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Strong convergence of iterative algorithms with variable coefficients for generalized equilibrium problems, variational inequality problems and fixed point problems

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Abstract

In this paper, we propose some new iterative algorithms with variable coefficients for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping and the set of common fixed points of a finite family of asymptotically κ-strict pseudocontractive mappings in the intermediate sense. Some strong convergence theorems of these iterative algorithms are obtained without some boundedness conditions which are not easy to examine in advance. The results of the paper improve and extend some recent ones announced by many others. The algorithms with variable coefficients introduced in this paper are of independent interests.

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Keywords: generalized equilibrium problem; variational inequality; fixed point; asymptotically strict pseudocontractive mapping in the intermediate sense; algorithm with variable coefficients

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H.

Recall that a mapping $S: C\to C$ is called nonexpansive if

$$\|Sx-Sy\|\le\|x-y\|,\quad \forall x,y\in C.$$

A mapping $S: C\to C$ is called asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $k_n\to1$ as $n\to\infty$ such that

$$\|S^nx-S^ny\|\le k_n\|x-y\|\quad\text{for all } x,y\in C, \text{ and all integers } n\ge1.$$

$S: C\to C$ is called asymptotically nonexpansive in the intermediate sense [2] if it is continuous, and the following inequality holds:

$$\limsup_{n\to\infty}\sup_{x,y\in C}\left(\|S^nx-S^ny\|-\|x-y\|\right)\le0. \tag{1.1}$$

In fact, we see that (1.1) is equivalent to

$$\|S^n x - S^n y\|^2 \le \|x-y\|^2 + c_n \quad \text{for all } x,y \in C, \text{ and all integers } n \ge 1, \tag{1.2}$$

where $c_n \in [0,\infty)$ with $c_n \to 0$ as $n \to \infty$.

Recall that S is called an asymptotically κ-strict pseudocontractive mapping with the sequence $\{\gamma_n\}$ [3] if there exists a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\} \subset [0,\infty)$ with $\gamma_n \to 0$ as $n \to \infty$ such that

$$\|S^n x - S^n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + \kappa \|(I-S^n)x - (I-S^n)y\|^2 \tag{1.3}$$

for all $x,y \in C$, and all integers $n \ge 1$.

A mapping S is called an asymptotically κ-strict pseudocontraction in the intermediate sense with the sequence $\{\gamma_n\}$ [4] if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \{\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa \|(I-S^n)x - (I-S^n)y\|^2\} \le 0, \tag{1.4}$$

where $\kappa \in [0,1)$ and $\gamma_n \in [0,\infty)$ such that $\gamma_n \to 0$ as $n \to \infty$. In fact, (1.4) is reduced to the following:

$$\|S^n x - S^n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + \kappa \|(I-S^n)x - (I-S^n)y\|^2 + c_n, \quad \forall x,y \in C, \tag{1.5}$$

where $c_n \in [0,\infty)$ with $c_n \to 0$ as $n \to \infty$.

Example 1.1 [4] Let $X = \mathbb{R}$ and $C = [0,1]$, where $\mathbb{R}$ is the set of real numbers. For each $x \in C$, we define $T: C \to C$ by

$$Tx = \begin{cases} kx, & \text{if } x \in [0, \frac{1}{2}], \\ 0, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then:

(1) T is an asymptotically κ-strict pseudocontraction in the intermediate sense.
(2) T is not continuous. Therefore, T is not an asymptotically κ-strict pseudocontractive and asymptotically nonexpansive in the intermediate sense.

Recall that a mapping A of C into H is said to be L-Lipschitz-continuous if there exists a positive constant L such that

$$\|Ax - Ay\| \le L\|x-y\|, \quad \forall x,y \in C.$$

A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x-y\rangle \ge 0, \quad \forall x,y \in C.$$

A mapping A of C into H is said to be β-inverse strongly monotone if there exists a positive constant β such that

$$\langle Ax - Ay, x-y\rangle \ge \beta\|Ax-Ay\|^2, \quad \forall x,y \in C.$$

It is obvious that if A is β-inverse-strongly monotone, then A is monotone and Lipschitz-continuous.

Let mapping A from C to H be monotone and Lipschitz-continuous. The variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C.$$

The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for the bifunction F is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of the equilibrium problem for the bifunction F is denoted by $\mathrm{EP}(F)$.

Let $B : C \to H$ be a nonlinear mapping. Then Blum and Oettli [5], Moudafi and Thera [6] and Takahashi and Takahashi [7] considered the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions of (1.7) is denoted by $\mathrm{GEP}(F, B)$. In the case of $B = 0$, $\mathrm{GEP}(F, B) = \mathrm{EP}(F)$. In the case of $F \equiv 0$, $\mathrm{GEP}(F, B) = \mathrm{VI}(C, B)$.

Problem (1.7) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problems in noncooperative games, and other; see, for instance, [5–7].

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (*cf.* [5, 8]):

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} F\bigl(tz + (1 - t)x, y\bigr) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

In 2009, Kangtunyakarn and Suantai [9] introduced the following mapping the sequence $\{K_n\}$ generated by a finite family of nonexpansive mappings $T_1, T_2, \ldots, T_N$ and the sequence $\{\lambda_{n,i}\}_{i=1}^N$ in $[0, 1]$

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})U_{n,1}, \\
U_{n,3} &= \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})U_{n,2}, \\
&\ldots, \\
U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})U_{n,N-2}, \\
K_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})U_{n,N-1}.
\end{aligned} \tag{1.8}$$

Recently, utilizing K_n-mapping in (1.8), Jaiboon *et al.* [10] introduced the following iterative algorithm based on a hybrid relaxed extragradient method for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of the variational inequality problem for an inverse-strongly monotone mapping and the set of common fixed points of a finite family of nonexpansive mappings. To be more precise, they obtained the following theorem.

Theorem 1.2 [10, Theorem 3.1] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *let $\{T_i\}_{i=1}^N$ be a finite family of a nonexpansive mapping from H into itself, let A be a β-inverse-strongly monotone mapping of C into H, and let B be a ξ-inverse-strongly monotone mapping of C into H such that $\Theta = \bigcap_{i=1}^N \operatorname{Fix}(T_i) \cap \operatorname{GEP}(F,A) \cap \operatorname{VI}(C,B) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be the sequences generated by $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$, and let*

$$\begin{cases} F(u_n,y) + \langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \delta_n B u_n), \\ v_n = \epsilon_n x_n + (1-\epsilon_n)P_C(y_n - \lambda_n B y_n), \\ z_n = \alpha_n x_n + (1-\alpha_n)K_n v_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $\{K_n\}$ is the sequence generated by (1.8), *and $\alpha_n \subset (0,1)$ satisfy the following conditions*:

(i) $\{\epsilon_n\} \subset [0,e]$ *for some e with $0 \leq e < 1$ and $\lim_{n\to\infty} \alpha_n = 0$;*
(ii) $\{\delta_n\}, \{\lambda_n\} \subset [a,b]$ *for some a, b with $0 < a < b < 2\xi$;*
(iii) $\{r_n\} \subset [c,d]$ *for some c, d with $0 < c < d < 2\beta$.*

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{\bigcap_{i=1}^N \operatorname{Fix}(T_i)\cap\operatorname{GEP}(F,A)\cap\operatorname{VI}(C,B)}x_0$.

Considering the common fixed point problems of a finite family of asymptotically κ-strict pseudocontractive mappings, Qin *et al.* [11] introduced the following algorithm. Let $x_0 \in C$ and $\{\alpha_n\}_{n=0}^\infty$ be a sequence in $(0,1)$. The sequence $\{x_n\}$ is generated in the following way:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0)S_1 x_0, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)S_2 x_1, \\ &\ldots, \\ x_N &= \alpha_{N-1}x_{N-1} + (1-\alpha_{N-1})S_N x_{N-1}, \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)S_1^2 x_N, \\ &\ldots, \\ x_{2N} &= \alpha_{2N-1}x_{2N-1} + (1-\alpha_{2N-1})S_N^2 x_{2N-1}, \\ x_{2N+1} &= \alpha_{2N}x_{2N} + (1-\alpha_{2N})S_1^3 x_{2N}, \\ &\ldots. \end{aligned} \tag{1.9}$$

Since for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \to \infty$, as $n \to \infty$. Hence, we can rewrite the table above in the following compact form:

$$x_n = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})S_{i(n)}^{h(n)}x_{n-1}, \quad \forall n \geq 1.$$

For finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of a finite family of asymptotically κ-strict pseudocontractive mappings in the intermediate sense, utilizing the method in (1.9) and some hybrid method, Hu and Cai [12] got the following strong convergence theorem with the help of some boundedness assumptions.

Theorem 1.3 [12, Theorem 4.1] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $N \geq 1$ be an integer. Let ϕ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let A be an α-inverse-strongly monotone mapping of C into H. Let, for each $1 \leq i \leq N$, $T_i : C \to C$ be a uniformly continuous asymptotically κ_i-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa_i < 1$ with the sequences $\{\gamma_{n,i}\} \subset [0,\infty)$ such that $\lim_{n\to\infty} \gamma_{n,i} = 0$ and $\{c_{n,i}\} \subset [0,\infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$, $\gamma_n = \max\{\gamma_{n,i} : 1 \leq i \leq N\}$ and $c_n = \max\{c_{n,i} : 1 \leq i \leq N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{GEP}(\phi, A)$ is nonempty and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences in $[0,1]$ such that $0 < a \leq \alpha_n \leq 1$, $0 < \delta \leq \beta_n \leq 1-\kappa$ for all $n \in \mathbb{N}$ and $0 < b \leq r_n \leq c < 2\alpha$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by the following algorithm*:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrary}, \\ u_n \in C \quad \text{such that } \phi(u_n, y) + \langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = (1-\beta_n)u_n + \beta_n T_{i(n)}^{h(n)}u_n, \\ y_n = (1-\alpha_n)u_n + \alpha_n z_n, \\ C_n = \{v \in H : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_0, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\theta_n = \gamma_{h(n)}\rho_n^2 + c_{h(n)} \to 0$, as $n \to \infty$, where $\rho_n = \sup\{\|x_n - v\| : v \in \mathcal{F}\} < \infty$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{\mathcal{F}}x_0$.

Motivated and inspired by Jaiboon *et al.* [10], Hu and Cai [12], Hu and Wang [13] and Ge [14, 15], we introduce some new algorithms with variable coefficients based on the hybrid-type method and extragradient-type method for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping and the set of common fixed points of a finite family of asymptotically κ-strict pseudocontractive mappings in the intermediate sense in real Hilbert spaces. Some strong convergence theorems of these iterative algorithms are obtained without some boundedness conditions. The results of the paper improve and extend some recent ones announced by Inchan [8], Jaiboon *et al.* [10], Hu and Cai [12], Ceng and Yao [16], Kumam *et al.* [17] and others. The algorithms with variable coefficients introduced in this paper are of independent interests.

2 Preliminaries

Throughout this paper,

- $x_n \to x$ means that $\{x_n\}$ converges strongly to x;
- $\mathrm{Fix}(S) = \{x \in C : Sx = x\}$ denotes the set of fixed points of a self-mapping S on a set C;
- $B_r(x_1) := \{x \in H : \|x - x_1\| \leq r\}$;
- $\mathbb{N}$ is the set of positive integers;
- $\mathbb{R}$ is the set of real numbers.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. Recall that the inequality holds

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \quad \forall x \in H, y \in C. \tag{2.1}$$

Moreover, it is easy to see that it is equivalent to

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

It is also equivalent to

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C. \tag{2.2}$$

Lemma 2.1 [18] *Let C be a nonempty closed convex subsets of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C$$

holds.

Lemma 2.2 [16] *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem, the characterization of projection* (2.1) *implies that*

$$u \in \Omega \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \forall \lambda > 0.$$

Lemma 2.3 [19] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x, y, z \in H$ and given also a real number a, the set*

$$\left\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\right\}$$

is convex and closed.

Lemma 2.4 [20] *Let H be a real Hilbert space. Then for all $x, y, z \in H$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.5 [4] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be an asymptotically κ-strict pseudocontraction in the intermediate sense with the sequence $\{\gamma_n\}$. Then*

$$\|S^n x - S^n y\| \leq \frac{1}{1-\kappa}\left(\kappa\|x-y\| + \sqrt{\left(1+(1-\kappa)\gamma_n\right)\|x-y\|^2 + (1-\kappa)c_n}\right)$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 2.6 [5] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 [21] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Define a mapping $T_r(x) : H \to C$ as follows*:

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\right\}$$

for all $z \in H$. Then the following hold:

(1) *T_r is single-valued*;
(2) *T_r is firmly nonexpansive, i.e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle$ for all $x, y \in H$*;
(3) $\mathrm{Fix}(T_r) = \mathrm{EP}(F)$;
(4) $\mathrm{EP}(F)$ *is closed and convex.*

By Ibaraki *et al.* [22, Theorem 4.1], we have the following lemma.

Lemma 2.8 [14] *Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of a real Hilbert space H such that $K_{n+1} \subset K_n$ for each $n \in \mathbb{N}$. If $K^* = \bigcap_{n=0}^{\infty} K_n$ is nonempty, then for each $x \in H$, $\{P_{K_n} x\}$ converges strongly to $P_{K^*} x$.*

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply that $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for all $(y,g) \in G(T)$ implies that $f \in Tx$. Let $A : C \to H$ be a monotone and Lipschitz-continuous mapping, and let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*, $N_C = \{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

It is known that in this case, T is maximal monotone, and $0 \in Tv$ if and only if $v \in \Omega$, see [23].

3 Results and proofs

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $N \geq 1$ be an integer. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *let $A : C \to H$ be a monotone, L-Lipschitz-continuous mapping, and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a uniformly continuous asymptotically κ_i-strict pseudocontractive mapping in the intermediate sense with the sequences $\{\gamma_{n,i}\} \subset [0,\infty)$ such that $\lim_{n\to\infty} \gamma_{n,i} = 0$ and $\{c_{n,i}\} \subset [0,\infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$, $\gamma_n = \max\{\gamma_{n,i} : 1 \leq i \leq N\}$ and $c_n = \max\{c_{n,i} : 1 \leq i \leq N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C,A) \cap \mathrm{GEP}(F,B) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ be the sequences generated by the following algorithm with variable coefficients*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary}, \\ u_n \in C \quad \text{such that } F(u_n,y) + \langle Bx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ z_n = (1-\alpha_n-\hat{\beta}_n)x_n + \alpha_n t_n + \hat{\beta}_n S_{i(n)}^{h(n)} t_n, \\ C_0 = C, \\ C_n = \{z \in C_{n-1} : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (\alpha_n - \kappa)\hat{\beta}_n \|t_n - S_{i(n)}^{h(n)} t_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n} x_1 \end{cases} \tag{3.1}$$

for every $n \in \mathbb{N}$, where $\hat{\beta}_n = \frac{\beta_n}{1+\|x_n-x_1\|^2}$, $\theta_n = \beta_n(2\gamma_{h(n)}(1+r_0^2) + c_{h(n)})$, $\{\alpha_n\} \subset (a,1)$, $\{\beta_n\} \subset (b,1-a)$, $\{\lambda_n\} \subset (b/L,(1-a)/L)$ and $\{r_n\} \subset [d,e]$ for some $a \in (\kappa,1)$, $b \in (0,1-a)$ and $0 < d < e < 2\beta$, the positive real number r_0 is chosen so that $\mathcal{F} \cap B_{r_0}(x_1) \neq \emptyset$. Then the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ converge strongly to a point of $\mathcal{F}$.

Proof We divide the proof into eight steps.

Step 1. We claim that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ are well defined.

Indeed, by Lemma 2.6, we have $u_n = T_{r_n}(x_n - r_n B x_n)$, where $\{T_{r_n}\}$ is a sequence defined as in Lemma 2.7. From the definition of C_n and Lemma 2.3, it is easy to see that C_n is convex and closed for each $n \in \mathbb{N}$. So, it is sufficient to prove that $\mathcal{F} \cap B_{r_0}(x_1) \subset C_n$ for each $n \in \mathbb{N}$.

Let $p \in \mathcal{F} \cap B_{r_0}(x_1)$ be an arbitrary element. Then we see that $p = T_{r_n}(p - r_n Bp)$. Since $B : C \to H$ is a β-inverse-strongly monotone mapping and $r_n < 2\beta$, it follows from $u_n = T_{r_n}(x_n - r_n B x_n)$ and Lemma 2.7 that

$$\begin{aligned} \|u_n - p\|^2 &= \big\|T_{r_n}(x_n - r_n B x_n) - T_{r_n}(p - r_n Bp)\big\|^2 \\ &\leq \big\|(x_n - r_n B x_n) - (p - r_n Bp)\big\|^2 \\ &= \big\|(x_n - p) - r_n(Bx_n - Bp)\big\|^2 \\ &= \|x_n - p\|^2 - 2r_n\langle x_n - p, Bx_n - Bp\rangle + r_n^2\|Bx_n - Bp\|^2 \\ &\leq \|x_n - p\|^2 - r_n(2\beta - r_n)\|Bx_n - Bp\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.2}$$

Putting $x = u_n - \lambda_n Ay_n$ and $y = p$ in (2.2), we have

$$\begin{aligned}\|t_n - p\|^2 &\le \|u_n - \lambda_n Ay_n - p\|^2 - \|u_n - \lambda_n Ay_n - t_n\|^2 \\ &= \|u_n - p\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, p - t_n\rangle \\ &= \|u_n - p\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\big(\langle Ay_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle\big) \\ &\quad + 2\lambda_n\langle Ay_n, y_n - t_n\rangle.\end{aligned}$$

Since $A : C \to H$ is a monotone mapping and $p \in \mathrm{VI}(C, A)$, further, we have

$$\begin{aligned}\|t_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|u_n - p\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\quad + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.\end{aligned} \tag{3.3}$$

Since $y_n = P_C(u_n - \lambda_n Au_n)$ and A is L-Lipschitz-continuous, by Lemma 2.1, we have

$$\begin{aligned}&\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\ &\quad = \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle + \lambda_n\langle Au_n - Ay_n, t_n - y_n\rangle \\ &\quad \le \lambda_n\langle Au_n - Ay_n, t_n - y_n\rangle \\ &\quad \le \lambda_n L\|u_n - y_n\|\|t_n - y_n\|.\end{aligned} \tag{3.4}$$

So, it follows from (3.3), (3.4) and $\{\lambda_n\} \subset (b/L, (1-a)/L)$, we obtain

$$\begin{aligned}\|t_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n L\|u_n - y_n\|\|t_n - y_n\| \\ &\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 L^2\|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\ &= \|u_n - p\|^2 - \big(1 - \lambda_n^2 L^2\big)\|u_n - y_n\|^2 \\ &\le \|u_n - p\|^2.\end{aligned} \tag{3.5}$$

By the definition of S_i, for all $n \in \mathbb{N}$, $x \in C$, $1 \le i \le N$ we have

$$\begin{aligned}\big\|S_i^n x - p\big\|^2 &\le (1 + \gamma_{n,i})\|x - p\|^2 + \kappa_i\big\|x - S_i^n x\big\|^2 + c_{n,i} \\ &\le (1 + \gamma_n)\|x - p\|^2 + \kappa\big\|x - S_i^n x\big\|^2 + c_n,\end{aligned} \tag{3.6}$$

where $c_n \in [0, \infty)$ with $c_n \to 0$ as $n \to \infty$. So, from $z_n = (1 - \alpha_n - \hat{\beta}_n)x_n + \alpha_n t_n + \hat{\beta}_n S_{i(n)}^{h(n)} t_n$, (3.2), (3.5), (3.6) and Lemma 2.4, we deduce that

$$\begin{aligned}\|z_n - p\|^2 &= \big\|(1 - \alpha_n - \hat{\beta}_n)(x_n - p) + \alpha_n(t_n - p) + \hat{\beta}_n\big(S_{i(n)}^{h(n)} t_n - p\big)\big\|^2 \\ &\le (1 - \alpha_n - \hat{\beta}_n)\|x_n - p\|^2 + \alpha_n\|t_n - p\|^2 + \hat{\beta}_n\big\|S_{i(n)}^{h(n)} t_n - p\big\|^2 \\ &\quad - \alpha_n\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)} t_n\big\|^2 \\ &\le (1 - \alpha_n - \hat{\beta}_n)\|x_n - p\|^2 + \alpha_n\|t_n - p\|^2\end{aligned}$$

$$
\begin{aligned}
&\quad + \hat{\beta}_n\big((1+\gamma_{h(n)})\|t_n - p\|^2 + \kappa\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 + c_{h(n)}\big) - \alpha_n\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 \\
&= (1-\alpha_n-\hat{\beta}_n)\|x_n - p\|^2 + (\alpha_n+\hat{\beta}_n)\|t_n - p\|^2 \\
&\quad + \hat{\beta}_n\big(\gamma_{h(n)}\|t_n - p\|^2 + c_{h(n)}\big) - (\alpha_n - \kappa)\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 \\
&\le (1-\alpha_n-\hat{\beta}_n)\|x_n - p\|^2 + (\alpha_n+\hat{\beta}_n)\|x_n - p\|^2 \\
&\quad - (\alpha_n - \kappa)\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 + \hat{\beta}_n\big(\gamma_{h(n)}\|t_n - p\|^2 + c_{h(n)}\big) \\
&= \|x_n - p\|^2 - (\alpha_n - \kappa)\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 + \hat{\beta}_n\big(\gamma_{h(n)}\|x_n - p\|^2 + c_{h(n)}\big). \qquad (3.7)
\end{aligned}
$$

Further, it follows from the definition of $\hat{\beta}_n$ that

$$
\begin{aligned}
\|z_n - p\|^2 &\le \|x_n - p\|^2 - (\alpha_n - \kappa)\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 \\
&\quad + \beta_n\frac{2\gamma_{h(n)}(\|x_n - x_1\|^2 + \|p - x_1\|^2) + c_{h(n)}}{1 + \|x_n - x_1\|^2} \\
&\le \|x_n - p\|^2 - (\alpha_n - \kappa)\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 + \beta_n\big(2\gamma_{h(n)}\big(1 + r_0^2\big) + c_{h(n)}\big) \\
&\le \|x_n - p\|^2 - (\alpha_n - \kappa)\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 + \theta_n, \qquad (3.8)
\end{aligned}
$$

where $\theta_n = \beta_n(2\gamma_{h(n)}(1 + r_0^2) + c_{h(n)})$. Therefore, we have

$$\mathcal{F} \cap B_{r_0}(x_1) \subset C_n, \quad \forall n \in \mathbb{N}.$$

Step 2. We claim that the sequence $\{x_n\}$ converges strongly to an element in C, say x^*.

Since $\{C_n\}$ is a decreasing sequence of closed convex subset of H such that $C^* = \bigcap_{n=0}^{\infty} C_n$ is a nonempty and closed convex subset of H, it follows from Lemma 2.8 that $\{x_{n+1}\} = \{P_{C_n}x_1\}$ converges strongly to $P_{C^*}x_1$, say x^*.

Step 3. We claim that $\lim_{n\to\infty} z_n = x^*$, $\lim_{n\to\infty} t_n = x^*$ and $\lim_{n\to\infty}\|t_n - S_{i(n)}^{h(n)}t_n\| = 0$.

Indeed, the definition of x_{n+1} shows that $x_{n+1} \in C_n$, *i.e.*,

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 - (\alpha_n - \kappa)\hat{\beta}_n\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 + \theta_n. \qquad (3.9)$$

Note that $\gamma_{h(n)} \to 0$, $c_{h(n)} \to 0$, $x_n \to x^*$ as $n \to \infty$ and $\alpha_n > a > \kappa$, $\forall n \in \mathbb{N}$, we have $\theta_n \to 0$, $\|z_n - x_{n+1}\| \to 0$, $\|z_n - x_n\| \to 0$ and $z_n \to x^*$ as $n \to \infty$. Further, it follows from (3.9) that

$$(a - \kappa)\frac{b}{1 + \|x_n - x_1\|^2}\big\|t_n - S_{i(n)}^{h(n)}t_n\big\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n.$$

Thus, $\lim_{n\to\infty}\|t_n - S_{i(n)}^{h(n)}t_n\| = 0$. Since $z_n = (1 - \alpha_n - \hat{\beta}_n)x_n + \alpha_n t_n + \hat{\beta}_n S_{i(n)}^{h(n)}t_n$, we have

$$z_n - x_n = (\alpha_n + \hat{\beta}_n)(t_n - x_n) + \hat{\beta}_n\big(S_{i(n)}^{h(n)}t_n - t_n\big).$$

That is,

$$t_n - x_n = \frac{1}{\alpha_n + \hat{\beta}_n}(z_n - x_n) - \frac{\hat{\beta}_n}{\alpha_n + \hat{\beta}_n}\big(S_{i(n)}^{h(n)}t_n - t_n\big).$$

Considering $0 < a < \alpha_n + \hat{\beta}_n$, $\forall n \in \mathbb{N}$, we have

$$t_n - x_n \to 0, \qquad t_n \to x^*, \quad \text{as } n \to \infty. \tag{3.10}$$

Step 4. We claim that $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(S_i)$.

Indeed, for each $n \in \mathbb{N}$, $1 \le i \le N$, we have

$$\|S_i^n x^* - x^*\| \le \|S_i^n x^* - S_i^n t_{(n-1)N+i}\| + \|S_i^n t_{(n-1)N+i} - t_{(n-1)N+i}\| + \|t_{(n-1)N+i} - x^*\|.$$

This together with Step 3 and Lemma 2.5 implies that

$$S_i^n x^* - x^* \to 0, \quad \text{as } n \to \infty, \tag{3.11}$$

where $1 \le i \le N$. Since $S_i : C \to C$ is uniformly continuous, by (3.11), we have

$$S_i^{n+1} x^* = S_i(S_i^n x^*) \to S_i x^*, \quad \text{as } n \to \infty.$$

Hence, $S_i x^* = x^*$, *i.e.*, $x^* \in \mathrm{Fix}(S_i)$. Thus, we obtain $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(S_i)$.

Step 5. We claim that $t_n - y_n \to 0$, $t_n - u_n \to 0$, $y_n \to x^*$ and $u_n \to x^*$, as $n \to \infty$.

By (3.5), for $p \in \mathcal{F} \cap B_{r_0}(x_1)$, we have

$$\|t_n - p\|^2 \le \|u_n - p\|^2 - (1 - \lambda_n^2 L^2)\|u_n - y_n\|^2.$$

Therefore, from (3.2), we have

$$\begin{aligned}
(1 - \lambda_n^2 L^2)\|u_n - y_n\|^2 &\le \|u_n - p\|^2 - \|t_n - p\|^2 \\
&\le \|x_n - p\|^2 - \|t_n - p\|^2 \\
&\le (\|x_n - p\| + \|t_n - p\|)(\|x_n - p\| - \|t_n - p\|) \\
&\le (\|x_n - p\| + \|t_n - p\|)(\|x_n - t_n\|).
\end{aligned}$$

This together with (3.10) and $0 < 1 - (1-a)^2 < 1 - \lambda_n^2 L^2$ implies that

$$u_n - y_n \to 0, \quad \text{as } n \to \infty. \tag{3.12}$$

On the other hand, it follows from (3.5) that for $p \in \mathcal{F} \cap B_{r_0}(x_1)$,

$$\begin{aligned}
\|t_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n L\|u_n - y_n\|\|t_n - y_n\| \\
&\le \|u_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + \lambda_n^2 L^2\|t_n - y_n\|^2 \\
&= \|u_n - p\|^2 - (1 - \lambda_n^2 L^2)\|t_n - y_n\|^2.
\end{aligned}$$

Further, from (3.2), we have

$$\begin{aligned}
(1 - \lambda_n^2 L^2)\|t_n - y_n\|^2 &\le \|u_n - p\|^2 - \|t_n - p\|^2 \\
&\le \|x_n - p\|^2 - \|t_n - p\|^2 \\
&\le (\|x_n - p\| + \|t_n - p\|)(\|x_n - p\| - \|t_n - p\|) \\
&\le (\|x_n - p\| + \|t_n - p\|)(\|x_n - t_n\|).
\end{aligned}$$

This together with (3.10) and $0 < 1 - (1-a)^2 < 1 - \lambda_n^2 L^2$ implies that $t_n - y_n \to 0$, as $n \to \infty$. Further, from (3.12), Step 2 and Step 3, we have $t_n - u_n \to 0$, $y_n \to x^*$ and $u_n \to x^*$, as $n \to \infty$.

Step 6. We claim that $x^* \in \mathrm{VI}(C, A)$.

Indeed, let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned in Section 2 that in this case, T is maximal monotone, and $0 \in Tv$ if and only if $v \in \Omega$, see [23].

Let $(v, w) \in G(T)$, the graph of T. Then we have $w \in Tv = Av + N_C v$, and hence, $w - Av \in N_C v$. So, we have

$$\langle v - t, w - Av \rangle \geq 0, \quad \forall t \in C. \tag{3.13}$$

Noticing $t_n = P_C(u_n - \lambda_n A y_n)$ and $v \in C$, by (2.1), we have

$$\langle u_n - \lambda_n A y_n - t_n, t_n - v \rangle \geq 0,$$

and hence,

$$\left\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + A y_n \right\rangle \geq 0. \tag{3.14}$$

From (3.13), (3.14) and $t_n \in C$, we have

$$\begin{aligned} \langle v - t_n, w \rangle &\geq \langle v - t_n, Av \rangle \\ &\geq \langle v - t_n, Av \rangle - \left\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + A y_n \right\rangle \\ &\geq \langle v - t_n, Av - A t_n \rangle + \langle v - t_n, A t_n - A y_n \rangle - \left\langle v - t_n, \frac{t_n - u_n}{\lambda_n} \right\rangle. \end{aligned} \tag{3.15}$$

Letting $n \to \infty$ in (3.15), considering $A : C \to H$ is monotone, L-Lipschitz-continuous, $\{\lambda_n\} \subset (b/L, (1-a)/L)$ and Step 5, we have $\langle v - x^*, w \rangle \geq 0$. Since T is maximal monotone, we have $0 \in Tx^*$, and hence, $x^* \in \mathrm{VI}(C, A)$.

Step 7. We claim that $x^* \in \mathrm{GEP}(F, B)$.

Since $u_n = T_{r_n}(x_n - r_n B x_n)$, for any $y \in C$, we have

$$F(u_n, y) + \langle B x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0.$$

From (A2), we have

$$\langle B x_n, y - u_n \rangle + \left\langle y - u_n, \frac{u_n - x_n}{r_n} \right\rangle \geq F(y, u_n). \tag{3.16}$$

Put $y_t = ty + (1-t)x^*$ for all $t \in (0,1]$ and $y \in C$. Thus, we have $y_t \in C$. So, from (3.16), we have

$$\begin{aligned}\langle y_t - u_n, By_t\rangle &\geq \langle y_t - u_n, By_t\rangle - \langle Bx_n, y_t - u_n\rangle \\ &\quad - \left\langle y_t - u_n, \frac{u_n - x_n}{r_n}\right\rangle + F(y_t, u_n) \\ &= \langle y_t - u_n, By_t - Bu_n\rangle + \langle y_t - u_n, Bu_n - Bx_n\rangle \\ &\quad - \left\langle y_t - u_n + \frac{u_n - x_n}{r_n}\right\rangle + F(y_t, u_n). \end{aligned} \tag{3.17}$$

Since $B : C \to H$ is a β-inverse-strongly monotone mapping, letting $n \to \infty$, it follows from Step 3, Step 5, (A4) and $0 < d < r_n$ that

$$\langle y_t - x^*, By_t\rangle \geq F(y_t, x^*), \quad \forall t \in (0,1]. \tag{3.18}$$

From (A1), (A4) and (3.18), we also have

$$\begin{aligned}0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, x^*) \\ &\leq tF(y_t, y) + (1-t)\langle y_t - x^*, By_t\rangle \\ &= tF(y_t, y) + t(1-t)\langle y - x^*, By_t\rangle,\end{aligned}$$

and hence,

$$0 \leq F(y_t, y) + (1-t)\langle y - x^*, By_t\rangle.$$

Letting $t \to 0^+$, we have, for each $y \in C$,

$$0 \leq F(x^*, y) + \langle y - x^*, Bx^*\rangle.$$

This implies that $x^* \in \mathrm{GEP}(F, B)$.

Step 8. We claim that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ converge strongly to $x^* \in \mathcal{F}$.

From Step 4, 6, 7, we have $x^* \in \mathcal{F}$. Therefore, it follows from Step 2, Step 3 and Step 5 that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ converge strongly to $x^* \in \mathcal{F}$. This completes the proof. □

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $N \geq 1$ be an integer. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a uniformly continuous asymptotically κ_i-strict pseudocontractive mapping in the intermediate sense with the sequences $\{\gamma_{n,i}\} \subset [0,\infty)$ such that* $\lim_{n\to\infty} \gamma_{n,i} = 0$ *and* $\{c_{n,i}\} \subset [0,\infty)$ *such that* $\lim_{n\to\infty} c_{n,i} = 0$. *Let* $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$, $\gamma_n = \max\{\gamma_{n,i} : 1 \leq i \leq N\}$ *and* $c_n = \max\{c_{n,i} : 1 \leq i \leq N\}$. *Assume that* $\mathcal{F} = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GEP}(F, B) \neq \emptyset$. *Let* $\{x_n\}$, $\{u_n\}$ *and*

$\{z_n\}$ be the sequences generated by the following algorithm with variable coefficients

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary}, \\ u_n \in C \quad \text{such that } F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = (1 - \alpha_n - \hat{\beta}_n)x_n + \alpha_n u_n + \hat{\beta}_n S_{i(n)}^{h(n)} u_n, \\ C_0 = C, \\ C_n = \{z \in C_{n-1} : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (\alpha_n - \kappa)\hat{\beta}_n \|u_n - S_{i(n)}^{h(n)} u_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n} x_1 \end{cases}$$

for every $n \in \mathbb{N}$, where $\hat{\beta}_n = \frac{\beta_n}{1+\|x_n - x_1\|^2}$, $\theta_n = \beta_n(2\gamma_{h(n)}(1 + r_0^2) + c_{h(n)})$, $\{\alpha_n\} \subset (a, 1)$, $\{\beta_n\} \subset (b, 1 - a)$ and $\{r_n\} \subset [d, e]$ for some $a \in (\kappa, 1)$, $b \in (0, 1 - a)$ and $0 < d < e < 2\beta$, the positive real number r_0 is chosen so that $\mathcal{F} \cap B_{r_0}(x_1) \neq \emptyset$. Then the sequences $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge strongly to a point of $\mathcal{F}$.

Proof Putting $A = 0$, the conclusion of Corollary 3.2 can be obtained by Theorem 3.1 immediately. □

Remark 3.3 Corollary 3.2 improves and extends [12, Theorem 4.1] and [17, Theorem 4.3] since

(1) the boundedness assumptions that set $\mathcal{F}$ and the sequence $\{\rho_n\}$ are both bounded in [12, Theorem 4.1] are dispensed with,
(2) the boundedness condition on the sequence $\{\rho_n\}$ in [17, Theorem 4.3] is dropped off,
(3) a finite family of asymptotically strict pseudocontractive mapping in [17, Theorem 4.3] is extended to a finite family of asymptotically strict pseudocontractive mapping in the intermediate sense,
(4) the equilibrium problem in [17, Theorem 4.3] is extended to the generalized equilibrium problem.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $N \geq 1$ be an integer. Let $A : C \to H$ be a monotone, L-Lipschitz-continuous mapping. Let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a uniformly continuous asymptotically κ_i-strict pseudocontractive mapping in the intermediate sense with the sequences $\{\gamma_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} \gamma_{n,i} = 0$ and $\{c_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$, $\gamma_n = \max\{\gamma_{n,i} : 1 \leq i \leq N\}$ and $c_n = \max\{c_{n,i} : 1 \leq i \leq N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ be the sequences generated by the following algorithm with variable coefficients*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ z_n = (1 - \alpha_n - \hat{\beta}_n)x_n + \alpha_n t_n + \hat{\beta}_n S_{i(n)}^{h(n)} t_n, \\ C_0 = C, \\ C_n = \{z \in C_{n-1} : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (\alpha_n - \kappa)\hat{\beta}_n \|t_n - S_{i(n)}^{h(n)} t_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n} x_1 \end{cases}$$

for every $n \in \mathbb{N}$, *where* $\hat{\beta}_n = \frac{\beta_n}{1+\|x_n - x_1\|^2}$, $\theta_n = \beta_n(2\gamma_{h(n)}(1 + r_0^2) + c_{h(n)})$, $\{\alpha_n\} \subset (a, 1)$, $\{\beta_n\} \subset (b, 1 - a)$ *and* $\{\lambda_n\} \subset (b/L, (1 - a)/L)$ *for some* $a \in (\kappa, 1)$ *and some* $b \in (0, 1 - a)$, *the positive real number* r_0 *is chosen so that* $\mathcal{F} \cap B_{r_0}(x_1) \neq \emptyset$. *Then the sequences* $\{x_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to a point of* $\mathcal{F}$.

Proof Putting $F = 0$, $B = 0$, respectively, the conclusion of Corollary 3.4 can be obtained by Theorem 3.1 immediately. □

Remark 3.5 Corollary 3.4 improves and extends [16, Theorem 3.1] since

(1) the convergence condition that $\liminf_{n\to\infty} \langle Ax_n, y - x_n\rangle \geq 0$ for all $y \in C$ in [16, Theorem 3.1] is removed,

(2) the boundedness assumptions that the intersection $F(S) \cap \Omega$ and the sequence $\{\Delta_n\}$ are both bounded in [16, Theorem 3.1] are dispensed with,

(3) the requirement $(I - A)(C) \subset C$ in [16, Theorem 3.1] is dropped off,

(4) an asymptotically κ_i-strict pseudocontractive mapping in the intermediate sense in [16, Theorem 3.1] is extended to a finite family of ones.

Corollary 3.6 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *and let* $N \geq 1$ *be an integer. Let, for each* $1 \leq i \leq N$, $S_i : C \to C$ *be a uniformly continuous asymptotically* κ_i*-strict pseudocontractive mapping in the intermediate sense with the sequences* $\{\gamma_{n,i}\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} \gamma_{n,i} = 0$ *and* $\{c_{n,i}\} \subset [0, \infty)$ *such that* $\lim_{n\to\infty} c_{n,i} = 0$. *Let* $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$, $\gamma_n = \max\{\gamma_{n,i} : 1 \leq i \leq N\}$ *and* $c_n = \max\{c_{n,i} : 1 \leq i \leq N\}$. *Assume that* $\mathcal{F} = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be the sequences generated by the following algorithm with variable coefficients*

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary,} \\ z_n = (1 - \alpha_n - \hat{\beta}_n)x_n + \alpha_n x_n + \hat{\beta}_n S_{i(n)}^{h(n)} x_n, \\ C_0 = C, \\ C_n = \{z \in C_{n-1} : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (\alpha_n - \kappa)\hat{\beta}_n \|x_n - S_{i(n)}^{h(n)} x_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n} x_1 \end{cases}$$

for every $n \in \mathbb{N}$, *where* $\hat{\beta}_n = \frac{\beta_n}{1+\|x_n - x_1\|^2}$, $\theta_n = \beta_n(2\gamma_{h(n)}(1 + r_0^2) + c_{h(n)})$, $\{\alpha_n\} \subset (a, 1)$ *and* $\{\beta_n\} \subset (b, 1 - a)$ *for some* $a \in (\kappa, 1)$ *and some* $b \in (0, 1 - a)$, *the positive real number* r_0 *is chosen so that* $\mathcal{F} \cap B_{r_0}(x_1) \neq \emptyset$. *Then the sequences* $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to a point of* $\mathcal{F}$.

Proof Putting $F = 0$, $A = 0$, $B = 0$, respectively, the conclusion of Corollary 3.6 can be obtained by Theorem 3.1 immediately. □

By the careful analysis of the proof of Theorem 3.1, we can obtain the following result.

Theorem 3.7 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *and let* $N \geq 1$ *be an integer. Let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4), *let* $A : C \to H$ *be a monotone,* L*-Lipschitz-continuous mapping, and let* $B : C \to H$ *be a* β*-inverse-strongly monotone mapping. Let, for each* $1 \leq i \leq N$, $S_i : C \to C$ *be a uni-*

formly continuous asymptotically κ_i-strict pseudocontractive mapping in the intermediate sense with the sequences $\{\gamma_{n,i}\} \subset [0,\infty)$ such that $\lim_{n\to\infty}\gamma_{n,i} = 0$ and $\{c_{n,i}\} \subset [0,\infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $\kappa = \max\{\kappa_i : 1 \le i \le N\}$, $\gamma_n = \max\{\gamma_{n,i} : 1 \le i \le N\}$ and $c_n = \max\{c_{n,i} : 1 \le i \le N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{VI}(C,A) \cap \mathrm{GEP}(F,B)$ is nonempty and bounded. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ be the sequences generated by the following algorithm

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrary}, \\ u_n \in C \quad \text{such that } F(u_n,y) + \langle Bx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n t_n + \beta_n S_{i(n)}^{h(n)} t_n, \\ C_0 = C, \\ C_n = \{z \in C_{n-1} : \|z_n - z\|^2 \le \|x_n - z\|^2 - (\alpha_n - \kappa)\beta_n \|t_n - S_{i(n)}^{h(n)} t_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n} x_1 \end{cases}$$

for every $n \in \mathbb{N}$, where $\theta_n = \beta_n(\gamma_{h(n)}\Delta_n + c_{h(n)})$, $\Delta_n = \sup_{p\in\mathcal{F}} \|x_n - p\|^2$, $\{\alpha_n\} \subset (a,1)$, $\{\beta_n\} \subset (b, 1-a)$, $\{\lambda_n\} \subset (b/L, (1-a)/L)$ and $\{r_n\} \subset [d,e]$ for some $a \in (\kappa,1)$, $b \in (0,1-a)$ and $0 < d < e < 2\beta$. Then the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ converge strongly to a point of $\mathcal{F}$.

Proof Following the reasoning in the proof of Theorem 3.1, we use $\mathcal{F}$ instead of $\mathcal{F} \cap B_{r_0}(x_1)$. Considering that $\mathcal{F}$ is bounded, we take $\Delta_n = \sup_{p\in\mathcal{F}} \|x_n - p\|^2$, $\theta_n = \beta_n(\gamma_n\Delta_n + c_n)$ in (3.7), so the assertion of Step 1 holds. From Step 2, we have that the sequence $\{\Delta_n\}$ is bounded, and hence, $\theta_n = \beta_n(\gamma_n\Delta_n + c_n) \to 0$ as $n \to \infty$. The remainder of the proof of Theorem 3.7 is similar to Theorem 3.1. The conclusion, therefore, follows. This completes the proof. □

Remark 3.8 Theorem 3.7 improves and extends [12, Theorem 4.1] since

(1) the requirement that the sequence $\{\rho_n\}$ is bounded in [12, Theorem 4.1] is dispensed with,

(2) Theorem 4.1 of [12] is a special case, in which mapping $A = 0$ in Theorem 3.7.

Theorem 3.9 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $N \ge 1$ be an integer. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *let $A : C \to H$ be a monotone, L-Lipschitz-continuous mapping, and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let, for each $1 \le i \le N$, $S_i : C \to C$ be a uniformly continuous asymptotically nonexpansive mapping in the intermediate sense with the sequence $\{c_{n,i}\} \subset [0,\infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $c_n = \max\{c_{n,i} : 1 \le i \le N\}$. Assume that $\mathcal{F} = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{VI}(C,A) \cap \mathrm{GEP}(F,B) \ne \emptyset$. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ be the se-*

quences generated by the following algorithm

$$\begin{cases} x_1 \in C \quad \textit{chosen arbitrary}, \\ u_n \in C \quad \textit{such that } F(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n t_n + \beta_n S_{i(n)}^{h(n)} t_n, \\ C_0 = C, \\ C_n = \{z \in C_{n-1} : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n \beta_n \|t_n - S_{i(n)}^{h(n)} t_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_n} x_1 \end{cases}$$

for every $n \in \mathbb{N}$, where $\theta_n = \beta_n c_{h(n)}$, $\{\alpha_n\} \subset (a, 1)$, $\{\beta_n\} \subset (b, 1 - a)$, $\{\lambda_n\} \subset (b/L, (1 - a)/L)$ and $\{r_n\} \subset [d, e]$ for some $a \in (0, 1)$, $b \in (0, 1 - a)$ and $0 < d < e < 2\beta$. Then the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ converge strongly to a point of $\mathcal{F}$.

Proof In Theorem 3.1, whenever $S_i : C \to C$ is an asymptotically nonexpansive mapping in the intermediate sense, we have $\gamma_{n,i} = 0$, $\kappa_i = 0$ for all $n \in \mathbb{N}$, $1 \leq i \leq N$. From (3.7), we have

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - (\alpha_n - \kappa)\beta_n \left\| t_n - S_{i(n)}^{h(n)} t_n \right\|^2 + \beta_n \left(\gamma_{h(n)} \|t_n - p\|^2 + c_{h(n)} \right). \tag{3.19}$$

Since $\kappa = \max\{\kappa_i : 1 \leq i \leq N\} = 0$, $\gamma_{h(n)} = \max\{\gamma_{h(n),i} : 1 \leq i \leq N\} = 0$ and $c_{h(n)} = \max\{c_{h(n),i} : 1 \leq i \leq N\}$, thus, (3.19) is reduced to

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \alpha_n \beta_n \left\| t_n - S_{i(n)}^{h(n)} t_n \right\|^2 + \theta_n,$$

where $\theta_n = \beta_n c_{h(n)}$. So, we have

$$\mathcal{F} \subset C_n, \quad \forall n \in \mathbb{N},$$

and hence, the result of Step 1 holds.

Next, following the reasoning in the proof of Theorem 3.1 and using $\mathcal{F}$ instead of $\mathcal{F} \cap B_{r_0}(x_1)$, we deduce the conclusion of Theorem 3.9. □

Remark 3.10 Theorem 3.9 improves and extends [8, Theorem 3.1] and [10, Theorem 3.1] since

(1) a finite family of nonexpansive mappings is extended to a finite family of asymptotically nonexpansive mapping in the intermediate sense,
(2) inverse-strongly monotone mapping A is extended to monotone L-Lipschitz-continuous mapping.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed significantly in writing this paper. All authors read and approved the final manuscript.

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