Kangtunyakarn *Fixed Point Theory and Applications* 2011, **2011**:23
http://www.fixedpointtheoryandapplications.com/content/2011/1/23

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Strong convergence theorem for a generalized equilibrium problem and system of variational inequalities problem and infinite family of strict pseudo-contractions

Atid Kangtunyakarn

Correspondence:
beawrock@hotmail.com
Department of Mathematics,
Faculty of Science, King Mongkut's
Institute of Technology
Ladkrabang, Bangkok 10520,
Thailand

Abstract

In this article, we introduce a new mapping generated by an infinite family of κ_i-strict pseudo-contractions and a sequence of positive real numbers. By using this mapping, we consider an iterative method for finding a common element of the set of a generalized equilibrium problem of the set of solution to a system of variational inequalities, and of the set of fixed points of an infinite family of strict pseudo-contractions. Strong convergence theorem of the purposed iteration is established in the framework of Hilbert spaces.

Keywords: nonexpansive mappings, strongly positive operator, generalized equilibrium problem, strict pseudo-contraction, fixed point

1 Introduction

Let C be a closed convex subset of a real Hilbert space H, and let $G : C \times C \to \mathbb{R}$ be a bifunction. We know that the equilibrium problem for a bifunction G is to find $x \in C$ such that

$$G(x, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(G)$. Given a mapping $T : C \to H$, let $G(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in$. Then, $z \in EP(G)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Let $A : C \to H$ be a nonlinear mapping. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. Now, we consider the following generalized equilibrium problem:

$$\text{Find } z \in C \text{ such that } G(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of such $z \in C$ is denoted by $EP(G, A)$, i.e.,

$$EP(G, A) = \{z \in C : G(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C.$$

In the case of $A \equiv 0$, $EP(G, A)$ is denoted by $EP(G)$. In the case of $G \equiv 0$, $EP(G, A)$ is also denoted by $VI(C, A)$. Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and economics reduce to find a solution of (1.3) (see, for instance, [1]-[3]).

A mapping A of C into H is called *inverse-strongly monotone* (see [4]), if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha||Ax - Ay||^2$$

for all $x, y \in C$.

A mapping T with domain $D(T)$ and range $R(T)$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y|| \tag{1.4}$$

for all $x, y \in D(T)$ and T is said to be *κ-strict pseudo-contration* if there exist $\kappa \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + \kappa||(I - T)x - (I - T)y||^2, \quad \forall x, y \in D(T). \tag{1.5}$$

We know that the class of κ-strict pseudo-contractions includes class of nonexpansive mappings. If $\kappa = 1$, then T is said to be *pseudo-contractive*. T is *strong pseudo-contraction* if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contractive. In a real Hilbert space H (1.5) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 - \frac{1-\kappa}{2}||(I - T)x - (I - T)y||^2 \quad \forall x, y \in D(T). \tag{1.6}$$

T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 \quad \forall x, y \in D(T).$$

Then, T is strongly pseudo-contractive, if there exists a positive constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in D(T).$$

The class of κ-strict pseudo-contractions fall into the one between classes of nonexpansive mappings and pseudo-contractions, and the class of strong pseudo-contractions is independent of the class of κ-strict pseudo-contractions.

We denote by $F(T)$ the set of fixed points of T. If $C \subset H$ is bounded, closed and convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty; for instance, see [5]. Recently, Tada and Takahashi [6] and Takahashi and Takahashi [7] considered iterative methods for finding an element of $EP(G) \cap F(T)$. Browder and Petryshyn [8] showed that if a κ-strict pseudo-contraction T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula:

$$x_{n+1} = \alpha x_n + (1 - \alpha)Tx_n, \tag{1.7}$$

where α is a constant such that $0 < \alpha < 1$, converges weakly to a fixed point of T. Marino and Xu [9] extended Browder and Petryshyn's above mentioned result by proving that the sequence $\{x_n\}$ generated by the following Manns algorithm [10]:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.8}$$

converges weakly to a fixed point of T provided the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies the conditions that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - \kappa)(1-\alpha_n) = \infty$.

Recently, in 2009, Qin et al. [11] introduced a general iterative method for finding a common element of $EP(F, T)$, $F(S)$, and $F(D)$. They defined $\{x_n\}$ as follows:

$$\begin{cases} x_1, \ u \in C, \\ F\langle u_n, \ y\rangle + \langle Tx_n, \ y - u_n\rangle + \dfrac{1}{r}\langle y - u_n, \ u_n - x_n\rangle, \quad \forall y \in C, \\ y_n = P_C(x_n - \eta Bx_n), \\ v_n = P_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(\mu_1 S_k x_n + \mu_2 u_n + \mu_3 v_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.9}$$

where the mapping $D : C \to C$ is defined by $D(x) = P_C(P_C(x - \eta Bx) - \lambda AP_C(x - \eta Bx))$, S_k is the mapping defined by $S_k x = kx + (1 - k)Sx$, $\forall x \in C$, $S : C \to C$ is a κ-strict pseudo-contraction, and A, $B : C \in H$ are a-inverse-strongly monotone mapping and b-inverse-strongly monotone mappings, respectively. Under suitable conditions, they proved strong convergence of $\{x_n\}$ defined by (1.9) to $z = P_{EP(F,\ T)\cap F(S)\ \cap F(D)}u$.

Let C be a nonempty convex subset of a real Hilbert space. Let T_i, $i = 1, 2, \ldots$ be mappings of C into itself. For each $j = 1, 2, \ldots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$, we define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned} U_{n,n+1} &= I \\ U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I \\ U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I \\ &\ \vdots \\ U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I \\ U_{n,k} &= \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I \\ &\ \vdots \\ U_{n,2} &= \alpha_1^2 T_2 U_{n,1} + \alpha_2^2 U_{n,1} + \alpha_3^2 I \\ S_n = U_{n,1} &= \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I. \end{aligned}$$

This mapping is called *S-mapping* generated by T_n, ..., T_1 and α_n, α_{n-1}, ..., α_1.

Question. How can we define an iterative method for finding an element in $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i, \ A_i) \bigcap \bigcap_{i=1}^{N} F(G_i)$?

In this article, motivated by Qin et al. [11], by using S-mapping, we introduce a new iteration method for finding a common element of the set of a generalized equilibrium problem of the set of solution to a system of variational inequalities, and of the set of fixed points of an infinite family of strict pseudo-contractions. Our iteration scheme is define as follows.

For u, $x_1 \in C$, let $\{x_n\}$ be generated by

$$\begin{cases} F_i\langle v_n^i, \ v\rangle + \langle A_i x_n, \ v - v_n^i\rangle + \dfrac{1}{r_i}\langle v - v_n^i, \ v_n^i - x_n\rangle, \quad \forall v \in C, \ i = 1, 2, \ldots, N. \\ y_n = \Sigma_{i=1}^N \delta_i v_n^i \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(a_n S_n x_n + b_n Bx_n + c_n y_n), \quad \forall n \in \mathbb{N}. \end{cases}$$

For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction, $A_i : C \to H$ be α_i-inverse strongly monotone and let $G_i : C \to C$ be defined by $G_i(y) = P_C(I - \lambda_i A_i)y$, $\forall y \in C$ with

$(0, 1] \subset (0, 2\alpha_i)$ such that $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i, A_i) \bigcap \bigcap_{i=1}^{N} F(G_i) \neq \emptyset$, where B is the K-mapping generated by $G_1, G_2, ..., G_N$ and $\beta_1, \beta_2, ..., \beta_N$.

We prove a strong convergence theorem of purposed iterative sequence $\{x_n\}$ to a point $z \in \mathbb{F}$ and z is a solution of (1.10)

$$\begin{cases} \langle x - z, A_1 z\rangle \geq 0 \\ \langle x - z, A_2 z\rangle \geq 0 \\ \vdots \\ \langle x - z, A_N z\rangle \geq 0, \quad \forall x \in C \text{ and } \lambda_i \in (0, 1]\ i = 1, 2, \ldots, N. \end{cases} \tag{1.10}$$

2 Preliminaries

In this section, we collect and provide some useful lemmas that will be used for our main result in the next section.

Let C be a closed convex subset of a real Hilbert space H, and let P_C be the metric projection of H onto C i.e., so that for $x \in H$, $P_C x$ satisfies the property:

$$|x - P_C x|| = \min_{y \in C} ||x - y||.$$

The following characterizes the projection P_C.

Lemma 2.1 [5]. *Given $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2 [12]. *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

$$(1)\ \{\alpha_n\} \subset [0, 1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2)\ \limsup_{n\to\infty} \beta_n \leq 0 \ or \ \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 [13]. *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\Sigma_{n=1}^{\infty} \lambda_n = 1$. Then, a mapping S on C defined by*

$$S(x) = \Sigma_{n=1}^{\infty} \lambda_n T_n x_n$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$hold.

Lemma 2.4 [14]. *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping. Then, I - S is demi-closed at zero.*

Lemma 2.5 [15]. *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in 0*[1]*with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

for all integer $n \geq 0$ *and*

$$\limsup_{n\to\infty} (||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Then $\lim_{n\to\infty} ||x_n - z_n|| = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0 \ \forall x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;
(A3) $\forall x, y, z \in C$,

$$\lim_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) $\forall x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.6 [1]. *Let C be a nonempty closed convex subset of H, and let F be a bifunction of C × C into* $\mathbb{R}$ *satisfying* (A1) - (A4). *Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \tag{2.1}$$

for all $x \in C$.

Lemma 2.7 [16]. *Assume that* $F : C \times C \to \mathbb{R}$ *satisfies* (A1) - (A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows.*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}.$$

for all $z \in H$. *Then, the following hold.*

(1) T_r *is single-valued,*

(2) T_r *is firmly nonexpansive i.e*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle \quad \forall x, y \in H;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Definition 2.1 [17]. *Let C be a nonempty convex subset of real Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpanxive mappings of C into itself, and let* $\lambda_1, ..., \lambda_N$ *be real numbers such that* $0 \leq \lambda_i \leq 1$ *for every* $i = 1, ..., N$. *We define a mapping* $K : C \to C$ *as follows.*

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\ K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}. \end{aligned} \tag{2.3}$$

Such a mapping K is called the *K-mapping* generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$.

Lemma 2.8 [17]. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpanxive mappings of C into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $\lambda_1, ..., \lambda_N$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, ..., N - 1$ *and* $0 < \lambda_N \leq 1$. *Let K be the K-mapping generated by* $T_1, ..., T_N$ *and* $\lambda_1, ..., \lambda_N$. *Then* $F(K) = \bigcap_{i=1}^N F(T_i)$.

Lemma 2.9 [9]. *Let C be a nonempty closed convex subset of a real Hilbert space H and* $S : C \to C$ *be a self-mapping of C. If S is a* κ*-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition.*

$$||Sx - Sy|| \leq \frac{1+\kappa}{1-\kappa}||x - y||, \quad \forall x, y \in C.$$

Lemma 2.10. *Let C be a nonempty closed convex subset of a real Hilbert space. Let* $\{T_i\}_{i=1}^N$ *be* κ_i*-strict pseudo-contraction mappings of C into itself with* $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ *and* $\kappa = \sup_i \kappa_i$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j \leq b < 1, \alpha_1^j + \alpha_2^j \leq b < 1$, *and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ *for all* $j = 1, 2, ...$ *For every* $n \in \mathbb{N}$, *let* S_n *be S-mapping generated by* $T_n, ..., T_1$ *and* $\alpha_n, \alpha_{n-1}, ..., \alpha_1$. *Then, for every* $x \in C$ *and* $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n,k}x$ *exists.*

Proof. Let $x \in C$ and $y \in \bigcap_{i=1}^\infty F(T_i)$. Fix $k \in \mathbb{N}$, then for every $n \in \mathbb{N}$ with $n \geq k$, we have

$$\begin{aligned}
\|U_{n+1,k}x - U_{n,k}x\|^2 &= \|\alpha_1^k T_k U_{n+1,k+1}x + \alpha_2^k U_{n+1,k+1}x + \alpha_3^k x - \alpha_1^k T_k U_{n,k+1}x \\
&\quad - \alpha_2^k U_{n,k+1}x - \alpha_3^k x\|^2 \\
&= \|\alpha_1^k (T_k U_{n+1,k+1}x - T_k U_{n,k+1}x) + \alpha_2^k (U_{n+1,k+1}x - U_{n,k+1}x)\|^2 \\
&\leq \alpha_1^k \|T_k U_{n+1,k+1}x - T_k U_{n,k+1}x\|^2 + \alpha_2^k \|U_{n+1,k+1}x - U_{n,k+1}x\|^2 \\
&\quad - \alpha_1^k \alpha_2^k \|T_k U_{n+1,k+1}x - T_k U_{n,k+1}x - U_{n+1,k+1}x + U_{n,k+1}x\|^2 \\
&\leq \alpha_1^k (\|U_{n+1,k+1}x - U_{n,k+1}x\|^2 + \kappa \|(I - T_k)U_{n+1,k+1}x \\
&\quad - (I - T_k)U_{n,k+1}x\|^2) + \alpha_2^k \|U_{n+1,k+1}x - U_{n,k+1}x\|^2 \\
&\quad - \alpha_1^k \alpha_2^k \|(I - T_k)U_{n,k+1}x - (I - T_k)U_{n+1,k+1}x\|^2 \\
&\leq (1 - \alpha_3^k)\|U_{n+1,k+1}x - U_{n,k+1}x\|^2 \\
&\ \ \vdots \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)\|U_{n+1,n+1}x - U_{n,n+1}x\|^2 \\
&= \Pi_{j=k}^n (1 - \alpha_3^j)\|\alpha_1^{n+1} T_{n+1} U_{n+1,n+2}x + \alpha_2^{n+1} U_{n+1,n+2}x + \alpha_3^{n+1}x - x\|^2 \\
&= \Pi_{j=k}^n (1 - \alpha_3^j)\|\alpha_1^{n+1} T_{n+1}x + (1 - \alpha_1^{n+1})x - x\|^2 \\
&= \Pi_{j=k}^n (1 - \alpha_3^j)\|\alpha_1^{n+1}(T_{n+1}x - x)\|^2 \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)(\|T_{n+1}x - y\| + \|y - x\|)^2 \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)\left(\frac{1+\kappa}{1-\kappa}\|x - y\| + \|y - x\|\right)^2 \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)\left(\frac{2}{1-\kappa}\|x - y\|\right)^2 \\
&\leq b^{n-(k-1)}\left(\frac{2}{1-\kappa}\|x - y\|\right)^2.
\end{aligned}$$

It follows that

$$||U_{n+1,k}x - U_{n,k}x|| \leq b^{\frac{n-(k-1)}{2}}\left(\frac{2}{1-\kappa}||x - y||\right)$$

$$= \frac{b^{\frac{n}{2}}}{b^{\frac{k-1}{2}}}\left(\frac{2}{1-\kappa}||x-y||\right) \\ = \frac{a^n}{a^{k-1}}M, \tag{2.4}$$

where $a = b^{\frac{1}{2}} \in (0, 1)$ and $M = \frac{2}{1-\kappa}||x-y||$

For any k, n, $p \in \mathbb{N}$, $p > 0$, $n \geq k$, we have

$$\begin{aligned} \|U_{n+p,k}x - U_{n,k}x\| &\leq \|U_{n+p,k}x - U_{n+p-1,k}x\| + \|U_{n+p-1,k}x - U_{n+p-2,k}x\| + \ . \ . \ . \\ &\quad +\|U_{n+1,k}x - U_{n,k}x\| \\ &= \Sigma_{j=n}^{n+p-1}\|U_{j+1,k}x - U_{j,k}x\| \\ &\leq \Sigma_{j=n}^{n+p-1}\frac{a^j}{a^{k-1}}M \\ &\leq \frac{a^n}{(1-a)a^{k-1}}M. \end{aligned} \tag{2.5}$$

Since $a \in (0, 1)$, we have $\lim_{n\to\infty} a^n = 0$. From (2.5), we have that $\{U_{n,k}x\}$ is a Cauchy sequence. Hence $\lim_{n\to\infty} U_{n,k}x$ exists. □

For every $k \in \mathbb{N}$ and $x \in C$, we define mapping $U_{\infty,k}$ and $S : C \in C$ as follows:

$$\lim_{n\to\infty} U_{n,k}x = U_{\infty,k}x \tag{2.6}$$

and

$$\lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x = Sx \tag{2.7}$$

Such a mapping S is called S-mapping generated by T_n, T_{n-1}, ... and α_n, α_{n-1}, ...

Remark 2.11. For each $n \in \mathbb{N}$, S_n is nonexpansive and $\lim_{n\to\infty} sup_{x\in D}||S_n x - Sx|| = 0$ for every bounded subset D of C. To show this, let x, $y \in C$ and D be a bounded subset of C. Then, we have

$$\begin{aligned} \|S_n x - S_n y\|^2 &= \|\alpha_1^1(T_1U_{n,2}x - T_1U_{n,2}y) + \alpha_2^1(U_{n,2}x - U_{n,2}y) + \alpha_3^1(x-y)\|^2 \\ &\leq \alpha_1^1\|T_1U_{n,2}x - T_1U_{n,2}y\|^2 + \alpha_2^1\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2 \\ &\quad -\alpha_1^1\alpha_2^1\|T_1U_{n,2}x - T_1U_{n,2}y - U_{n,2}x + U_{n,2}y\|^2 \\ &\leq \alpha_1^1(\|U_{n,2}x - U_{n,2}y\|^2 + \kappa\|(I-T_1)U_{n,2}x - (I-T_1)U_{n,2}y\|^2) \\ &\quad +\alpha_2^1\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2 - \alpha_1^1\alpha_2^1\|(I-T_1)U_{n,2}y - (I-T_1)U_{n,2}x\|^2 \\ &\leq (1-\alpha_3^1)\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2 \\ &\leq (1-\alpha_3^1)((1-\alpha_3^2)\|U_{n,3}x - U_{n,3}y\|^2 + \alpha_3^2\|x-y\|^2) + \alpha_3^1\|x-y)\|^2 \\ &= (1-\alpha_3^1)(1-\alpha_3^2)\|U_{n,3}x - U_{n,3}y\|^2 + \alpha_3^2(1-\alpha_3^1)\|x-y\|^2 + \alpha_3^1\|x-y)\|^2 \end{aligned}$$

$$\begin{aligned} &= \Pi_{j=1}^2(1-\alpha_3^j)||U_{n,3}x - U_{n,3}y||^2 + (1 - \Pi_{j=1}^2(1-\alpha_3^j))||x-y||^2 \\ &\vdots \\ &\leq \Pi_{j=1}^n(1-\alpha_3^j)||U_{n,n+1}x - U_{n,n+1}y||^2 + (1 - \Pi_{j=1}^n(1-\alpha_3^j))||x-y||^2 \\ &= ||x-y||^2. \end{aligned}$$

Then, we have that $S : C \to C$ is also nonexpansive indeed, observe that for each $x, y \in C$

$$|Sx - Sy|| = \lim_{n\to\infty}||S_n x - S_n y|| \leq ||x-y||.$$

By (2.8), we have

$$\begin{aligned} ||S_{n+1}x - S_n x|| &= ||U_{n+1,1}x - U_{n,1}x|| \\ &\leq a^n M. \end{aligned}$$

This implies that for $m > n$ and $x \in D$,

$$\begin{aligned}||S_m x - S_n x|| &\le \Sigma_{j=n}^{m-1} ||S_{j+1}x - S_j x|| \\ &\le \Sigma_{j=n}^{m-1} a^j M \\ &\le \frac{a^n}{1-a} M.\end{aligned}$$

By letting $m \to \infty$, for any $x \in D$, we have

$$||Sx - S_n x|| \le \frac{a^n}{1-a} M. \tag{2.8}$$

It follows that

$$\lim_{n\to\infty} sup_{x\in D} ||S_n x - Sx|| = 0. \tag{2.9}$$

Lemma 2.12. *Let C be a nonempty closed convex subset of a real Hilbert space. Let* $\{T_i\}_{i=1}^{\infty}$*be* κ_i*-strict pseudo-contraction mappings of C into itself with* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$*and* $\kappa = \sup_{i\in} \kappa_i$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$*and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$*for all* $j = 1, \ldots$. *For every* $n \in \mathbb{N}$, *let* S_n *and S be S-mappings generated by* $T_n, \ldots, T_1$ *and* $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ *and* $T_n, T_{n-1}, \ldots,$ *and* $\alpha_n, \alpha_{n-1}, \ldots,$ *respectively. Then* $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.

Proof. It is evident that $\bigcap_{i=1}^{\infty} F(T_i) \subseteq F(S)$. For every $n, k \in \mathbb{N}$, *with* $n \ge k$, let $x_0 \in F(S)$ and $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, we have

$$\begin{aligned}
\|S_n x_0 - x^*\|^2 &= \|\alpha_1^1(T_1U_{n,2}x_0 - x^*) + \alpha_2^1(U_{n,2}x_0 - x^*) + \alpha_3^1(x_0 - x^*)\|^2 \\
&\le \alpha_1^1\|T_1U_{n,2}x_0 - x^*\|^2 + \alpha_2^1\|U_{n,2}x_0 - x^*\|^2 + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^1\alpha_2^1\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\le \alpha_1^1(\|U_{n,2}x_0 - x^*\|^2 + \kappa\|(I - T_1)U_{n,2}x_0\|^2) + \alpha_2^1\|U_{n,2}x_0 - x^*\|^2 \\
&\quad +\alpha_3^1\|x_0 - x^*\|^2 - \alpha_1^1\alpha_2^1\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= (1 - \alpha_3^1)\|U_{n,2}x_0 - x^*\|^2 + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\le (1 - \alpha_3^1)((1 - \alpha_3^2)\|U_{n,3}x_0 - x^*\|^2 + \alpha_3^2\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2\|U_{n,3}x_0 - x_0\|^2) + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= (1 - \alpha_3^1)(1 - \alpha_3^2)\|U_{n,3}x_0 - x^*\|^2 + \alpha_3^2(1 - \alpha_3^1)\|x_0 - x^*\|^2 + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= \Pi_{j=1}^2(1 - \alpha_3^j)\|U_{n,3}x_0 - x^*\|^2 + (1 - \Pi_{j=1}^2(1 - \alpha_3^j))\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\le \Pi_{j=1}^2(1 - \alpha_3^j)((1 - \alpha_3^3)\|U_{n,4}x_0 - x^*\|^2 + \alpha_3^3\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^3(\alpha_2^3 - \kappa)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 - \alpha_2^3\alpha_3^3\|U_{n,4}x_0 - x_0\|^2) \\
&\quad +(1 - \Pi_{j=1}^2(1 - \alpha_3^j))\|x_0 - x^*\|^2 - \alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 \\
&\quad -\alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 - \alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 \\
&\quad -\alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= \Pi_{j=1}^2(1 - \alpha_3^j)(1 - \alpha_3^3)\|U_{n,4}x_0 - x^*\|^2 + \alpha_3^3\Pi_{j=1}^2(1 - \alpha_3^j)\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^3(\alpha_2^3 - \kappa)\Pi_{j=1}^2(1 - \alpha_3^j)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 \\
&\quad -\alpha_2^3\alpha_3^3\Pi_{j=1}^2(1 - \alpha_3^j)\|U_{n,4}x_0 - x_0\|^2 + (1 - \Pi_{j=1}^2(1 - \alpha_3^j))\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= \Pi_{j=1}^3(1 - \alpha_3^j)\|U_{n,4}x_0 - x^*\|^2 + (1 - \Pi_{j=1}^3(1 - \alpha_3^j))\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^3(\alpha_2^3 - \kappa)\Pi_{j=1}^2(1 - \alpha_3^j)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 \\
&\quad -\alpha_2^3\alpha_3^3\Pi_{j=1}^2(1 - \alpha_3^j)\|U_{n,4}x_0 - x_0\|^2 - \alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 \\
&\quad -\alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 - \alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 \\
&\quad -\alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\;\;\vdots
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\vdots \\
\leq\ & \Pi_{j=1}^{k+1}(1-\alpha_3^j)||U_{n,k+2}x_0 - x^*||^2 + (1-\Pi_{j=1}^{k+1}(1-\alpha_3^j)||x_0 - x^*||^2 \\
& - \alpha_1^{k+1}(\alpha_2^{k+1} - \kappa)\Pi_{j=1}^{k}(1-\alpha_3^j)||T_{k+1}U_{n,k+2}x_0 - U_{n,k+2}x_0||^2 \\
& - \alpha_2^{k+1}\alpha_3^{k+1}\Pi_{j=1}^{k}(1-\alpha_3^j)||U_{n,k+2}x_0 - x_0||^2 \\
& - \alpha_1^{k}(\alpha_2^{k} - \kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)||T_kU_{n,k+1}x_0 - U_{n,k+1}x_0||^2 \\
& - \alpha_2^{k}\alpha_3^{k}\Pi_{j=1}^{k-1}(1-\alpha_3^j)||U_{n,k+1}x_0 - x_0||^2
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\vdots \\
&-\alpha_1^3(\alpha_2^3-\kappa)\Pi_{j=1}^2(1-\alpha_3^j)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 - \alpha_2^3\alpha_3^3\Pi_{j=1}^2(1-\alpha_3^j)\|U_{n,4}x_0 - x_0\|^2 \\
&-\alpha_1^2(\alpha_2^2-\kappa)(1-\alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1-\alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&-\alpha_1^1(\alpha_2^1-\kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\vdots \\
&\leq \Pi_{j=1}^{n}(1-\alpha_3^j)\|U_{n,n+1}x_0 - x^*\|^2 + (1-\Pi_{j=1}^{n}(1-\alpha_3^j)\|x_0 - x^*\|^2 \\
&-\alpha_1^n(\alpha_2^n-\kappa)\Pi_{j=1}^{n-1}(1-\alpha_3^j)\|T_nU_{n,n+1}x_0 - U_{n,n+1}x_0\|^2 \\
&-\alpha_2^n\alpha_3^n\Pi_{j=1}^{n-1}(1-\alpha_3^j)\|U_{n,n+1}x_0 - x_0\|^2 \\
&\vdots \\
&-\alpha_1^{k+1}(\alpha_2^{k+1}-\kappa)\Pi_{j=1}^{k}(1-\alpha_3^j)\|T_{k+1}U_{n,k+2}x_0 - U_{n,k+2}x_0\|^2 \\
&-\alpha_2^{k+1}\alpha_3^{k+1}\Pi_{j=1}^{k}(1-\alpha_3^j)\|U_{n,k+2}x_0 - x_0\|^2 \\
&-\alpha_1^{k}(\alpha_2^{k}-\kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|T_kU_{n,k+1}x_0 - U_{n,k+1}x_0\|^2 \\
&-\alpha_2^{k}\alpha_3^{k}\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|U_{n,k+1}x_0 - x_0\|^2 \\
&-\alpha_1^{k-1}(\alpha_2^{k-1}-\kappa)\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|T_{k-1}U_{n,k}x_0 - U_{n,k}x_0\|^2 \\
&-\alpha_2^{k-1}\alpha_3^{k-1}\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|U_{n,k}x_0 - x_0\|^2 \\
&\vdots \\
&-\alpha_1^3(\alpha_2^3-\kappa)\Pi_{j=1}^2(1-\alpha_3^j)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 - \alpha_2^3\alpha_3^3\Pi_{j=1}^2(1-\alpha_3^j)\|U_{n,4}x_0 - x_0\|^2 \\
&-\alpha_1^2(\alpha_2^2-\kappa)(1-\alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1-\alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&-\alpha_1^1(\alpha_2^1-\kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&= \|x_0 - x^*\|^2 \\
&-\alpha_1^n(\alpha_2^n-\kappa)\Pi_{j=1}^{n-1}(1-\alpha_3^j)\|T_nU_{n,n+1}x_0 - U_{n,n+1}x_0\|^2 \\
&\vdots \\
&-\alpha_1^{k+1}(\alpha_2^{k+1}-\kappa)\Pi_{j=1}^{k}(1-\alpha_3^j)\|T_{k+1}U_{n,k+2}x_0 - U_{n,k+2}x_0\|^2 \\
&-\alpha_2^{k+1}\alpha_3^{k+1}\Pi_{j=1}^{k}(1-\alpha_3^j)\|U_{n,k+2}x_0 - x_0\|^2 \\
&-\alpha_1^{k}(\alpha_2^{k}-\kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|T_kU_{n,k+1}x_0 - U_{n,k+1}x_0\|^2 \\
&-\alpha_2^{k}\alpha_3^{k}\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|U_{n,k+1}x_0 - x_0\|^2 \\
&-\alpha_1^{k-1}(\alpha_2^{k-1}-\kappa)\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|T_{k-1}U_{n,k}x_0 - U_{n,k}x_0\|^2 \\
&-\alpha_2^{k-1}\alpha_3^{k-1}\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|U_{n,k}x_0 - x_0\|^2 \\
&\vdots \\
&-\alpha_1^3(\alpha_2^3-\kappa)\Pi_{j=1}^2(1-\alpha_3^j)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 - \alpha_2^3\alpha_3^3\Pi_{j=1}^2(1-\alpha_3^j)\|U_{n,4}x_0 - x_0\|^2 \\
&-\alpha_1^2(\alpha_2^2-\kappa)(1-\alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1-\alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&-\alpha_1^1(\alpha_2^1-\kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2.
\end{aligned} \tag{2.13}$$

For $k \in \mathbb{N}$ and (2.12), we have

$$\alpha_2^{k-1}\alpha_3^{k-1}\Pi_{j=1}^{k-2}(1-\alpha_3^j)||U_{n,k}x_0 - x_0||^2 \leq ||x_0 - x^*||^2 - ||S_n x_0 - x^*||^2, \tag{2.14}$$

as $n \to \infty$. This implies that $U_{\infty,k}x_0 = x_0$, $\forall k \in \mathbb{N}$.

Again by (2.12), we have

$$\alpha_1^k(\alpha_2^k - \kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)||T_k U_{n,k+1}x_0 - U_{n,k+1}x_0||^2 \leq ||x_0 - x^*||^2 - ||S_n x_0 - x^*||^2, \tag{2.15}$$

as $n \to \infty$. Hence

$$\alpha_1^k(\alpha_2^k - \kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)||T_k U_{\infty,k+1}x_0 - U_{\infty,k+1}x_0||^2 \leq 0. \tag{2.16}$$

From $U_{\infty,k}x_0 = x_0$, $\forall k \in \mathbb{N}$, and (2.15), we obtain that $T_k x_0 = x_0$, $\forall k \in \mathbb{N}$. This implies that $x_0 \in \bigcap_{i=1}^{\infty} F(T_i)$. □

Lemma 2.13. *Let C be a closed convex subset of Hilbert space H. Let $A_i : C \to H$ be mappings and let $G_i : C \to C$ be defined by $G_i(y) = P_C(I - \lambda_i A_i)y$ with $\lambda_i > 0$, $\forall_i = 1, 2, \ldots N$. Then $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$ if and only if $x^* \in \bigcap_{i=1}^{N} F(G_i)$.*

Proof. For given $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$, we have $x^* \in VI(C, A_i)$, $\forall_i = 1, 2, \ldots, N$. Since $\langle A_i x^*, x - x^*\rangle \geq 0$, we have $\langle \lambda_i A_i x^*, x - x^*\rangle \geq 0$, $\forall \lambda_i > 0$, $i = 1, 2, \ldots, N$. It follows that

$$\langle x^* - (I - \lambda_i A_i)x^*, x - x^*\rangle = \langle \lambda_i A_i x^*, x - x^*\rangle \geq 0, \quad \forall x \in C, i = 1, 2, \ldots, N. \tag{2.17}$$

Hence, $x^* = P_C(I - \lambda_i A_i)x^* = G_i(x^*)$, $\forall x \in C$, $i = 1, 2, \ldots, N$. Therefore, we have $x^* \in \bigcap_{i=1}^{N} F(G_i)$. For the converse, let $x^* \in \bigcap_{i=1}^{N} F(G_i)$; then, we have for every $i = 1, \ldots, N$, $x^* = G_i(x^*) = P_C(I - \lambda_i A_i)x^*$, $\forall \lambda_i > 0$, $i = 1, 2, \ldots, N$. It implies that

$$\langle x^* - (I - \lambda_i A_i)x^*, x - x^*\rangle = \langle \lambda_i A_i x^*, x - x^*\rangle \geq 0, \quad \forall i = 1, 2, \ldots, N, \quad \forall x \in C. \tag{2.18}$$

Hence, $\langle A_i x^*, x - x^*\rangle \geq 0$, $\forall x \in C$, so $x^* \in VI(C, A_i)$, $\forall i = 1, 2, \ldots, N$. Hence, $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$.

□

3 Main results

Theorem 3.1. *Let C be a closed convex subset of Hilbert space H. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1) - (A_4), let $A_i : C \to H$ be α_i-inverse strongly monotone and let $G_i : C \to C$ be defined by $G_i(y) = P_C(I - \lambda_i A_i)y$, $\forall y \in C$ with $\lambda_i \in (0, 1] \subset (0, 2\alpha_i)$. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \ldots, G_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ where $\beta_i \in (0, 1)$, $\forall i = 1, 2, 3, \ldots, N - 1$, $\beta_N \in (0, 1]$ and let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strict pseudo-contraction mappings of C into itself with $\kappa = \sup_i \kappa_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S are S-mapping generated by $T_n, \ldots, T_1$ and $\rho_n, \rho_{n-1}, \ldots, \rho_1$ and $T_n, T_{n-1}, \ldots,$ and $\rho_n, \rho_{n-1}, \ldots,$ respectively. Assume that $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i, A_i) \bigcap \bigcap_{i=1}^{N} F(G_i) \neq \emptyset$. For every $n \in \mathbb{N}$, $i = 1, 2, \ldots, N$, let $\{x_n\}$ and $\{v_n^i\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} F_i\langle v_n^i, v\rangle + \langle A_i x_n, v - v_n^i\rangle + \dfrac{1}{r_i}\langle v - v_n^i, v_n^i - x_n\rangle \geq 0, \quad \forall v \in C, \\ i = 1, 2, \ldots, N. \\ y_n = \Sigma_{i=1}^{N}\delta_i v_n^i \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(a_n S_n x_n + b_n B x_n + c_n y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\} \subset (0, 1)$, $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = \Sigma_{i=1}^N \delta_i = 1$, *and* $\{r_i\}_{i=1}^N \subset (\varsigma, \tau) \subset (0, 2\alpha_i)$, *satisfy the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=0}^\infty \alpha_n = \infty$,
(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(*iii*) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$, *with* $a, b, c \in (0, 1)$.

Then, the sequence $\{x_n\}$, $\{y_n\}$, $\{v_n^i\}$, $\forall i = 1, 2, ..., N$, *converge strongly to* $z = P_{\Im}u$ *and z is a solution of (1.10).*

Proof. First, we show that $(I - \lambda_i A_i)$ is nonexpansive mapping for every $i = 1, 2, ..., N$. For $x, y \in C$, we have

$$\begin{aligned} \left\|(I-\lambda_iA_i)x-(I-\lambda_iA_i)y\right\|^2 &= ||x-y-\lambda_i(A_ix-A_iy)||^2 \\ &= ||x-y||^2 - 2\lambda_i\langle x-y, A_ix-A_iy\rangle + \lambda_i^2||A_ix-A_iy||^2 \\ &\le ||x-y||^2 - 2\alpha_i\lambda_i||A_ix-A_iy||^2 + \lambda_i^2||A_ix-A_iy||^2 \\ &= ||x-y||^2 + \lambda_i(\lambda_i-2\alpha_i)||A_ix-A_iy||^2 \\ &\le ||x-y||^2. \end{aligned} \tag{3.2}$$

Thus, $(I - \lambda_i A_i)$ is nonexpansive, and so are B and G_i, for all $i = 1, 2, ..., N$.

Now, we shall divide our proof into five steps.

Step 1. We shall show that the sequence $\{x_n\}$ is bounded. Since

$$F(v_n^i, v) + \langle A_ix_n, v-v_n^i\rangle + \frac{1}{r_i}\langle v-v_n^i, v_n^i-x_n\rangle \ge 0, \qquad \forall v \in C,\ i = 1, 2, \dots, N, \tag{3.3}$$

we have

$$F(v_n^i, v) + \frac{1}{r_i}\langle v-v_n^i, v_n^i-(I-r_iA_i)x_n\rangle \ge 0, \qquad \forall v \in C,\ i = 1, 2, \dots, N.$$

By Lemma 2.7, we have $v_n^i = T_{r_i}(I-r_iA_i)x_n$.

Let $z = \Im$. Then $F(z, y) + \langle y - z, A_iz\rangle \ge 0$ $\forall y \in C$, so we have

$$F(z, y) + \frac{1}{r_i}\langle y-z, z-z+r_iA_iz\rangle \ge 0, \quad \forall i = 1, 2, \dots, N.$$

Again by Lemma 2.7, we have $z = T_{r_i}(I-r_iA_i)z$, $\forall i = 1, 2, ..., N$. Since B is K-mapping generated by $G_1, G_2, ..., G_N$ and $\beta_1, \beta_2, ..., \beta_N$ and $\bigcap_{i=1}^N F(G_i) \neq \emptyset$. By Lemma 2.8, we have $\bigcap_{i=1}^N F(G_i) = F(B)$. Since $z = \Im$, we have $z \in F(B)$. Setting $e_n = a_nS_nx_n + b_nBx_n + c_ny_n$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{n+1}-z\| &= ||\alpha_n(u-z)+\beta_n(x_n-z)+\gamma_n(e_n-z)|| \\ &\le \alpha_n||u-z||+\beta_n||x_n-z||+\gamma_n||e_n-z|| \\ &= \alpha_n||u-z||+\beta_n||x_n-z||+\gamma_n||a_n(S_nx_n-z)+b_n(Bx_n-z)+c_n(y_n-z)|| \\ &\le \alpha_n||u-z||+\beta_n||x_n-z||+\gamma_n((1-c_n)||x_n-z||+c_n||y_n-z||) \\ &= \alpha_n||u-z||+\beta_n||x_n-z||+\gamma_n((1-c_n)||x_n-z||+c_n||\Sigma_{i=1}^N\delta_i(v_n^i-z)||) \\ &\le \alpha_n||u-z||+\beta_n||x_n-z||+\gamma_n((1-c_n)||x_n-z||+ c_n\Sigma_{i=1}^N\delta_i||v_n^i-z||) \\ &\le \alpha_n||u-z||+\beta_n||x_n-z||+\gamma_n((1-c_n)||x_n-z||+c_n||x_n-z||) \\ &= \alpha_n||u-z||+(1-\alpha_n)||x_n-z||, \\ &\le \max\{||u-z||,\ ||x_n-z||\}. \end{aligned} \tag{3.4}$$

By induction, we can prove that $\{x_n\}$ is bounded, and so are $\{v_n^i\}$, $\{y_n\}$, $\{Bx_n\}$ $\{S_n x_n\}$, $\{e_n\}$.

Step 2. We will show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Let $d_n = \dfrac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, and then we have

$$x_{n+1} = (1-\beta_n)d_n + \beta_n x_n, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

From definition of d_n, we have

$$\begin{aligned} \|d_{n+1} - d_n\| &= ||\frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}|| \\ &= ||\frac{\alpha_{n+1}u + \gamma_{n+1}e_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n e_n}{1-\beta_n}|| \\ &= ||\frac{\alpha_{n+1}u + (1-\beta_{n+1}-\alpha_{n+1})e_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\beta_n - \alpha_n)e_n}{1-\beta_n}|| \\ &= ||\frac{\alpha_{n+1}}{1-\beta_{n+1}}(u - e_{n+1}) - \frac{\alpha_n}{1-\beta_n}(u-e_n) + e_{n+1} - e_n|| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}||u - e_{n+1}|| + \frac{\alpha_n}{1-\beta_n}||u-e_n|| \\ &\quad +||e_{n+1} - e_n||. \end{aligned} \tag{3.6}$$

By definition of e_n, we have

$$\begin{aligned} \|e_{n+1} - e_n\| &= \|a_{n+1}S_{n+1}x_{n+1} + b_{n+1}Bx_{n+1} + c_{n+1}y_{n+1} - a_nS_nx_n - b_nBx_n - c_ny_n\| \\ &= \|a_{n+1}S_{n+1}x_{n+1} - a_nS_{n+1}x_{n+1} + a_nS_{n+1}x_{n+1} + b_{n+1}Bx_{n+1} - b_nBx_{n+1} \\ &\quad + b_nBx_{n+1} + c_{n+1}y_{n+1} - c_ny_{n+1} + c_ny_{n+1} - a_nS_nx_n - b_nBx_n - c_ny_n\| \\ &= \|(a_{n+1} - a_n)S_{n+1}x_{n+1} + a_n(S_{n+1}x_{n+1} - S_nx_n) + (b_{n+1} - b_n)Bx_{n+1} \\ &\quad + b_n(Bx_{n+1} - Bx_n) + (c_{n+1} - c_n)y_{n+1} + c_n(y_{n+1} - y_n)\| \\ &\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n\|S_{n+1}x_{n+1} - S_nx_n\| + |b_{n+1} - b_n|\|Bx_{n+1}\| \\ &\quad + b_n\|Bx_{n+1} - Bx_n\| + |c_{n+1} - c_n|\|y_{n+1}\| + c_n\|y_{n+1} - y_n\| \\ &\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n(\|S_{n+1}x_{n+1} - S_{n+1}x_n\| + \|S_{n+1}x_n - S_nx_n\|) \\ &\quad + |b_{n+1} - b_n|\|Bx_{n+1}\| + b_n\|Bx_{n+1} - Bx_n\| + |c_{n+1} - c_n|\|y_{n+1}\| \\ &\quad + c_n\Sigma_{i=1}^N \delta_i \|T_{r_i}(I - r_iA_i)x_{n+1} - T_{r_i}(I - r_iA_i)x_n\| \\ &\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n(\|x_{n+1} - x_n\| + \|S_{n+1}x_n - S_nx_n\|) \\ &\quad + |b_{n+1} - b_n|\|Bx_{n+1}\| + b_n\|x_{n+1} - x_n\| + |c_{n+1} - c_n|\|y_{n+1}\| \\ &\quad + c_n\|x_{n+1} - x_n\| \\ &\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n\|x_{n+1} - x_n\| + \|S_{n+1}x_n - S_nx_n\| \\ &\quad + |b_{n+1} - b_n|\|Bx_{n+1}\| + b_n\|x_{n+1} - x_n\| + |c_{n+1} - c_n|\|y_{n+1}\| \\ &\quad + c_n\|x_{n+1} - x_n\| \\ &= \|x_{n+1} - x_n\| + |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + \|S_{n+1}x_n - S_nx_n\| \\ &\quad + |b_{n+1} - b_n|\|Bx_{n+1}\| + |c_{n+1} - c_n|\|y_{n+1}\|. \end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we have

$$\begin{aligned} ||d_{n+1} - d_n|| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}||u - e_{n+1}|| + \frac{\alpha_n}{1-\beta_n}||u - e_n|| \\ &\quad + ||e_{n+1} - e_n|| \end{aligned} \tag{3.8}$$

$$\begin{aligned}\leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}||u-e_{n+1}|| + \frac{\alpha_n}{1-\beta_n}||u-e_n|| \\ & + ||x_{n+1}-x_n|| + |a_{n+1}-a_n|\ ||S_{n+1}x_{n+1}|| + ||S_{n+1}x_n - S_nx_n|| \\ & + |b_{n+1}-b_n|\ ||Bx_{n+1}|| + |c_{n+1}-c_n|\ ||y_{n+1}||.\end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}\|d_{n+1}-d_n\| - \|x_{n+1}-x_n\| \leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-e_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-e_n\| \\ & +|a_{n+1}-a_n|\|S_{n+1}x_{n+1}\| + \|S_{n+1}x_n - S_nx_n\| \\ & +|b_{n+1}-b_n|\|Bx_{n+1}\| + |c_{n+1}-c_n|\|y_{n+1}\| \\ \leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-e_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-e_n\|\end{aligned} \tag{3.10}$$

$$\begin{aligned}& +|a_{n+1}-a_n|\ ||S_{n+1}x_{n+1}|| + ||S_{n+1}x_n - Sx_n|| + ||Sx_n - S_nx_n|| \\ & +|b_{n+1}-b_n|\ ||Bx_{n+1}|| + |c_{n+1}-c_n|\ ||y_{n+1}||.\end{aligned} \tag{3.11}$$

From Remark 2.11 and conditions (i)-(iii), we have

$$\limsup_{n\to\infty}(||d_{n+1}-d_n|| - ||x_{n+1}-x_n||) \leq 0. \tag{3.12}$$

From (3.5), (3.12) and Lemma 2.5, we have

$$\lim_{n\to\infty} ||d_n - x_n|| = 0. \tag{3.13}$$

We can rewrite (3.5) as

$$x_{n+1} - x_n = (1-\beta_n)(d_n - x_n), \quad \forall n \in \mathbb{N}. \tag{3.14}$$

By (3.13) and (3.14), we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.15}$$

Step. 3. Show that $\lim_{n\to\infty} ||x_n - e_n|| = 0$. From (3.1), we have

$$x_{n+1} - x_n + \alpha_n(x_n - u) = \gamma_n(e_n - x_n).$$

It implies that

$$\gamma_n||e_n - x_n|| \leq ||x_{n+1} - x_n|| + \alpha_n||x_n - u||.$$

By conditions (i), (ii), and (3.15), we have

$$\lim_{n\to\infty} ||e_n - x_n|| = 0. \tag{3.16}$$

Step. 4. We show that $\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \leq 0$, where $z = P_{\mathfrak{F}}u$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z,\ x_n - z\rangle = \limsup_{j\to\infty}\langle u-z,\ x_{n_j} - z\rangle \tag{3.17}$$

Without loss of generality, we may assume that $\{x_{n_j}\}$ converges weakly to some q in H. Next, we will show that

$$q \in \mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i,\ A_i) \bigcap \bigcap_{i=1}^{N} F(G_i). \tag{3.18}$$

First, we define a mapping $A : C \to C$ by

$$Ax = \Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x, \quad \forall x \in C.$$

Since $F(T_{r_i}(I - r_i A_i)) = EF(F_i, A_i), \ \forall i = 1, 2, \ldots, N,$ we have $\bigcap_{i=1}^{N} F(T_{r_i}(I - r_i A_i)) = \bigcap_{i=1}^{N} EF(F_i, A_i) \neq \emptyset.$ By Lemma 2.3, we have $F(A) = \bigcap_{i=1}^{N} F(T_{r_i}(I - r_i A_i)).$

Next, we define $Q : C \to C$ by

$$Qx = aSx + bBx + cAx \quad \forall x \in C. \tag{3.19}$$

Again, by Lemma 2.3, we have

$$F(Q) = F(S) \bigcap F(B) \bigcap F(A) = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} F(G_i) \bigcap \bigcap_{i=1}^{N} EF(F_i, A_i).$$

By (3.19), we have

$$\begin{aligned} \|Qx_n - e_n\| &= \|aSx_n + bBx_n + cAx_n - a_n S_n x_n - b_n Bx_n - c_n y_n\| \\ &= \|aSx_n - aS_n x_n + aS_n x_n + bBx_n + c\Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x_n - a_n S_n x_n \\ &\quad - b_n Bx_n - c_n \Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x_n\| \\ &= \|a(Sx_n - S_n x_n) + (a - a_n)S_n x_n + (b - b_n)Bx_n + (c - c_n)\Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x_n\| \\ &\leq a\|Sx_n - S_n x_n\| + |a - a_n|\|S_n x_n\| + |b - b_n|\|Bx_n\| \\ &\quad + |c - c_n|\Sigma_{i=1}^{N} \delta_i \|T_{r_i}(I - r_i A_i)x_n\|. \end{aligned} \tag{3.20}$$

By condition (iii), (3.20), and (2.11), we have

$$\lim_{n\to\infty} ||Qx_n - e_n|| = 0. \tag{3.21}$$

Since

$$||Qx_n - x_n|| \leq ||Qx_n - e_n|| + ||e_n - x_n||.$$

by (3.16) and (3.21), we have

$$\lim_{n\to\infty} ||Qx_n - x_n|| = 0. \tag{3.22}$$

From, (3.22), we have

$$\lim_{j\to\infty} ||Qx_{n_j} - x_{n_j}|| = 0. \tag{3.23}$$

By Lemma 2.4, we obtain that

$$q \in F(Q) = \mathfrak{F}. \tag{3.24}$$

From (3.17)

$$\limsup_{n\to\infty} \langle u - z, x_n - z\rangle = \limsup_{j\to\infty} \langle u - z, x_{n_j} - z\rangle = \langle u - z, q - z\rangle \leq 0. \tag{3.25}$$

Step. 5. Finally, we show that $\lim_{n\to\infty} x_n = z$, where $z = P_{\mathfrak{F}}u$.

By nonexpansiveness of S_n and B, we can show that $||e_n - z|| \leq ||x_n - z||$. Then,

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n(e_n - z)\|^2 \\ &= \alpha_n\langle u - z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle + \gamma_n\langle e_n - z, x_{n+1} - z\rangle \\ &\leq \alpha_n\langle u - z, x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|e_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\langle u - z, x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|x_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\langle u - z, x_{n+1} - z\rangle + (1 - \alpha_n)\|x_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\langle u - z, x_{n+1} - z\rangle + \frac{(1 - \alpha_n)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2). \end{aligned}$$

It follows that

$$\|x_{n+1} - z\|^2 \leq 2\alpha_n \langle u - z,\ x_{n+1} - z\rangle + (1 - \alpha_n)\|x_n - z\|^2. \tag{3.26}$$

From Step 4, (3.26), and Lemma 2.2, we have $\lim_{n\to\infty} x_n = z$, where $z = P_{\mathfrak{F}}u$. The proof is complete. □

4 Applications

From Theorem 3.1, we obtain the following strong convergence theorems in a real Hilbert space:

Theorem 4.1. *Let C be a closed convex subset of Hilbert space H. For every* $i = 1, 2, ..., N$, *let* $F_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A_1) - (A_4) *and let* $\{T_i\}_{i=1}^{\infty}$ *be* κ_i*-strict pseudo-contraction mappings of C into itself with* $\kappa = \sup_i \kappa_i$ *and let* $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, *and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ *for all* $j = 2, \ldots$ *.. For every* $n \in \mathbb{N}$, *let* S_n *and S are S-mappings generated by* $T_n, ..., T_1$ *and* $\rho_n, \rho_{n-1}, ..., \rho_1$ *and* $T_n, T_{n-1}, ...,$ *and* $\rho_n, \rho_{n-1}, ...,$ *respectively. Assume that* $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i) \neq \emptyset$. *For every* $n \in \mathbb{N}$, $i = 1, 2, ..., N$, *let* $\{x_n\}$ *and* $\{v_n^i\}$ *be generated by* $x_1, u \in C$ *and*

$$\begin{cases} F_i\langle v_n^i,\ v\rangle + \dfrac{1}{r_i}\langle v - v_n^i,\ v_n^i - x_n\rangle \geq 0, \quad \forall v \in C,\ i = 1,\ 2,\ \ldots,\ N. \\ y_n = \Sigma_{i=1}^{N}\delta_i v_n^i \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(a_n S_n x_n + b_n x_n + c_n y_n), \qquad \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset (0, 1)$, $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = \Sigma_{i=1}^{N}\delta_i = 1$, *and* $\{r_i\}_{i=1}^{N} \subset (\varsigma, \tau) \subset (0, 2\alpha_i)$, *satisfy the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(*ii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(*iii*) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$, *with* $a, b, c \in (0, 1)$,

Then, the sequence $\{x_n\}, \{y_n\}, \{v_n^i\}$, $\forall i = 1, 2, ..., N$, *converge strongly to* $z = P_{\mathfrak{F}}u$, *and z is solution of (1.10)*

Proof. From Theorem 3.1, let $A_i \equiv 0$; then we have $G_i(y) = P_{Cy} = y\ \forall y \in C$. Then, we get $Bx_n = x_n\ \forall n \in \mathbb{N}$. Then, from Theorem 3.1, we obtain the desired conclusion. □

Next theorem is derived from Theorem 3.1, and we modify the result of [11] as follows:

Theorem 4.2. *Let C be a closed convex subset of Hilbert space H and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A_1)-(A_4), *let* $A : C \to H$ *be* α*-inverse strongly monotone mapping, and let T be* κ*-strict pseudo-contraction mappings of C into itself. Define a mapping* T_κ *by* $T_\kappa x = \kappa x + (1 - \kappa)Tx$, $\forall x \in C$. *Assume that* $\mathfrak{F} = F(T) \bigcap EF(F, A) \bigcap VI(C, A) \neq \emptyset$. *For every* $n \in \mathbb{N}$, *let* $\{x_n\}$ *and* $\{v_n\}$ *be generated by* $x_1, u \in C$ *and*

$$\begin{cases} F\langle v_n,\ v\rangle + \langle Ax_n,\ v - v_n\rangle + \dfrac{1}{r}\langle v - v_n,\ v_n - x_n\rangle \geq 0, \quad \forall v \in C \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(aT_\kappa x_n + bP_C(I - \lambda A)x_n + cv_n), \qquad \forall n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a, b, c\} \subset (0, 1)$, $\alpha_n + \beta_n + \gamma_n = a + b + c = 1$, *and* $\{r, \lambda\} \subset (\varsigma, \tau) \subset (0, 2\alpha)$ *satisfy the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$*and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

Then, the sequence $\{x_n\}$ *and* $\{v_n\}$ *converge strongly to* $z = P_{\Im}u$.

Proof. From Theorem 3.1, choose $N = 1$ and let $A_1 = A$, $\lambda_1 = \lambda$. Then, we have $B(y) = G_1(y) = P_C(I - \lambda A)y$, $\forall y \in C$. Choose $v_n^1 = v_n$, $a = a_n$, $b = b_n$, $c = c_n$ for all $n \in \mathbb{N}$, and let $T_\kappa \equiv S_1 : C \to C$ be S-mapping generated by T_1 and ρ_1 with $T_1 = T$ and $\alpha_1^1 = \kappa$, and then we obtain the desired result from Theorem 3.1 □

Acknowledgements
The authors would like to thank Professor Dr. Suthep Suantai for his valuable suggestion in the preparation and improvement of this article.

Competing interests
The author declares that they have no competing interests.

Received: 21 February 2011 Accepted: 29 July 2011 Published: 29 July 2011

References
1. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
2. Moudafi, A, Thera, M: Proximal and Dynamical Approaches to Equilibrium Problems. In Lecture Notes in Economics and Mathematical Systems, vol. 477, pp. 187–201. Springer (1999)
3. Takahashi, S, Takahashi, W: Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. Nonlinear Anal. **69**, 1025–1033 (2008). doi:10.1016/j.na.2008.02.042
4. Iiduka, H, Takahashi, W: Weak convergence theorem by Ces'aro means for nonexpansive mappings and inverse-strongly monotone mappings. J Nonlinear Convex Anal. **7**, 105–113 (2006)
5. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yoko-hama (2000)
6. Tada, A, Takahashi, W: Strong convergence theorem for an equilibrium problem and a nonexpansive mapping. J Optim Theory Appl. **133**, 359–370 (2007). doi:10.1007/s10957-007-9187-z
7. Takahashi, S, Takahashi, W: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J Math Anal Appl. **331**, 506–515 (2007). doi:10.1016/j.jmaa.2006.08.036
8. Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert spaces. J Math Anal Appl. **20**, 197–228 (1967). doi:10.1016/0022-247X(67)90085-6
9. Marino, G, Xu, HK: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J Math Anal Appl. **329**, 336–346 (2007). doi:10.1016/j.jmaa.2006.06.055
10. Mann, WR: Mean value methods in iteration. Proc Am Math Soc. **4**, 506–510 (1953). doi:10.1090/S0002-9939-1953-0054846-3
11. Qin, X, Chang, SS, Cho, YJ: Iterative methods for generalized equilibrium problems and fixed point problems with applications. Nonlinear Analysis: Real World Applications. **11**, 2963–2972 (2010)
12. Xu, HK: Iterative algorithms for nonlinear operators. J Lond Math Soc. **66**, 240–256 (2002). doi:10.1112/S0024610702003332
13. Bruck, RE: Properties of fixed point sets of nonexpansive mappings in Banach spaces. Trans Am Math Soc. **179**, 251–262 (1973)
14. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc Sympos Pure Math. **18**, 78–81 (1976)
15. Suzuki, T: Strong convergence of Krasnoselskii and Manns type sequences for one-parameter nonexpansive semigroups without Bochner integrals. J Math Anal Appl. **305**, 227–239 (2005). doi:10.1016/j.jmaa.2004.11.017
16. Combettes, PL, Hirstoaga, A: Equilibrium programming in Hilbert spaces. J Nonlinear Convex Anal. **6**, 117–136 (2005)
17. Kangtunyakarn, A, Suantai, S: A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings. Nonlinear Anal. **71**, 4448–4460 (2009). doi:10.1016/j.na.2009.03.003

doi:10.1186/1687-1812-2011-23
Cite this article as: Kangtunyakarn: **Strong convergence theorem for a generalized equilibrium problem and system of variational inequalities problem and infinite family of strict pseudo-contractions.** *Fixed Point Theory and Applications* 2011 **2011**:23.