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Convergence analysis of projection methods for a new system of general nonconvex variational inequalities

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Abstract
In this article, we introduce and consider a new system of general nonconvex variational inequalities defined on uniformly prox-regular sets. We establish the equivalence between the new system of general nonconvex variational inequalities and the fixed point problems to analyze an explicit projection method for solving this system. We also consider the convergence of the projection method under some suitable conditions. Results presented in this article improve and extend the previously known results for the variational inequalities and related optimization problems.

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1 Introduction
Variational inequalities theory, which was introduced by Stampacchia [1], has emerged as an interesting and fascinating branch of mathematical and engineering sciences. The ideas and techniques of variational inequalities are being applied in structural analysis, economics, optimization, operations research fields. It has been shown that variational inequalities provide the most natural, direct, simple, and efficient framework for a general treatment of some unrelated problems arising in various fields of pure and applied sciences. In recent years, there have been considerable activities in the development of numerical techniques including projection methods, Wiener-Hopf equations, auxiliary principle, and descent framework for solving variational inequalities; see [2-17] and the references therein. These activities have motivated us to generalize and extend the variational inequalities and related optimization problems in several directions using novel techniques.

Projection technique has played a significant role in the numerical solution of variational inequalities based on the convergence analysis. It is worth mentioning that almost all the results regarding the existence and iterative schemes for variational inequalities, which have been investigated and considered, if the underlying set is a convex set. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. Recently, Clarke et al. [8] and Poliquin et al. [9] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. It is known that uniformly prox-regular sets are nonconvex sets and include convex sets as a special case. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamical systems, and differential inclusions.

In 2009, Noor [10] introduced a nonconvex variational inequalities based on the uniformly prox-regular sets. Moreover, he discussed the existence and algorithm of the solution for the nonconvex variational inequalities, which shows projection technique can be extended to nonconvex sets. Noor [11] proposed some iterative methods for solving a general nonconvex variational inequalities with projection methods and Wiener-Hopf equations technique. On the other hand, Verma [13] and Noor and Noor [14] proposed explicit projection methods for solving systems of variational inequalities and general variational inequalities on a closed convex subset of Hilbert space, respectively. Very recently, Wen [15] modified projection methods to a generalized system of nonconvex variational inequalities with different nonlinear operators. However, only iterative sequences $\{g(x_n)\}$, $\{g(y_n)\}$ come from the projection methods, which requires that mapping g must be injective in order to arrive at a solution of the generalized system. Furthermore, the property defined on the underlying operator T depends on the mapping g in convergence analysis. These strict conditions rule out many applications of the projection type methods for the generalized system of nonconvex variational inequalities.

In this article, motivated and inspired by the research going on in this direction, we introduce and consider a more general system, which is called a new general nonconvex variational inequalities. The new system includes the system of variational inequalities involving two different nonlinear operators, the general nonconvex variational inequalities and the systems of variational inequalities defined on closed convex sets as special cases.

The purpose of this article is not only to show that projection technique can be extended to the new system of general nonconvex variational inequalities on uniformly prox-regular sets, but also to get rid of the dependence of T on the mapping g and the injective property defined on g in convergence analysis of the projection method for solving the new system of general nonconvex variational inequalities. Our results extend and improve the corresponding results of [7,10-15].

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K be a nonempty and convex subset in H.

First of all, we recall the following well-known concepts from nonlinear convex analysis and non-smooth analysis [8,9].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is $d_K(u) = \inf_{v\in K} \|v - u\|$. The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty, closed, and convex subset in H. Then $\varsigma \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\zeta, u) > 0$ such that

$$\langle \varsigma, v - u\rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K.$$

Definition 2.2. The Clarke normal cone, denoted by $N_K^C(u)$, is defined as

$$N_K^C(u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}$means the closure of the convex hull. Clearly $N_K^P(u) \subset N_K^C(u)$, but the converse is not true. Note that $N_K^C(u)$ is always closed and convex, whereas $N_K^P(u)$ is convex, but may not be closed [9].

Definition 2.3. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$\left\langle \frac{\xi}{\|\xi\|}, v - u\right\rangle \leq \frac{1}{2r}\|v - u\|^2, \quad \forall v \in K_r.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [8,9]. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N_{K_r}^P(u)$ is closed as a set-valued mapping. Thus, we have $N_{K_r}^P(u) = N_{K_r}^C(u)$.

Remark 2.1. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K, that is, $K_r = K$.

Let K_r be a uniformly r-prox-regular (nonconvex) set, and $T_1, T_2 : K_r \times K_r \to K_r$ and $g, h : H \to K_r$ be different nonlinear operators, respectively. For any given constants $\rho > 0$ and $\eta > 0$, we consider the problem of finding $x^*, y^* \in K_r$ such that

$$\langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(x) - x^*\rangle + \frac{1}{2r}\left\|g(x) - x^*\right\|^2 \geq 0, \;\; \forall x \in H : g(x) \in K_r, \quad (2.1a)$$

$$\langle \eta T_2(x^*, y^*) + y^* - h(x^*), h(x) - y^*\rangle + \frac{1}{2r}\left\|h(x) - y^*\right\|^2 \geq 0, \;\; \forall x \in H : h(x) \in K_r, \quad (2.1b)$$

which is called a new system of general nonconvex variational inequalities.

If $g = h = I$, the identity operator, then problem (2.1) is equivalent to finding $x^*, y^* \in K_r$ such that

$$\langle \rho T_1(y^*, x^*) + x^* - y^*, x - x^*\rangle + \frac{1}{2r}\left\|x - x^*\right\|^2 \geq 0, \quad \forall x \in K_r, \; \rho > 0, \quad (2.2a)$$

$$\langle \eta T_2(x^*, y^*) + y^* - x^*, x - y^*\rangle + \frac{1}{2r}\left\|x - y^*\right\|^2 \geq 0, \quad \forall x \in K_r, \; \eta > 0, \quad (2.2b)$$

which appears to be the other new system of nonconvex variational inequalities.

We note that, if $r = \infty$, $K_r = K$, the convex subset in H, then problem (2.1) is equivalent to finding $x^*, y^* \in K$ such that

$$\langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(x) - x^* \rangle \geq 0, \quad \forall x \in H : g(x) \in K, \ \rho > 0, \tag{2.3a}$$

$$\langle \eta T_2(x^*, y^*) + y^* - h(x^*), h(x) - y^* \rangle \geq 0, \quad \forall x \in H : h(x) \in K, \ \eta > 0, \tag{2.3b}$$

which is known as the system of general variational inequalities involving four different nonlinear operators, introduced, and studied by Noor and Noor [14].

If $g = h = I$, T_1, $T_2 : K \to K$ are two univariate nonlinear operators, then problem (2.3) is equivalent to finding x^*, $y^* \in K$ such that

$$\langle \rho T_1(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K, \ \rho > 0, \tag{2.4a}$$

$$\langle \eta T_2(x^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in K, \ \eta > 0, \tag{2.4b}$$

which is known as the system of nonlinear variational inequalities involving two different nonlinear operators. If $T_1 = T_2$, problem (2.4) reduces to the system of variational inequalities, which was introduced and studied by Verma [13].

It is worth mentioning that if $T_1 = T_2 = T : K_r \to K_r$ is a univariate nonlinear operator, and $x^* = y^* = u$, then problem (2.2) reduces to finding $u \in K_r$ such that

$$\langle Tu, v - u \rangle + \frac{1}{2r}\|v - u\|^2 \geq 0, \quad \forall v \in K_r, \tag{2.5}$$

which is more general than the normal nonconvex variational inequality, introduced and studied by Bounkhel et al. [3] and Noor [7,10], that is

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K_r. \tag{2.6}$$

It is well known that problem (2.6) is equivalent to finding $u \in K_r$ such that

$$0 \in Tu + N^P_{K_r}(u), \tag{2.7}$$

where $N^P_{K_r}(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis. Problem (2.7) is called the variational inclusion associated with nonconvex variational inequality (2.6), which implies that the nonconvex variational inequality is equivalent to finding a zero of the sum of two monotone operators. This equivalent formulation plays a crucial and basic part in this article, which allows us to use the projection operator technique for solving the general system of nonconvex variational inequalities (2.1).

We now recall the well-known properties of the uniform prox-regular sets [8-10,15].

Lemma 2.2. Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K_r is uniformly prox-regular, then

(i) $\forall u \in K_r, \ P_{K_r}(u) \neq 0$..

(ii) $\forall r' \in (0, r), \ P_{K_r}$ is Lipschitz continuous with constant $\delta = \frac{r}{r - r'}$ on $K_{r'}$.

(iii) The proximal normal cone is closed as a set-valued mapping.

Lemma 2.3 [18]. Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\sigma_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n/\gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Definition 2.4. An operator $T : H \to H$ is said to be r-strongly monotone, if there exists a constant $r > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in H.$$

Definition 2.5. An operator $T : H \to H$ is said to be μ-Lipschitz continuous, if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu \|x - y\|, \quad \forall x, y \in H.$$

Remark 2.2. As $T = I$, the identity operator is 1-strongly monotone and 1-Lipschitz continuous.

Remark 2.3. Obviously, whenever operator T is r-strongly monotone and μ-Lipschitz continuous, it follows that $\mu \geq r$.

3 Projection methods

In this section, we establish the equivalence between the new system of general nonconvex variational inequalities (2.1) and the fixed point problem with the projection technique. This alternative formulation enable us to suggest and analyze an explicit projection method for solving system (2.1).

Lemma 3.1. $x^*, y^* \in K_r$ is a solution of the system of general nonconvex variational inequalities (2.1), if and only if

$$x^* = P_{K_r}[g(y^*) - \rho T_1(y^*, x^*)], \tag{3.1a}$$

$$y^* = P_{K_r}[h(x^*) - \eta T_2(x^*, y^*)], \tag{3.1b}$$

where P_{K_r} is the projection of H onto the uniformly prox-regular set K_r.

Proof. Let $x^*, y^* \in K_r$ be a solution of (2.1). From (2.7), we have that the problem (2.1a) is equivalent to that

$$0 \in \rho T_1(y^*, x^*) + x^* - g(y^*) + N^P_{K_r}(x^*). \tag{3.2}$$

where $N^P_{K_r}(x^*)$ is proximal normal cone of K_r at x^* in the sense of nonconvex analysis. Indeed, if $\rho T_1(y^*, x^*) + x^* - g(y^*) = 0$, because the vector zero always belongs to any normal cone, then (3.2) is valid. If $\rho T_1(y^*, x^*) + x^* - g(y^*) \neq 0$, then for all $x \in H : g(x) \in K_r$, it follows from (2.1a) that

$$\langle -(\rho T_1(y^*, x^*) + x^* - g(y^*)), g(x) - x^*\rangle \leq \frac{1}{2r}\|g(x) - x^*\|^2.$$

By using Lemma 2.1, we obtain

$$-(\rho T_1(y^*, x^*) + x^* - g(y^*)) \in N^P_{K_r}(x^*)$$

and so (3.2) holds also. Consequently, the general nonconvex variational inequality (2.1a) is equivalent to (3.2), which is called variational inclusion associated with the problem (2.1a).

On the other hand, (3.2) can be written as

$$g(y^*) - \rho T_1(y^*, x^*) \in x^* + N^P_{K_r}(x^*) = (I + N^P_{K_r})x^*,$$

where I is identity operator. Moreover, we have

$$x^* = P_{K_r}[g(y^*) - \rho T_1(y^*, x^*)],$$

where we have used the well-known fact that $P_{K_r} = (I + \rho N^P_{K_r})^{-1}$. In a similar way, we can obtain (3.1b). This proves our assertions. □

Algorithm 3.1. For arbitrarily chosen initial points x_0, $y_0 \in K_r$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_{K_r}[g(y_n) - \rho T_1(y_n, x_n)], \quad \rho > 0, \tag{3.3a}$$

$$y_{n+1} = (1 - \beta_n)x_{n+1} + \beta_n P_{K_r}[h(x_{n+1}) - \eta T_2(x_{n+1}, y_n)], \quad \eta > 0, \tag{3.3b}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1].

If $\beta_n = 1$, $K_r = K$, then Algorithm 3.1 reduces to the following explicit projection method for solving the system of variational inequalities (2.3), which is mainly due to Noor and Noor [14]:

Algorithm 3.2. For arbitrarily chosen initial points x_0, $y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n, x_n)], \quad \rho > 0,$$
$$y_{n+1} = P_K[h(x_{n+1}) - \eta T_2(x_{n+1}, y_n)], \quad \eta > 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1].

If $g = h = I$, $K_r = K$, and T_1, $T_2 : K \to K$ are two univariate nonlinear operators, then Algorithm 3.1 reduces to the following explicit projection method for solving the system of variational inequalities (2.4):

Algorithm 3.3. For arbitrarily chosen initial points x_0, $y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K[y_n - \rho T_1(y_n)], \quad \rho > 0,$$
$$y_{n+1} = (1 - \beta_n)x_{n+1} + \beta_n P_K[x_{n+1} - \eta T_2(x_{n+1})], \quad \eta > 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1]. Algorithm 3.3 extends and improves the two-step projection methods of Verma [13].

If $g = h$, $T_1 = T_2 = T$ is the univariate nonlinear operator, we again use the fixed point formulation (3.1) to suggest and analyze the following explicit projection method, known as Mann iteration:

Algorithm 3.4. For arbitrarily chosen initial points $x_0 \in K_r$, compute the sequence $\{x_n\}$ in the following way:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_{K_r}[g(x_n) - \rho T x_n], \quad \rho > 0,$$

where $\{\alpha_n\}$ is a sequence in [0,1].

Remark 3.1. Algorithm 3.4 includes the projection methods of Noor [10] as special cases.

4 Main results

We now consider the convergence analysis of Algorithm 3.1, and this is the main motivation of our next result. In a similar way, we consider the convergence criteria of other algorithms.

Theorem 4.1. Let P_{K_r} be a Lipschitz continuous operator with constant $\delta = \dfrac{r}{r - r'}$. Let $T_i : K_r \times K_r \to K_r$ be r_i-strongly monotone and μ_i-Lipschitz continuous in the first variable, $i = 1, 2$, and $g, h : K_r \to K_r$ be strongly monotone with constants r_3, r_4 and Lipschitz continuous with constants μ_3, μ_4, respectively. If there exist constants ρ, $\eta > 0$ such that

$$\left|\rho - \frac{r_1}{\mu_1^2}\right| < \frac{\sqrt{\delta^2 r_1^2 - \mu_1^2[\delta^2 - (1 - \delta k_1)^2]}}{\delta \mu_1^2}, \quad \delta r_1 > \mu_1 \sqrt{\delta^2 - (1 - \delta k_1)^2},\ \delta k_1 < 1, \tag{4.1}$$

$$\left|\eta - \frac{r_2}{\mu_2^2}\right| < \frac{\sqrt{\delta^2 r_2^2 - \mu_2^2[\delta^2 - (1 - \delta k_2)^2]}}{\delta \mu_2^2}, \quad \delta r_2 > \mu_2 \sqrt{\delta^2 - (1 - \delta k_2)^2},\ \delta k_2 < 1, \tag{4.2}$$

where

$$k_1 = \sqrt{1 - 2r_3 + \mu_3^2}, \quad k_2 = \sqrt{1 - 2r_4 + \mu_4^2},$$

and $\alpha_n, \beta_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} (1 - \beta_n) < \infty$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.1 converges to a solution of the system of general non-convex variational inequalities (2.1), respectively.

Proof. Let $x^*, y^* \in K_r$ be a solution of (2.1). From (3.1a) and (3.3a) and the Lipschitz continuous property of operator P_{K_r}, we can obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n\{P_{K_r}[g(y_n) - \rho T_1(y_n, x_n)] - P_{K_r}[g(y^*) - \rho T_1(y^*, x^*)]\}\| \\
&\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|P_{K_r}[g(y_n) - \rho T_1(y_n, x_n)] - P_{K_r}[g(y^*) - \rho T_1(y^*, x^*)]\| \\
&\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \delta \|g(y_n) - g(y^*) - \rho[T_1(y_n, x_n) - T_1(y^*, x^*)]\| \\
&\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \delta \|g(y_n) - g(y^*) - (y_n - y^*)\| + \alpha_n \delta \|(y_n - y^*) - \\
&\quad \rho[T_1(y_n, x_n) - T_1(y^*, x^*)]\| .
\end{aligned} \tag{4.3}$$

Since the operator T_1 is r_1-strongly monotone and μ_1-Lipschitz continuous definition in the first variable, it follows that

$$\begin{aligned}
&\|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x^*)]\|^2 \\
&= \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n, x_n) - T_1(y^*, x^*), y_n - y^*\rangle + \rho^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
&\le \|y_n - y^*\|^2 - 2\rho r_1\|y_n - y^*\|^2 + \rho^2\|T_1(y_n, x_n) - T_1(y^*, x^*)\|^2 \\
&\le (1 - 2\rho r_1 + \rho^2\mu_1^2)\|y_n - y^*\|^2,
\end{aligned} \tag{4.4}$$

which implies that

$$\|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x^*)]\| \le \sqrt{1 - 2\rho r_1 + \rho^2\mu_1^2}\, \|y_n - y^*\| . \tag{4.5}$$

In a similar way, we have (note that $\mu_3 \ge r_3$, from Remark 2.3)

$$\|g(y_n) - g(y^*) - (y_n - y^*)\| \le \sqrt{1 - 2r_3 + \mu_3^2}\, \|y_n - y^*\| . \tag{4.6}$$

Consequently, from (4.3), (4.5), and (4.6), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \delta \left(k_1 + \sqrt{1 - 2\rho r_1 + \rho^2\mu_1^2}\right)\|y_n - y^*\| \\
&= (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \theta_1 \|y_n - y^*\| ,
\end{aligned} \tag{4.7}$$

where $\theta_1 = \delta(k_1 + \sqrt{1 - 2\rho r_1 + \rho^2\mu_1^2})$, $k_1 = \sqrt{1 - 2r_3 + \mu_3^2}$. From (4.1), we obtain that $\theta_1 \in (0, 1)$.

On the other hand, it follows from (3.1b) and (3.3b) that

$$\begin{aligned}\|y_{n+1} - y^*\| &\le (1-\beta_n)\|x_{n+1} - y^*\| + \beta_n \|P_{K_r}[h(x_{n+1}) - \eta T_2(x_{n+1}, y_n)] - P_{K_r}[h(x^*) - \eta T_2(x^*, y^*)]\| \\ &\le (1-\beta_n)\|x_{n+1} - y^*\| + \beta_n\delta \|h(x_{n+1}) - h(x^*) - \eta[T_2(x_{n+1}, y_n) - T_2(x^*, y^*)]\| \\ &\le (1-\beta_n)\|x_{n+1} - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n\delta\|h(x_{n+1}) - h(x^*) - (x_{n+1} - x^*)\| + \\ &\quad \beta_n\delta\|x_{n+1} - x^* - \eta[T_2(x_{n+1}, y_n) - T_2(x^*, y^*)]\|.\end{aligned} \tag{4.8}$$

Similarly, from the properties defined on T_2 in the first variable, we have

$$\begin{aligned}&\|x_{n+1} - x^* - \eta[T_2(x_{n+1}, y_n) - T_2(x^*, y^*)]\|^2 \\ &= \|x_{n+1} - x^*\|^2 - 2\eta\langle T_2(x_{n+1}, y_n) - T_2(x^*, y^*), x_{n+1} - x^*\rangle + \eta^2\|T_2(x_{n+1}, y_n) - T_2(x^*, y^*)\|^2 \\ &\le \|x_{n+1} - x^*\|^2 - 2\eta r_2\|x_{n+1} - x^*\|^2 + \eta^2\|T_2(x_{n+1}, y_n) - T_2(x^*, y^*)\|^2 \\ &\le (1 - 2\eta r_2 + \eta^2\mu_2^2)\|x_{n+1} - x^*\|^2,\end{aligned} \tag{4.9}$$

and

$$\|h(x_{n+1}) - h(x^*) - (x_{n+1} - x^*)\|^2 \le (1 - 2r_4 + \mu_4^2)\|x_{n+1} - x^*\|^2 \tag{4.10}$$

Moreover, from (4.8)-(4.10), we have

$$\begin{aligned}\|y_{n+1} - y^*\| &\le (1-\beta_n)\|x_{n+1} - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n\delta\left(k_2 + \sqrt{1 - 2\eta r_2 + \eta^2\mu_2^2}\right)\|x_{n+1} - x^*\| \\ &\le (1-\beta_n)\|x_{n+1} - x^*\| + (1-\beta_n)\|x^* - y^*\| + \beta_n\theta_2\|x_{n+1} - x^*\|,\end{aligned} \tag{4.11}$$

where $\theta_2 = \delta(k_2 + \sqrt{1 - 2\eta r_2 + \eta^2\mu_2^2})$, $k_2 = \sqrt{1 - 2r_4 + \mu_4^2}$. From (4.2), we obtain that $\theta_2 \in (0, 1)$. For all $n \ge 1$, it follows from (4.11) that

$$\begin{aligned}\|y_n - y^*\| &\le (1-\beta_{n-1})\|x_n - x^*\| + (1-\beta_{n-1})\|x^* - y^*\| + \beta_{n-1}\theta_2\|x_n - x^*\| \\ &= [1 - (1-\theta_2)\beta_{n-1}]\|x_n - x^*\| + (1-\beta_{n-1})\|x^* - y^*\|.\end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.7), we have (note that $\theta_1, \theta_2 \in (0, 1)$)

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\{[1 - (1-\theta_2)\beta_{n-1}]\|x_n - x^*\| + (1-\beta_{n-1})\|x^* - y^*\|\} \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|x_n - x^*\| + \alpha_n(1-\beta_{n-1})\theta_1\|x^* - y^*\| \\ &\le [1 - (1-\theta_1)\alpha_n]\|x_n - x^*\| + \alpha_n(1-\beta_{n-1})\|x^* - y^*\|.\end{aligned} \tag{4.13}$$

Since $1 - \theta_1 > 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\sum_{n=0}^{\infty}(1-\beta_n) < \infty$, we apply Lemma 2.3 to get

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{4.14}$$

Combining (4.12) and (4.14), we have

$$\lim_{n\to\infty}\|y_n - y^*\| = 0. \tag{4.15}$$

It follows that $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$, satisfying the general system of nonconvex variational inequalities (2.1). This completes the proof. □

Theorem 4.2. Let K be a nonempty and convex subset of Hilbert space H. Let $T_i : K \times K \to K$ be r_i-strongly monotone and μ_i-Lipschitz continuous in the first variable, $i = 1, 2$, and $g, h : K \to K$ be strongly monotone with constants r_3, r_4 and Lipschitz continuous with constants μ_3, μ_4, respectively. If there exist constants $\rho, \eta > 0$ such that

$$\left|\rho - \frac{r_1}{\mu_1^2}\right| < \frac{\sqrt{r_1^2 - \mu_1^2\left(2k_1 - k_1^2\right)}}{\mu_1^2}, \quad r_1 > \mu_1\sqrt{2k_1 - k_1^2}, k_1 < 1, \tag{4.16}$$

$$\left|\eta - \frac{r_2}{\mu_2^2}\right| < \frac{\sqrt{r_2^2 - \mu_2^2\left(2k_2 - k_2^2\right)}}{\mu_2^2}, \quad r_2 > \mu_2\sqrt{2k_2 - k_2^2}, k_2 < 1, \tag{4.17}$$

where

$$k_1 = \sqrt{1 - 2r_3 + \mu_3^2}, \quad k_2 = \sqrt{1 - 2r_4 + \mu_4^2},$$

and $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.2 converges to a solution of the system of general variational inequalities (2.3), respectively.

Proof. If $K_r = K$, $\beta_n = 1$, Algorithm 3.1 reduces to Algorithm 3.2. Moreover, we can obtain $r = \infty$ and $\delta = 1$ from Remark 2.1, and $\sum_{n=0}^{\infty}(1 - \beta_n) = 0$. Then the conclusion follows immediately from Theorem 4.1. This completes the proof. □

Theorem 4.3. Let K be a nonempty and convex subset of Hilbert space H, and $T_i : K \to K$ be r_i-strongly monotone and μ_i-Lipschitz continuous, $i = 1, 2$. If there exist constants ρ, η such that

$$0 < \rho < \frac{2r_1}{\mu_1^2}, \quad 0 < \eta < \frac{2r_2}{\mu_2^2}. \tag{4.18}$$

and $\alpha_n, \beta_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty}(1 - \beta_n) < \infty$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.3 converges to a solution of the system of variational inequalities (2.4), respectively.

Proof. If $g = h = I$, $K_r = K$, Algorithm 3.1 reduces to Algorithm 3.3. Moreover, we can obtain $r = \infty$ and $\delta = 1$ from Remark 2.1, and $k_1 = k_2 = 0$ from Remark 2.2 ($r_3 = \mu_3 = r_4 = \mu_4 = 1$). Then the conclusion follows immediately from Theorem 4.1. This completes the proof. □

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Authors' contributions
DJW carried out the primary studies for the new system of general nonconvex variational inequalities, participated in the design of projection methods and drafted the manuscript. XJL participated in the design of the study and performed the nonconvex analysis. QFG participated in the convergence analysis and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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