge 2t/\pi.
\end{aligned}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$
\begin{aligned}
I_{4.1.2} &= O\left\{\left(\frac{\log(n+1)}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}}\left\{\frac{1}{t^{\beta s}}\right\}dt\right]^{\frac{1}{s}} \quad \text{for some } <\in< \frac{1}{n+1} \\
&= O\left\{\left(\frac{\log(n+1)}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-\beta s+1}}{-\beta s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\
&= O\left[\left\{\left(\frac{\log(n+1)}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta-1+\left(1-\frac{1}{s}\right)}\right\}\right] \\
&= O\left[\left\{\frac{\log(n+1)}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.
\end{aligned}
\tag{5.19}
$$

Next we consider,

$$
\begin{aligned}
I_{4.1.3} &= O\left[\int_0^{\frac{1}{n+1}} e^{-\frac{\lambda}{2}t^2\log(n+1)}\left|\sin\frac{t}{2}\right||\phi(t)|dt\right] \\
&= O\left[\int_0^{\frac{1}{n+1}} t|\psi(t)|dt\right].
\end{aligned}
$$

Using Hölder's inequality and the fact that $\psi(t) \in W(L_r, \xi(t))$,

$$
\begin{aligned}
I_{4.1.3} &= O\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t|\psi(t)|\sin^{\beta}(t)}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}} \quad \text{by (3.4)}
\end{aligned}
$$

Since, $\sin t \geq 2t/\pi$,

$$
I_{4.1.3} = O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t^{\beta}}\right\}^s dt\right]^{\frac{1}{s}}.
$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$
\begin{aligned}
I_{4.1.3} &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}}\left\{\frac{1}{t^{\beta s}}\right\}dt\right]^{\frac{1}{s}} \\
&= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-\beta s+1}}{-\beta s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\
&= O\left[\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta-\frac{1}{s}}\right\}\right] \\
&= O\left[\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta-1+\left(1-\frac{1}{s}\right)}\right\}\right] \\
&= O\left[\left\{\frac{1}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.
\end{aligned}
\tag{5.20}
$$

Combining from (5.17) to (5.20),

$$
\begin{aligned}
I_{4.1} &= O\left[\left\{\frac{1}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \\
&\quad + O\left[\left\{\frac{\log(n+1)}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \\
&\quad + O\left[\left\{\frac{1}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right].
\end{aligned}
\tag{5.21}
$$

Using Hölder's inequality, $|\sin t| \leq 1$, $\sin t \geq 2t/\pi$, conditions (3.3), (3.5) and second mean value theorem for integrals and the fact $\psi(t) \in W(L_r,\xi(t))$,

$$
\begin{aligned}
I_{4.2} &= O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{1}{\Gamma(\lambda+n)\,t}|\psi(t)|dt\right) \\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|\sin^{\beta}(t)}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{\Gamma(\lambda+n)}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}} \\
&= O\left[\left\{\frac{1}{\Gamma(\lambda+n)}\right\}\left\{(n+1)^{\delta}\right\}\right]\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}} \\
&= O\left[\left\{\frac{1}{\Gamma(\lambda+n)}\right\}\left\{(n+1)^{\delta}\right\}\right]\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+\beta+1}}\right\}^{s}dt\right]^{\frac{1}{s}}.
\end{aligned}
$$

Putting $t = \frac{1}{y}$

$$
\begin{aligned}
I_{4.2} &= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\right\}\left[\int_{\pi}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1'}}\right\}^{s}\frac{dy}{y^2}\right]^{\frac{1}{s}} \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\left\{\frac{dy}{y^{s(\delta-\beta-1)+2}}\right\}dt\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi} \le \eta \le n+1 \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1}\left\{\frac{dy}{y^{s(\delta-\beta-1)+2}}\right\}dt\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi} \le 1 \le n+1 \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(\beta+1-\delta)-1}}{s(\beta+1-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}} \\
&= O\left\{\frac{(n+1)^{\delta}}{\Gamma(\lambda+n)}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1+\beta-\delta-\frac{1}{s}}\right\} \\
&= O\left\{\frac{\xi\left(\frac{1}{n+1}\right)}{\Gamma(\lambda+n)}\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\} \quad \text{since } \frac{1}{r}+\frac{1}{s}=1.
\end{aligned}
\tag{5.22}
$$

Combining from (5.16), (5.21) and (5.22)

$$
\begin{aligned}
|S_m(x)-f(x)| &= O\left[\left\{\frac{1}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \\
&\quad + O\left[\left\{\frac{\log(n+1)}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \\
&\quad + O\left[\left\{\frac{1}{(n+1)^2}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\}\right] \\
&\quad + O\left\{\frac{\xi\left(\frac{1}{n+1}\right)}{\Gamma(\lambda+n)}\right\}\left\{(n+1)^{\beta+\frac{1}{r}}\right\} \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{2}{(n+1)^2}+\frac{\log(n+1)}{(n+1)^2}+\frac{1}{\Gamma(\lambda+n)}\right].
\end{aligned}
$$

Now using L_r-norm, we get

$$
\begin{aligned}
\| S_m(x)-f(x) \| &= \left\{\int_0^{2\pi}|S_m(x)-f(x)|^r dx\right\}^{\frac{1}{r}} \\
&= O\left[\int_0^{2\pi}(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right. \\
&\qquad \left.\cdot\left\{\frac{2}{(n+1)^2}+\frac{\log(n+1)}{(n+1)^2}+\frac{1}{\Gamma(\lambda+n)}\right\}dx\right]^{\frac{1}{r}} \\
&= \left[\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\right. \\
&\qquad \left.\cdot\left\{\frac{2}{(n+1)^2}+\frac{\log(n+1)}{(n+1)^2}+\frac{1}{\Gamma(\lambda+n)}\right\}\right]\left[\left\{\int_0^{2\pi}dx\right\}^{\frac{1}{r}}\right] \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left[\frac{2}{(n+1)^2}+\frac{\log(n+1)}{(n+1)^2}+\frac{1}{\Gamma(\lambda+n)}\right] \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left[1+\frac{\log(n+1)e}{(n+1)^2}+\frac{1}{\Gamma(\lambda+n)}\right].
\end{aligned}
$$

This completes the proof of Theorem 4.

Authors' contributions

HK framed the problems. HK and KS carried out the results and wrote the manuscripts. All the authors read and approved the final manuscripts.

Competing interests

The authors declare that they have no competing interests.

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