Zeng *Journal of Inequalities and Applications* 2014, 2014:172
http://www.journalofinequalitiesandapplications.com/content/2014/1/172

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Inequalities and boundedness for commutators related to integral operator with general kernel

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Abstract

In this paper, we establish the sharp maximal function inequalities for the commutators related to some integral operator with general kernel and the *BMO* and Lipschitz functions. As an application, we obtain the boundedness of the commutators on Lebesgue, Morrey and Triebel-Lizorkin space. The operator includes Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

MSC: 42B20; 42B25

Keywords: commutator; Littlewood-Paley operator; Marcinkiewicz operator; Bochner-Riesz operator; sharp maximal function; Morrey space; Triebel-Lizorkin space; *BMO*; Lipschitz function

1 Introduction and preliminaries

As the development of singular integral operators (see [1–3]), their commutators have been well studied (see [4–6]). In [5–7], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [8]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [9–11], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [12], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by *BMO* and Lipschitz functions are obtained (see [12, 13]). The purpose of this paper is to prove the sharp maximal function inequalities for the commutator associated with some integral operator with general kernel and the *BMO* and Lipschitz functions. As an application, we obtain the boundedness of the commutator on Lebesgue, Morrey and Triebel-Lizorkin space. The operator includes Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy;$$

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here, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)\,dx$. It is well known that (see [1, 2])

$$M^{\#}(f)(x) \approx \sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q \bigl|f(y)-c\bigr|\,dy.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $\|f\|_{BMO} = \|M^{\#}(f)\|_{L^\infty}$. It has been known that (see [14])

$$\|f - f_{2^kQ}\|_{BMO} \le Ck\|f\|_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \bigl|f(y)\bigr|\,dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \le r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|^{1-r\eta/n}}\int_Q \bigl|f(y)\bigr|^r\,dy\right)^{1/r}.$$

The A_p weight is defined by (see [1])

$$A_p = \left\{w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty\right\},$$
$$1 < p < \infty,$$

and

$$A_1 = \bigl\{w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \le Cw(x),\ \text{a.e.}\bigr\}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the weighted homogeneous Triebel-Lizorkin space (see [11]).

For $\beta > 0$, the Lipschitz space $\mathrm{Lip}_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{\mathrm{Lip}_\beta} = \sup_{\substack{x,y\in R^n\\ x\ne y}}\frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

In this paper, we will study some integral operators as follows (see [12]).

Definition 1 Let $F_t(x,y)$ be defined on $R^n \times R^n \times [0,+\infty)$ and b be a locally integrable function on R^n, set

$$F_t(f)(x) = \int_{R^n} F_t(x,y)f(y)\,dy$$

and

$$F_t^b(f)(x) = \int_{R^n}\bigl(b(x)-b(y)\bigr)F_t(x,y)f(y)\,dy$$

for every bounded and compactly supported function f.

Let H be the Banach space $H = \{h : \|h\| < \infty\}$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ and $F_t^b(f)(x)$ as the mappings from $[0, +\infty)$ to H. Set

$$T(f)(x) = \left\|F_t(f)(x)\right\|,$$

which T is bounded on $L^2(R^n)$. The commutator related to F_t^b is defined by

$$T^b(f)(x) = \left\|F_t^b(f)(x)\right\|$$

and for F_t we find that there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|} \left(\left\|F_t(x,y) - F_t(x,z)\right\| + \left\|F_t(y,x) - F_t(z,x)\right\|\right) dx \leq C$$

and

$$\left(\int_{2^k|z-y|\leq|x-y|<2^{k+1}|z-y|} \left(\left\|F_t(x,y) - F_t(x,z)\right\| + \left\|F_t(y,x) - F_t(z,x)\right\|\right)^q dy\right)^{1/q} \\ \leq C_k\left(2^k|z-y|\right)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Definition 2 Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for } t \geq 0.$$

Let f be a locally integrable function on R^n. Set, for $0 \leq \eta < n$ and $1 \leq p < n/\eta$,

$$\|f\|_{L^{p,\eta,\varphi}} = \sup_{x\in R^n, d>0} \left(\frac{1}{\varphi(d)^{1-p\eta/n}} \int_{Q(x,d)} \left|f(y)\right|^p dy\right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized fractional Morrey space is defined by

$$L^{p,\eta,\varphi}\left(R^n\right) = \left\{f \in L^1_{\text{loc}}\left(R^n\right) : \|f\|_{L^{p,\eta,\varphi}} < \infty\right\}.$$

We write $L^{p,\eta,\varphi}(R^n) = L^{p,\varphi}(R^n)$ if $\eta = 0$, which is the generalized Morrey space. If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [14, 15]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n)$, which is the Lebesgue spaces.

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [7, 16–19]).

It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [5, 6]). In [6], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove the sharp maximal inequalities for the commutator. As the application, we obtain the L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the commutator.

2 Theorems

We shall prove the following theorems.

Theorem 1 *Let T be the integral operator as Definition* 1, *the sequence* $\{C_k\} \in l^1$, $0 < \beta < 1$, $q' \le s < \infty$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}\big(T^b(f)\big)(\tilde{x}) \le C\|b\|_{\mathrm{Lip}_\beta}\big(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}\big(T(f)\big)(\tilde{x})\big).$$

Theorem 2 *Let T be the integral operator as Definition* 1, *the sequence* $\{2^{k\beta}C_k\} \in l^1$, $0 < \beta < 1$, $q' \le s < \infty$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \inf_{c \in R^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q \big|T^b(f)(x) - c\big|\,dx \le C\|b\|_{\mathrm{Lip}_\beta}\big(M_s(f)(\tilde{x}) + M_s\big(T(f)\big)(\tilde{x})\big).$$

Theorem 3 *Let T be the integral operator as Definition* 1, *the sequence* $\{kC_k\} \in l^1$, $q' \le s < \infty$ *and* $b \in BMO(R^n)$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}\big(T^b(f)\big)(\tilde{x}) \le C\|b\|_{BMO}\big(M_s(f)(\tilde{x}) + M_s\big(T(f)\big)(\tilde{x})\big).$$

Theorem 4 *Let T be the integral operator as Definition* 1, *the sequence* $\{C_k\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *Then T^b is bounded from $L^p(R^n)$ to $L^r(R^n)$.*

Theorem 5 *Let T be the integral operator as Definition* 1, *the sequence* $\{C_k\} \in l^1$, $0 < D < 2^n$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *Then T^b is bounded from $L^{p,\beta,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.*

Theorem 6 *Let T be the integral operator as Definition* 1, *the sequence* $\{2^{k\beta}C_k\} \in l^1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *Then T^b is bounded from $L^p(R^n)$ to $\dot{F}_r^{\beta,\infty}(R^n)$.*

Theorem 7 *Let T be the integral operator as Definition* 1, *the sequence* $\{kC_k\} \in l^1$ *and* $b \in BMO(R^n)$. *Then T^b is bounded on $L^p(R^n)$ for $q' \le p < \infty$.*

3 Proofs of theorems

To prove the theorems, we need the following lemma.

Lemma 1 (see [12]) *Let T be the integral operator as Definition* 1, *the sequence* $\{C_k\} \in l^1$. *Then T is bounded on* $L^p(R^n)$ *for* $1 < p < \infty$.

Lemma 2 (see [11]) *For* $0 < \beta < 1$ *and* $1 < p < \infty$, *we have*

$$\begin{aligned}\|f\|_{\dot F_p^{\beta,\infty}} &\approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|f(x) - f_Q\bigr|\, dx \right\|_{L^p} \\ &\approx \left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|f(x) - c\bigr|\, dx \right\|_{L^p}.\end{aligned}$$

Lemma 3 (see [1]) *Let* $0 < p < \infty$ *and* $w \in \bigcup_{1 \le r < \infty} A_r$. *Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x)\, dx \le C \int_{R^n} M^\#(f)(x)^p w(x)\, dx.$$

Lemma 4 (see [8]) *Suppose that* $0 < \eta < n$, $1 < s < p < n/\eta$ *and* $1/q = 1/p - \eta/n$. *Then*

$$\bigl\|M_{\eta,s}(f)\bigr\|_{L^q} \le C\|f\|_{L^p}.$$

Lemma 5 *Let* $1 < p < \infty$, $0 < D < 2^n$. *Then, for any smooth function f for which the left-hand side is finite,*

$$\bigl\|M(f)\bigr\|_{L^{p,\varphi}} \le C\bigl\|M^\#(f)\bigr\|_{L^{p,\varphi}}.$$

Proof For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [20]. Noticing that $M(\chi_Q) \le 1$ and $M(\chi_Q)(x) \le d^n/(|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\begin{aligned}\int_Q M(f)(x)^p\, dx &= \int_{R^n} M(f)(x)^p \chi_Q(x)\, dx \\ &\le \int_{R^n} M(f)(x)^p M(\chi_Q)(x)\, dx \\ &\le C \int_{R^n} M^\#(f)(x)^p M(\chi_Q)(x)\, dx \\ &= C\biggl(\int_Q M^\#(f)(x)^p M(\chi_Q)(x)\, dx \\ &\quad + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p M(\chi_Q)(x)\, dx\biggr) \\ &\le C\biggl(\int_Q M^\#(f)(x)^p\, dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p \frac{|Q|}{|2^{k+1}Q|}\, dx\biggr) \\ &\le C\biggl(\int_Q M^\#(f)(x)^p\, dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^\#(f)(x)^p 2^{-kn}\, dy\biggr) \\ &\le C\bigl\|M^\#(f)\bigr\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn}\varphi\bigl(2^{k+1}d\bigr)\end{aligned}$$

$$
\begin{aligned}
&\leq C\big\|M^{\#}(f)\big\|_{L^{p,\varphi}}^{p}\sum_{k=0}^{\infty}\big(2^{-n}D\big)^{k}\varphi(d)\\
&\leq C\big\|M^{\#}(f)\big\|_{L^{p,\varphi}}^{p}\varphi(d),
\end{aligned}
$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M(f)(x)^p\,dx\right)^{1/p}\leq C\left(\frac{1}{\varphi(d)}\int_Q M^{\#}(f)(x)^p\,dx\right)^{1/p}$$

and

$$\big\|M(f)\big\|_{L^{p,\varphi}}\leq C\big\|M^{\#}(f)\big\|_{L^{p,\varphi}}.$$

This finishes the proof. □

Lemma 6 *Let T be the integral operator as Definition* 1, $0\leq\eta<n$, $0<D<2^n$ *and* $1\leq p<\infty$. *Then*

$$\big\|T(f)\big\|_{L^{p,\eta,\varphi}}\leq C\|f\|_{L^{p,\eta,\varphi}}.$$

Lemma 7 *Let* $0<D<2^n$, $0<\eta<n$, $1\leq s<p<n/\eta$ *and* $1/q=1/p-\eta/n$. *Then*

$$\big\|M_{\eta,s}(f)\big\|_{L^{q,\varphi}}\leq C\|f\|_{L^{p,\eta,\varphi}}.$$

The proofs of the two lemmas are similar to that of Lemma 5 by Lemmas 1 and 4, we omit the details.

Proof of Theorem 1 It suffices to prove for $f\in C_0^{\infty}(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q\big|T^b(f)(x)-C_0\big|\,dx\leq C\|b\|_{\mathrm{Lip}_\beta}\big(M_{\beta,s}(f)(\tilde{x})+M_{\beta,s}\big(T(f)\big)(\tilde{x})\big).$$

Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Write, for $f_1=f\chi_{2Q}$ and $f_2=f\chi_{(2Q)^c}$,

$$F_t^b(f)(x)=\big(b(x)-b_{2Q}\big)F_t(f)(x)-F_t\big((b-b_{2Q})f_1\big)(x)-F_t\big((b-b_{2Q})f_2\big)(x).$$

Then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q\big\|F_t^b(f)(x)-F_t\big((b_{2Q}-b)f_2\big)(x_0)\big\|\,dx\\
&\quad\leq\frac{1}{|Q|}\int_Q\big\|\big(b(x)-b_{2Q}\big)F_t(f)(x)\big\|\,dx+\frac{1}{|Q|}\int_Q\big\|F_t\big((b-b_{2Q})f_1\big)(x)\big\|\,dx\\
&\qquad+\frac{1}{|Q|}\int_Q\big\|F_t\big((b-b_{2Q})f_2\big)(x)-F_t\big((b-b_{2Q})f_2\big)(x_0)\big\|\,dx\\
&\quad=I_1+I_2+I_3.
\end{aligned}
$$

For I_1, by Hölder's inequality and Lemma 2, we obtain

$$\begin{aligned} I_1 &\le \frac{C}{|Q|} \sup_{x\in 2Q} \bigl|b(x) - b_{2Q}\bigr| |Q|^{1-1/s} \biggl(\int_Q \bigl|T(f)(x)\bigr|^s \,dx \biggr)^{1/s} \\ &\le C|Q|^{-1/s} \|b\|_{\mathrm{Lip}_\beta} |2Q|^{\beta/n} |2Q|^{1/s-\beta/n} \biggl(\frac{1}{|Q|^{1-s\beta/n}} \int_Q \bigl|T(f)(x)\bigr|^s \,dx \biggr)^{1/s} \\ &\le C\|b\|_{\mathrm{Lip}_\beta} M_{\beta,s}\bigl(T(f)\bigr)(\tilde{x}). \end{aligned}$$

For I_2, by the boundedness of T, we get

$$\begin{aligned} I_2 &\le \biggl(\frac{1}{|Q|} \int_{R^n} \bigl|T\bigl((b - b_{2Q})f_1\bigr)(x)\bigr|^s \,dx \biggr)^{1/s} \\ &\le C\biggl(\frac{1}{|Q|} \int_{R^n} \bigl|\bigl(b(x) - b_{2Q}\bigr)f_1(x)\bigr|^s \,dx \biggr)^{1/s} \\ &\le C|Q|^{-1/s} |2Q|^{\beta/n} ||2Q|^{1/s-\beta/n} \biggl(\frac{1}{|2Q|^{1-s\beta/n}} \int_{2Q} \bigl|f(x)\bigr|^s \,dx \biggr)^{1/s} \\ &\le C\|b\|_{\mathrm{Lip}_\beta} M_{\beta,s}(f)(\tilde{x}). \end{aligned}$$

For I_3, recalling that $s > q'$, we have

$$\begin{aligned} I_3 &\le \frac{1}{|Q|} \int_Q \int_{(2Q)^c} \bigl|b(y) - b_{2Q}\bigr| \bigl|f(y)\bigr| \bigl\|F_t(x,y) - F_t(x_0,y)\bigr\| \,dy\,dx \\ &\le \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\|F_t(x,y) - F_t(x_0,y)\bigr\| \bigl|b(y) - b_{2^{k+1}Q}\bigr| \bigl|f(y)\bigr| \,dy\,dx \\ &\quad + \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\|F_t(x,y) - F_t(x_0,y)\bigr\| \bigl|b_{2^{k+1}Q} - b_{2Q}\bigr| \bigl|f(y)\bigr| \,dy\,dx \\ &\le \frac{C}{|Q|} \int_Q \sum_{k=1}^{\infty} \biggl(\int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\|F_t(x,y) - F_t(x_0,y)\bigr\|^q \,dy \biggr)^{1/q} \\ &\quad \times \sup_{y\in 2^{k+1}Q} \bigl|b(y) - b_{2^{k+1}Q}\bigr| \biggl(\int_{2^{k+1}Q} \bigl|f(y)\bigr|^{q'} \,dy \biggr)^{1/q'} \,dx \\ &\quad + \frac{C}{|Q|} \int_Q \sum_{k=1}^{\infty} |b_{2^{k+1}Q} - b_{2Q}| \biggl(\int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\|F_t(x,y) - F_t(x_0,y)\bigr\|^q \,dy \biggr)^{1/q} \\ &\quad \times \biggl(\int_{2^{k+1}Q} \bigl|f(y)\bigr|^{q'} \,dy \biggr)^{1/q'} \,dx \\ &\le C \sum_{k=1}^{\infty} C_k \bigl(2^k d\bigr)^{-n/q'} \bigl|2^{k+1}Q\bigr|^{\beta/n} \|b\|_{\mathrm{Lip}_\beta} \bigl|2^{k+1}Q\bigr|^{1/q'-1/s} \bigl|2^{k+1}Q\bigr|^{1/s-\beta/n} \\ &\quad \times \biggl(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}} \int_{2^{k+1}Q} \bigl|f(y)\bigr|^s \,dy \biggr)^{1/s} \\ &\quad + C \sum_{k=1}^{\infty} \|b\|_{\mathrm{Lip}_\beta} \bigl|2^k Q\bigr|^{\beta/n} C_k \bigl(2^k d\bigr)^{-n/q'} \bigl|2^{k+1}Q\bigr|^{1/q'-1/s} \bigl|2^{k+1}Q\bigr|^{1/s-\beta/n} \end{aligned}$$

$$\begin{aligned}&\times\left(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}}\int_{2^{k+1}Q}\left|f(y)\right|^s\,dy\right)^{1/s}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}M_{\beta,s}(f)(\tilde{x})\sum_{k=1}^{\infty}C_k\\ &\le C\|b\|_{\mathrm{Lip}_\beta}M_{\beta,s}(f)(\tilde{x}).\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x)-C_0\right|\,dx \le C\|b\|_{\mathrm{Lip}_\beta}\bigl(M_s(f)(\tilde{x})+M_s\bigl(T(f)\bigr)(\tilde{x})\bigr).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_t^b(f)(x) = \bigl(b(x)-b_{2Q}\bigr)F_t(f)(x) - F_t\bigl((b-b_{2Q})f_1\bigr)(x) - F_t\bigl((b-b_{2Q})f_2\bigr)(x).$$

Then

$$\begin{aligned}&\frac{1}{|Q|^{1+\beta/n}}\int_Q\left\|F_t^b(f)(x)-F_t\bigl((b_{2Q}-b)f_2\bigr)(x_0)\right\|\,dx\\ &\quad\le\frac{1}{|Q|^{1+\beta/n}}\int_Q\left\|\bigl(b(x)-b_{2Q}\bigr)F_t(f)(x)\right\|\,dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q\left\|F_t\bigl((b-b_{2Q})f_1\bigr)(x)\right\|\,dx\\ &\qquad+\frac{1}{|Q|^{1+\beta/n}}\int_Q\left\|F_t\bigl((b-b_{2Q})f_2\bigr)(x)-F_t\bigl((b-b_{2Q})f_2\bigr)(x_0)\right\|\,dx\\ &\quad=II_1+II_2+II_3.\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned}II_1&\le\frac{C}{|Q|^{1+\beta/n}}\sup_{x\in 2Q}\left|b(x)-b_{2Q}\right||Q|^{1-1/s}\left(\int_Q\left|T(f)(x)\right|^s\,dx\right)^{1/s}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}|2Q|^{\beta/n}|Q|^{-1/s}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|}\int_Q\left|T(f)(x)\right|^s\,dx\right)^{1/s}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}M_s\bigl(T(f)\bigr)(\tilde{x}),\\ II_2&\le\frac{1}{|Q|^{1+\beta/n}}|Q|^{1-1/s}\left(\int_{R^n}\left|T\bigl((b-b_{2Q})f_1\bigr)(x)\right|^s\,dx\right)^{1/s}\\ &\le\frac{C}{|Q|^{1+\beta/n}}|Q|^{1-1/s}\left(\int_{R^n}\left|\bigl(b(x)-b_{2Q}\bigr)f_1(x)\right|^s\,dx\right)^{1/s}\\ &\le\frac{C}{|Q|^{1+\beta/n}}|Q|^{1-1/s}|2Q|^{\beta/n}|2Q|^{1/s}\left(\frac{1}{|2Q|}\int_{2Q}\left|f(x)\right|^s\,dx\right)^{1/s}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}M_s(f)(\tilde{x}),\\ II_3&\le\frac{1}{|Q|^{1+\beta/n}}\int_Q\int_{(2Q)^c}\left|b(y)-b_{2Q}\right|\left|f(y)\right|\left\|F_t(x,y)-F_t(x_0,y)\right\|\,dy\,dx\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\| F_t(x,y) - F_t(x_0,y) \bigr\| \bigl| b(y) - b_{2^{k+1}Q} \bigr| \bigl| f(y) \bigr| \, dy \, dx \\
&\quad + \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\| F_t(x,y) - F_t(x_0,y) \bigr\| \bigl| b_{2^{k+1}Q} - b_{2Q} \bigr| \bigl| f(y) \bigr| \, dy \, dx \\
&\le \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \biggl(\int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\| F_t(x,y) - F_t(x_0,y) \bigr\|^q \, dy \biggr)^{1/q} \\
&\quad \times \sup_{y \in 2^{k+1}Q} \bigl| b(y) - b_{2^{k+1}Q} \bigr| \biggl(\int_{2^{k+1}Q} \bigl| f(y) \bigr|^{q'} \, dy \biggr)^{1/q'} \, dx \\
&\quad + \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} |b_{2^{k+1}Q} - b_{2Q}| \biggl(\int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl\| F_t(x,y) - F_t(x_0,y) \bigr\|^q \, dy \biggr)^{1/q} \\
&\quad \times \biggl(\int_{2^{k+1}Q} \bigl| f(y) \bigr|^{q'} \, dy \biggr)^{1/q'} \, dx \\
&\le C|Q|^{-\beta/n} \sum_{k=1}^{\infty} C_k \bigl(2^k d\bigr)^{-n/q'} \bigl| 2^{k+1}Q \bigr|^{\beta/n} \|b\|_{\mathrm{Lip}_\beta} \bigl| 2^{k+1}Q \bigr|^{1/q'} \\
&\quad \times \biggl(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl| f(y) \bigr|^s \, dy \biggr)^{1/s} \\
&\quad + C|Q|^{-\beta/n} \sum_{k=1}^{\infty} \|b\|_{\mathrm{Lip}_\beta} \bigl| 2^k Q \bigr|^{\beta/n} C_k \bigl(2^k d\bigr)^{-n/q'} \bigl| 2^{k+1}Q \bigr|^{1/q'} \\
&\quad \times \biggl(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl| f(x) \bigr|^s \, dx \biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta} M_s(f)(\tilde{x}) \sum_{k=1}^{\infty} 2^{k\beta} C_k \\
&\le C\|b\|_{\mathrm{Lip}_\beta} M_s(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$
\frac{1}{|Q|} \int_Q \bigl| T^b(f)(x) - C_0 \bigr| \, dx \le C\|b\|_{BMO} \bigl(M_s(f)(\tilde{x}) + M_s\bigl(T(f)\bigr)(\tilde{x}) \bigr).
$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$
F_t^b(f)(x) = \bigl(b(x) - b_{2Q}\bigr) F_t(f)(x) - F_t\bigl((b - b_{2Q}) f_1\bigr)(x) - F_t\bigl((b - b_{2Q}) f_2\bigr)(x).
$$

Then

$$
\begin{aligned}
&\frac{1}{|Q|} \int_Q \bigl\| F_t^b(f)(x) - F_t\bigl((b_{2Q} - b) f_2\bigr)(x_0) \bigr\| \, dx \\
&\quad \le \frac{1}{|Q|} \int_Q \bigl\| \bigl(b(x) - b_{2Q}\bigr) F_t(f)(x) \bigr\| \, dx + \frac{1}{|Q|} \int_Q \bigl\| F_t\bigl((b - b_{2Q}) f_1\bigr)(x) \bigr\| \, dx
\end{aligned}
$$

$$\begin{aligned}&+\frac{1}{|Q|}\int_Q \bigl\|F_t\bigl((b-b_{2Q})f_2\bigr)(x)-F_t\bigl((b-b_{2Q})f_2\bigr)(x_0)\bigr\|\,dx\\&=III_1+III_2+III_3.\end{aligned}$$

For III_1, by Hölder's inequality, we get

$$\begin{aligned}III_1&\le\left(\frac{1}{|Q|}\int_Q\bigl|b(x)-b_{2Q}\bigr|^{s'}dx\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q\bigl|T(f)(x)\bigr|^{s}dx\right)^{1/s}\\&\le C\|b\|_{BMO}M_s\bigl(T(f)\bigr)(\tilde{x}).\end{aligned}$$

For III_2, choose $1<r<s$, by Hölder's inequality and the boundedness of T, we obtain

$$\begin{aligned}III_2&\le\left(\frac{1}{|Q|}\int_{R^n}\bigl|T\bigl((b-b_{2Q})f_1\bigr)(x)\bigr|^{r}dx\right)^{1/r}\\&\le C|Q|^{-1/r}\left(\int_{R^n}\bigl|(b-b_{2Q})f_1(x)\bigr|^{r}dx\right)^{1/r}\\&\le C|Q|^{-1/r}\left(\int_{2Q}\bigl|b(x)-b_{2Q}\bigr|^{sr/(s-r)}dx\right)^{(s-r)/sr}\left(\int_{2Q}\bigl|f(x)\bigr|^{s}dx\right)^{1/s}\\&\le C\|b\|_{BMO}\left(\frac{1}{|2Q|}\int_{2Q}\bigl|f(x)\bigr|^{s}dx\right)^{1/s}\\&\le C\|b\|_{BMO}M_s(f)(\tilde{x}).\end{aligned}$$

For III_3, recalling that $s>q'$, taking $1<p<\infty$, $1<r<s$ with $1/p+1/q+1/r=1$, by Hölder's inequality, we obtain

$$\begin{aligned}III_3&\le\frac{1}{|Q|}\int_Q\int_{(2Q)^c}\bigl|b(y)-b_{2Q}\bigr|\bigl|f(y)\bigr|\bigl\|F_t(x,y)-F_t(x_0,y)\bigr\|\,dy\,dx\\&\le\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\int_{2^kd\le|y-x_0|<2^{k+1}d}\bigl\|F_t(x,y)-F_t(x_0,y)\bigr\|\bigl|b(y)-b_{2Q}\bigr|\bigl|f(y)\bigr|\,dy\,dx\\&\le\frac{1}{|Q|}\int_Q\sum_{k=1}^{\infty}\left(\int_{2^kd\le|y-x_0|<2^{k+1}d}\bigl\|F_t(x,y)-F_t(x_0,y)\bigr\|^{q}dy\right)^{1/q}\\&\quad\times\left(\int_{2^{k+1}Q}\bigl|b(x)-b_{2Q}\bigr|^{p}dx\right)^{1/p}\left(\int_{2^{k+1}Q}\bigl|f(y)\bigr|^{r}dy\right)^{1/r}dx\\&\le C\sum_{k=1}^{\infty}k\bigl(2^kd\bigr)^{n/q'}\left(\int_{2^kd\le|y-x_0|<2^{k+1}d}\bigl\|K(x,y)-K(x_0,y)\bigr\|^{q}dy\right)^{1/q}\\&\quad\times\|b\|_{BMO}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\bigl|f(y)\bigr|^{s}dy\right)^{1/s}\\&\le C\|b\|_{BMO}M_s(f)(\tilde{x})\sum_{k=1}^{\infty}kC_k\\&\le C\|b\|_{BMO}M_s(f)(\tilde{x}).\end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4 Choose $q' < s < p$ in Theorem 1, we have, by Lemmas 1, 4, and 5,

$$\begin{aligned}\|T^b(f)\|_{L^r} &\le \|M(T^b(f))\|_{L^r} \le C\|M^\#(T^b(f))\|_{L^r}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left(\|M_{\beta,s}(T(f))\|_{L^r} + \|M_{\beta,s}(f)\|_{L^r}\right)\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left(\|T(f)\|_{L^p} + \|f\|_{L^p}\right)\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^p}.\end{aligned}$$

This completes the proof of Theorem 4. □

Proof of Theorem 5 Choose $q' < s < p$ in Theorem 1, we have, by Lemmas 5-7,

$$\begin{aligned}\|T^b(f)\|_{L^{r,\varphi}} &\le \|M(T^b(f))\|_{L^{r,\varphi}} \le C\|M^\#(T^b(f))\|_{L^{r,\varphi}}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left(\|M_{\beta,s}(T(f))\|_{L^{r,\varphi}} + \|M_{\beta,s}(f)\|_{L^{r,\varphi}}\right)\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left(\|T(f)\|_{L^{p,\beta,\varphi}} + \|f\|_{L^{p,\beta,\varphi}}\right)\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^{p,\varphi}}.\end{aligned}$$

This completes the proof of Theorem 5. □

Proof Theorem 6 Choose $q' < s < p$ in Theorem 2. By using Lemma 3, we obtain

$$\begin{aligned}\|T^b(f)\|_{\dot{F}_r^{\beta,\infty}} &\le C\left\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x) - T\left((b_{2Q}-b)f_2\right)(x_0)\right|\,dx\right\|_{L^r}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left(\|M_s(T(f))\|_{L^r} + \|M_s(f)\|_{L^r}\right)\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left(\|T(f)\|_{L^p} + \|f\|_{L^p}\right)\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^p}.\end{aligned}$$

This completes the proof of the theorem. □

Proof of Theorem 7 Choose $q' \le s < p$ in Theorem 3, we have

$$\begin{aligned}\|T^b(f)\|_{L^p} &\le \|M(T^b(f))\|_{L^p} \le C\|M^\#(T^b(f))\|_{L^p}\\ &\le C\|b\|_{BMO}\left(\|M_s(T(f))\|_{L^p} + \|M_s(f)\|_{L^p}\right)\\ &\le C\|b\|_{BMO}\left(\|T(f)\|_{L^p} + \|f\|_{L^p}\right)\\ &\le C\|b\|_{BMO}\|f\|_{L^p}.\end{aligned}$$

This completes the proof of the theorem. □

4 Applications

In this section we shall apply Theorems 1-6 of the paper to some particular operators such as the Littlewood-Paley operators, Marcinkiewicz operator and Bochner-Riesz operator.

Application 1 Littlewood-Paley operators.

Fixed $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n} \psi(x)\,dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley commutators are defined by

$$g_\psi^b(f)(x) = \left(\int_0^\infty \left|F_t^b(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^b(f)(x) = \left[\int\int_{\Gamma(x)} \left|F_t^b(f)(x,y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2},$$

and

$$g_\mu^b(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \left|F_t^b(f)(x,y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^b(f)(x) = \int_{R^n} \big(b(x) - b(y)\big)\psi_t(x-y)f(y)\,dy,$$

$$F_t^b(f)(x,y) = \int_{R^n} \big(b(x) - b(z)\big)f(z)\psi_t(y-z)\,dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define

$$g_\psi(f)(x) = \left(\int_0^\infty \left|F_t(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood-Paley operators (see [3]). Let H be the space

$$H = \left\{h : \|h\| = \left(\int_0^\infty \left|h(t)\right|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : \|h\| = \left(\int\int_{R_+^{n+1}} \left|h(y,t)\right|^2 dy\,dt/t^{n+1}\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^b(f)(x)$ and $F_t^b(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^b(f)(x) = \left\|F_t^b(f)(x)\right\|, \qquad g_\psi(f)(x) = \left\|F_t(f)(x)\right\|,$$

$$S_\psi^b(f)(x) = \left\| \chi_{\Gamma(x)} F_t^b(f)(x,y) \right\|, \qquad S_\psi(f)(x) = \left\| \chi_{\Gamma(x)} F_t(f)(y) \right\|$$

and

$$g_\mu^b(f)(x) = \left\| \left(\frac{t}{t + |x-y|} \right)^{n\mu/2} F_t^b(f)(x,y) \right\|, \qquad g_\mu(f)(x) = \left\| \left(\frac{t}{t + |x-y|} \right)^{n\mu/2} F_t(f)(y) \right\|.$$

It is easily to see that g_ψ^b, S_ψ^b, and g_μ^b satisfy the conditions of Theorems 1-6 (see [21–23]), thus Theorems 1-6 hold for g_ψ^b, S_ψ^b, and g_μ^b.

Application 2 Marcinkiewicz operators.

Fixed $\lambda > \max(1, 2n/(n+2))$ and $0 < \gamma \leq 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$. The Marcinkiewicz commutators are defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty \left| F_t^b(f)(x) \right|^2 \frac{dt}{t^3} \right)^{1/2},$$
$$\mu_S^b(f)(x) = \left[\int\int_{\Gamma(x)} \left| F_t^b(f)(x,y) \right|^2 \frac{dy\,dt}{t^{n+3}} \right]^{1/2},$$

and

$$\mu_\lambda^b(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \left| F_t^b(f)(x,y) \right|^2 \frac{dy\,dt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^b(f)(x) = \int_{|x-y| \leq t} \big(b(x) - b(y)\big) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy$$

and

$$F_t^b(f)(x,y) = \int_{|y-z| \leq t} \big(b(x) - b(z)\big) \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)\, dz.$$

Set

$$F_t(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy.$$

We also define

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| F_t(f)(x) \right|^2 \frac{dt}{t^3} \right)^{1/2},$$
$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} \left| F_t(f)(y) \right|^2 \frac{dy\,dt}{t^{n+3}} \right)^{1/2},$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \left| F_t(f)(y) \right|^2 \frac{dy\,dt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz operators (see [24]). Let H be the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty \left| h(t) \right|^2 dt/t^3 \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : \|h\| = \left(\int \int_{R_+^{n+1}} \left| h(y,t) \right|^2 dy\, dt/t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then it is clear that

$$\mu_\Omega^b(f)(x) = \left\| F_t^b(f)(x) \right\|, \qquad \mu_\Omega(f)(x) = \left\| F_t(f)(x) \right\|,$$
$$\mu_S^b(f)(x) = \left\| \chi_{\Gamma(x)} F_t^b(f)(x,y) \right\|, \qquad \mu_S(f)(x) = \left\| \chi_{\Gamma(x)} F_t(f)(y) \right\|,$$

and

$$\mu_\lambda^b(f)(x) = \left\| \left(\frac{t}{t + |x-y|} \right)^{n\lambda/2} F_t^b(f)(x,y) \right\|, \qquad \mu_\lambda(f)(x) = \left\| \left(\frac{t}{t + |x-y|} \right)^{n\lambda/2} F_t(f)(y) \right\|.$$

It is easy to see that μ_Ω^b, μ_S^b, and μ_λ^b satisfy the conditions of Theorems 1-6 (see [21–24]), thus Theorems 1-6 hold for μ_Ω^b, μ_S^b, and μ_λ^b.

Application 3 Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(\hat{f})(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^b(f)(x) = \int_{R^n} \big(b(x) - b(y) \big) B_t^\delta(x-y) f(y)\, dy.$$

The maximal Bochner-Riesz commutator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} \left| B_{\delta,t}^b(f)(x) \right|.$$

We also define

$$B_{\delta,*}(f)(x) = \sup_{t>0} \left| B_t^\delta(f)(x) \right|,$$

which is the maximal Bochner-Riesz operator (see [25]). Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^b(f)(x) = \left\| B_{\delta,t}^b(f)(x) \right\|, \qquad B_*^\delta(f)(x) = \left\| B_t^\delta(f)(x) \right\|.$$

It is easy to see that $B_{\delta,*}^b$ satisfies the conditions of Theorems 1-6 (see [21]), thus Theorems 1-6 hold for $B_{\delta,*}^b$.

Competing interests

The author declares that they have no competing interests.

Received: 9 March 2014 Accepted: 25 April 2014 Published: 10 May 2014

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10.1186/1029-242X-2014-172
Cite this article as: Zeng: Inequalities and boundedness for commutators related to integral operator with general kernel. *Journal of Inequalities and Applications* 2014, 2014:172