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Existence of a tripled coincidence point in ordered G_b-metric spaces and applications to a system of integral equations

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Abstract
In this paper, tripled coincidence points of mappings satisfying some nonlinear contractive conditions in the framework of partially ordered G_b-metric spaces are obtained. Our results extend the results of Aydi *et al.* (Fixed Point Theory Appl., 2012:101, 2012, doi:10.1186/1687-1812-2012-101). Moreover, some examples of the main result are given. Finally, some tripled coincidence point results for mappings satisfying some contractive conditions of integral type in complete partially ordered G_b-metric spaces are deduced.

MSC: Primary 47H10; secondary 54H25

Keywords: tripled fixed point; generalized weakly contraction; generalized metric spaces; partially ordered set

1 Introduction and preliminaries

The concepts of mixed monotone mapping and coupled fixed point were introduced in [1] by Bhaskar and Lakshmikantham. Also, they established some coupled fixed point theorems for a mixed monotone mapping in partially ordered metric spaces. For more details on coupled fixed point theorems and related topics in different metric spaces, we refer the reader to [2–13] and [14–25].

Also, Berinde and Borcut [26] introduced a new concept of tripled fixed point and obtained some tripled fixed point theorems for contractive-type mappings in partially ordered metric spaces. For a survey of tripled fixed point theorems and related topics, we refer the reader to [26–32].

Definition 1.1 [26] An element $(x,y,z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if $F(x,y,z) = x$, $F(y,x,y) = y$ and $F(z,y,x) = z$.

Definition 1.2 [27] An element $(x,y,z) \in X^3$ is called a tripled coincidence point of the mappings $F : X^3 \to X$ and $g : X \to X$ if $F(x,y,z) = g(x)$, $F(y,x,y) = gy$ and $F(z,y,x) = gz$.

Definition 1.3 [27] An element $(x,y,z) \in X^3$ is called a tripled common fixed point of $F : X^3 \to X$ and $g : X \to X$ if $x = g(x) = F(x,y,z)$, $y = g(y) = F(y,x,y)$ and $z = g(z) = F(z,y,x)$.

Definition 1.4 [29] Let X be a nonempty set. We say that the mappings $F : X^3 \to X$ and $g : X \to X$ are commutative if $g(F(x,y,z)) = F(gx,gy,gz)$ for all $x,y,z \in X$.

The notion of altering distance function was introduced by Khan *et al.* [10] as follows.

Definition 1.5 The function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if
1. ψ is continuous and nondecreasing.
2. $\psi(t) = 0$ if and only if $t = 0$.

The concept of generalized metric space, or G-metric space, was introduced by Mustafa and Sims [33]. Mustafa and others studied several fixed point theorems for mappings satisfying different contractive conditions (see [33–45]).

Definition 1.6 (G-metric space, [33]) Let X be a nonempty set and $G : X^3 \to R^+$ be a function satisfying the following properties:
(G1) $G(x,y,z) = 0$ iff $x = y = z$;
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).
Then the function G is called a G-metric on X and the pair (X,G) is called a G-metric space.

Example 1.7 If we think that $G(x,y,z)$ is measuring the perimeter of the triangle with vertices at x, y and z, then (G5) can be interpreted as

$$[x,y] + [x,z] + [y,z] \leq 2[x,a] + [a,y] + [a,z] + [y,z],$$

where $[x,y]$ is the 'length' of the side x, y. If we take $y = z$, we have

$$2[x,y] \leq 2[x,a] + 2[a,y].$$

Thus, (G5) embodies the triangle inequality. And so (G5) can be sharp.

In [46], Aydi *et al.* established some tripled coincidence point results for mappings $F : X^3 \to X$ and $g : X \to X$ involving nonlinear contractions in the setting of ordered G-metric spaces.

Theorem 1.8 [46] *Let $(X,\preceq)$ be a partially ordered set and (X,G) be a G-metric space such that (X,G) is G-complete. Let $F : X^3 \to X$ and $g : X \to X$. Assume that there exist $\psi,\phi : [0,\infty) \to [0,\infty)$ such that ψ is an altering distance function and ϕ is a lower-semicontinuous and nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$ and for all $x,y,z,u,v,w,r,s,t \in X$, with $gx \preceq gu \preceq gr$, $gy \succeq gv \succeq gs$ and $gz \preceq gw \preceq gt$, we have*

$$\begin{aligned}
&\psi\bigl(G\bigl(F(x,y,z),F(u,v,w),F(r,s,t)\bigr)\bigr)\\
&\quad\leq \psi\bigl(\max\bigl\{G(gx,gu,gr),G(gy,gv,gs),G(gz,gw,gt)\bigr\}\bigr)\\
&\qquad - \phi\bigl(\max\bigl\{G(gx,gu,gr),G(gy,gv,gs),G(gz,gw,gt)\bigr\}\bigr).
\end{aligned}$$

Assume that F and g satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,
(2) *F has the mixed g-monotone property,*
(3) *F is continuous,*
(4) *g is continuous and commutes with F.*

Let there exist $x_0, y_0, z_0 \in X$ *such that* $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$ *and* $gz_0 \preceq F(z_0, y_0, x_0)$. *Then F and g have a tripled coincidence point in X, i.e., there exist* $x, y, z \in X$ *such that* $F(x, y, z) = gx$, $F(y, x, y) = gy$ *and* $F(z, y, x) = gz$.

Also, they proved that the above theorem is still valid for F not necessarily continuous assuming the following hypothesis (see Theorem 19 of [46]).

(I) If $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.
(II) If $\{y_n\}$ is a nonincreasing sequence with $y_n \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.

A partially ordered G-metric space (X, G) with the above properties is called regular.

In this paper, we obtain some tripled coincidence point theorems for nonlinear (ψ, φ)-weakly contractive mappings in partially ordered G_b-metric spaces. This results generalize and modify several comparable results in the literature. First, we recall the concept of generalized b-metric spaces, or G_b-metric spaces.

Definition 1.9 [47] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X^3 \to \mathbb{R}^+$ satisfies:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,
(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
(G_b5) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or a G_b-metric space.

Obviously, each G-metric space is a G_b-metric space with $s = 1$. But the following example shows that a G_b-metric on X need not be a G-metric on X (see also [48]).

Example 1.10 If we think that $G_b(x, y, z)$ is the maximum of the squares of length sides of a triangle with vertices at x, y and z such that:

If $x \neq y \neq z$, then $G_b(x, y, z) = \max\{([x, y])^2, ([y, z])^2, ([z, x])^2\}$.
If $x \neq y = z$, then $G_b(x, y, y) = ([x, y])^2$,

where $[x, y]$ is the 'length' of the side x, y. Then it is easy to see that $G_b(x, y, z)$ is a G_b function with $s = 2$.

Since by the triangle inequality we have

$$[x, y] \leq [x, a] + [a, y], \qquad [z, x] \leq [z, a] + [a, x],$$

hence

$$\begin{aligned} G_b(x, y, z) &= \max\{([x, y])^2, ([y, z])^2, ([z, x])^2\} \\ &\leq \max\{([x, a] + [a, y])^2, ([y, z])^2, ([z, a] + [a, x])^2\} \end{aligned}$$

$$\begin{aligned}
&\le \max\{2(([x,a])^2 + ([a,y])^2), ([y,z])^2, 2(([z,a])^2 + ([a,x])^2)\} \\
&\le 2([x,a])^2 + \max\{2([a,y])^2, ([y,z])^2, 2([z,a])^2\} \\
&\le 2([x,a])^2 + \max\{2([a,y])^2, 2([y,z])^2, 2([z,a])^2\} \\
&= 2(G_b(x,a,a) + G_b(a,y,z)).
\end{aligned}$$

Example 1.11 [47] Let (X,G) be a G-metric space and $G_*(x,y,z) = G(x,y,z)^p$, where $p > 1$ is a real number. Then G_* is a G_b-metric with $s = 2^{p-1}$.

Also, in the above example, (X,G_*) is not necessarily a G-metric space. For example, let $X = \mathbb{R}$ and G-metric G be defined by

$$G(x,y,z) = \frac{1}{3}\bigl(|x-y| + |y-z| + |x-z|\bigr)$$

for all $x,y,z \in \mathbb{R}$ (see [33]). Then $G_*(x,y,z) = G(x,y,z)^2 = \frac{1}{9}(|x-y| + |y-z| + |x-z|)^2$ is a G_b-metric on $\mathbb{R}$ with $s = 2^{2-1} = 2$, but it is not a G-metric on $\mathbb{R}$.

Example 1.12 [47] Let $X = \mathbb{R}$ and $d(x,y) = |x-y|^2$. We know that (X,d) is a b-metric space with $s = 2$. Let $G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$, then (X,G) is not a G_b-metric space. Indeed, (G_b3) is not true for $x = 0$, $y = 2$ and $z = 1$. To see this, we have

$$G(0,0,2) = d(0,0) + d(0,2) + d(2,0) = 2d(0,2) = 8$$

and

$$G(0,2,1) = d(0,2) + d(2,1) + d(1,0) = 4 + 1 + 1 = 6.$$

So, $G(0,0,2) > G(0,2,1)$.

However, $G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$ is a G_b-metric on $\mathbb{R}$ with $s = 2$. Similarly, if $d(x,y) = |x-y|^p$ is selected with $p \ge 1$, then $G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$ is a G_b-metric on $\mathbb{R}$ with $s = 2^{p-1}$.

Now we present some definitions and propositions in a G_b-metric space.

Definition 1.13 [47] A G_b-metric G is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Definition 1.14 Let (X,G) be a G_b-metric space. Then, for $x_0 \in X$ and $r > 0$, the G_b-ball with center x_0 and radius r is

$$B_G(x_0,r) = \{y \in X \mid G(x_0,y,y) < r\}.$$

By some straight forward calculations, we can establish the following.

Proposition 1.15 [47] *Let X be a G_b-metric space. Then, for each $x,y,z,a \in X$, it follows that*:

(1) *if $G(x,y,z) = 0$, then $x = y = z$,*

(2) $G(x,y,z) \leq s(G(x,x,y) + G(x,x,z))$,
(3) $G(x,y,y) \leq 2sG(y,x,x)$,
(4) $G(x,y,z) \leq s(G(x,a,z) + G(a,y,z))$.

Definition 1.16 [47] Let X be a G_b-metric space. We define $d_G(x,y) = G(x,y,y) + G(x,x,y)$ for all $x,y \in X$. It is easy to see that d_G defines a b-metric d on X, which we call the b-metric associated with G.

Proposition 1.17 [47] *Let X be a G_b-metric space. Then, for any $x_0 \in X$ and $r > 0$, if $y \in B_G(x_0,r)$, then there exists $\delta > 0$ such that $B_G(y,\delta) \subseteq B_G(x_0,r)$.*

From the above proposition, the family of all G_b-balls

$$F = \{B_G(x,r) \mid x \in X,\ r > 0\}$$

is a base of a topology $\tau(G)$ on X, which we call the G_b-metric topology.

Now, we generalize Proposition 5 in [34] for a G_b-metric space as follows.

Proposition 1.18 [47] *Let X be a G_b-metric space. Then, for any $x_0 \in X$ and $r > 0$, we have*

$$B_G\left(x_0, \frac{r}{2s+1}\right) \subseteq B_{d_G}(x_0,r) \subseteq B_G(x_0,r).$$

Thus every G_b-metric space is topologically equivalent to a b-metric space. This allows us to readily transport many concepts and results from b-metric spaces into G_b-metric space setting.

Definition 1.19 [47] Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:
(1) G_b-Cauchy if for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m,n,l \geq n_0$, $G(x_n,x_m,x_l) < \varepsilon$;
(2) G_b-convergent to a point $x \in X$ if for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m,n \geq n_0$, $G(x_n,x_m,x) < \varepsilon$.

Proposition 1.20 [47] *Let X be a G_b-metric space. Then the following are equivalent*:
(1) *the sequence $\{x_n\}$ is G_b-Cauchy*;
(2) *for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$ for all $m,n \geq n_0$.*

Proposition 1.21 [47] *Let X be a G_b-metric space. The following are equivalent*:
(1) *$\{x_n\}$ is G_b-convergent to x*;
(2) *$G(x_n,x_n,x) \to 0$ as $n \to +\infty$*;
(3) *$G(x_n,x,x) \to 0$ as $n \to +\infty$.*

Definition 1.22 [47] A G_b-metric space X is called complete if every G_b-Cauchy sequence is G_b-convergent in X.

Definition 1.23 [47] Let (X,G) and (X',G') be two G_b-metric spaces. Then a function $f: X \to X'$ is G_b-continuous at a point $x \in X$ if and only if it is G_b-sequentially continuous at x, that is, whenever $\{x_n\}$ is G_b-convergent to x, $\{f(x_n)\}$ is G'_b-convergent to $f(x)$.

Mustafa and Sims proved that each G-metric function $G(x,y,z)$ is jointly continuous in all three of its variables (see Proposition 8 in [33]). But, in general, a G_b-metric function $G(x,y,z)$ for $s>1$ is not jointly continuous in all its variables. Now, we recall an example of a discontinuous G_b-metric.

Example 1.24 [49] Let $X=\mathbb{N}\cup\{\infty\}$ and let $D:X\times X\to\mathbb{R}$ be defined by

$$D(m,n)=\begin{cases}0 & \text{if } m=n,\\ |\frac{1}{m}-\frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty,\\ 5 & \text{if one of } m,n \text{ is odd and the other is odd (and } m\neq n) \text{ or } \infty,\\ 2 & \text{otherwise.}\end{cases}$$

Then it is easy to see that for all $m,n,p\in X$, we have

$$D(m,p)\leq\frac{5}{2}\big(D(m,n)+D(n,p)\big).$$

Thus, (X,D) is a b-metric space with $s=\frac{5}{2}$ (see corrected Example 3 in [9]).

Let $G(x,y,z)=\max\{D(x,y),D(y,z),D(z,x)\}$. It is easy to see that G is a G_b-metric with $s=5/2$. In [49], it is proved that $G(x,y,z)$ is not a continuous function.

So, from the above discussion, we need the following simple lemma about the G_b-convergent sequences in the proof of our main result.

Lemma 1.25 [49] *Let (X,G) be a G_b-metric space with $s>1$ and suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_b-convergent to x, y and z, respectively. Then we have*

$$\begin{aligned}\frac{1}{s^3}G(x,y,z)&\leq\liminf_{n\longrightarrow\infty}G(x_n,y_n,z_n)\\ &\leq\limsup_{n\longrightarrow\infty}G(x_n,y_n,z_n)\leq s^3G(x,y,z).\end{aligned}$$

In particular, if $x=y=z$, then we have $\lim_{n\longrightarrow\infty}G(x_n,y_n,z_n)=0$.

In this paper, we present some tripled coincidence point results in ordered G_b-metric spaces. Our results extend and generalize the results in [46].

2 Main results

Let $(X,\preceq,G)$ be an ordered G_b-metric space and $F:X^3\to X$ and $g:X\to X$. In the rest of this paper, unless otherwise stated, for all $x,y,z,u,v,w,r,s,t\in X$, let

$$\begin{aligned}M_F(x,y,z,u,v,w,r,s,t)=\max\big\{&G\big(F(x,y,z),F(u,v,w),F(r,s,t)\big),\\ &G\big(F(y,x,y),F(v,u,v),F(s,r,s)\big),\\ &G\big(F(z,y,x),F(w,v,u),F(t,s,r)\big)\big\}\end{aligned}$$

and

$$M_g(x,y,z,u,v,w,r,s,t)=\max\big\{G(gx,gu,gr),G(gy,gv,gs),G(gz,gw,gt)\big\}.$$

Now, the main result is presented as follows.

Theorem 2.1 *Let* $(X, \preceq, G)$ *be a partially ordered* G_b*-metric space and* $F : X^3 \to X$ *and* $g : X \to X$ *be such that* $F(X^3) \subseteq g(X)$. *Assume that*

$$\begin{aligned}&\psi\bigl(sM_F(x,y,z,u,v,w,r,s,t)\bigr)\\&\quad\le \psi\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr) - \varphi\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\end{aligned}\tag{2.1}$$

for every $x,y,z,u,v,w,r,s,t \in X$ *with* $gx \preceq gu \preceq gr$, $gy \succeq gv \succeq gs$ *and* $gz \preceq gw \preceq gt$, *or* $gr \preceq gu \preceq gx$, $gs \succeq gv \succeq gy$ *and* $gt \preceq gw \preceq gz$, *where* $\psi, \varphi : [0,\infty) \to [0,\infty)$ *are altering distance functions.*

Assume that

(1) *F has the mixed g-monotone property.*

(2) *g is* G_b*-continuous and commutes with F.*

Also suppose that

(a) *either F is* G_b*-continuous and* (X, G) *is* G_b*-complete, or*

(b) (X, G) *is regular and* $(g(X), G)$ *is* G_b*-complete.*

If there exist $x_0, y_0, z_0 \in X$ *such that* $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$ *and* $gz_0 \preceq F(z_0, y_0, x_0)$, *then F and g have a tripled coincidence point in X.*

Proof Let $x_0, y_0, z_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$. Define $x_1, y_1, z_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Then $gx_0 \preceq gx_1$, $gy_0 \succeq gy_1$ and $gz_0 \preceq gz_1$. Similarly, define $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, y_1)$ and $gz_2 = F(z_1, y_1, x_1)$. Since F has the mixed g-monotone property, we have $gx_0 \preceq gx_1 \preceq gx_2$, $gy_0 \succeq gy_1 \succeq gy_2$ and $gz_0 \preceq gz_1 \preceq gz_2$.

In this way, we construct the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ as

$$\begin{aligned}a_n &= gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}),\\ b_n &= gy_n = F(y_{n-1}, x_{n-1}, y_{n-1})\end{aligned}$$

and

$$c_n = gz_n = F(z_{n-1}, y_{n-1}, x_{n-1})$$

for all $n \ge 1$.

We will finish the proof in two steps.

Step I. We shall show that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are G_b-Cauchy.

Let

$$\delta_n = \max\bigl\{G(a_{n-1}, a_n, a_n), G(b_{n-1}, b_n, b_n), G(c_{n-1}, c_n, c_n)\bigr\}.$$

So, we have

$$\delta_n = M_F(x_{n-2}, y_{n-2}, z_{n-2}, x_{n-1}, y_{n-1}, z_{n-1}, x_{n-1}, y_{n-1}, z_{n-1})$$

and

$$\delta_n = M_g(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n, x_n, y_n, z_n).$$

As $gx_{n-1} \preceq gx_n$, $gy_{n-1} \succeq gy_n$ and $gz_{n-1} \preceq gz_n$, using (2.1) we obtain that

$$\begin{aligned} \psi(s\delta_{n+1}) &= \psi\bigl(sM_F(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n, x_n, y_n, z_n)\bigr) \\ &\leq \psi\bigl(M_g(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n, x_n, y_n, z_n)\bigr) \\ &\quad - \varphi\bigl(M_g(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n, x_n, y_n, z_n)\bigr) \\ &= \psi(\delta_n) - \varphi(\delta_n) \\ &\leq \psi(s\delta_n) - \varphi(\delta_n). \end{aligned} \tag{2.2}$$

Since ψ is an altering distance function, by (2.2) we deduce that

$$\delta_{n+1} \leq \delta_n,$$

that is, $\{\delta_n\}$ is a nonincreasing sequence of nonnegative real numbers. Thus, there is $r \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = r.$$

Letting $n \to \infty$ in (2.2), from the continuity of ψ and φ, we obtain that

$$\psi(sr) \leq \psi(sr) - \varphi(r),$$

which implies that $\varphi(r) = 0$ and hence $r = 0$.

Next, we claim that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are G_b-Cauchy.

We shall show that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that if $m, n \geq k$,

$$\max\bigl\{G(a_m, a_n, a_n), G(b_m, b_n, b_n), G(c_m, c_n, c_n)\bigr\} < \varepsilon.$$

Suppose that the above statement is false. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{a_{m(k)}\}$ and $\{a_{n(k)}\}$ of $\{a_n\}$, $\{b_{m(k)}\}$ and $\{b_{n(k)}\}$ of $\{b_n\}$ and $\{c_{m(k)}\}$ and $\{c_{n(k)}\}$ of $\{c_n\}$ such that $n(k) > m(k) > k$ and

$$\max\bigl\{G(a_{m(k)}, a_{n(k)}, a_{n(k)}), G(b_{m(k)}, b_{n(k)}, b_{n(k)}), G(c_{m(k)}, c_{n(k)}, c_{n(k)})\bigr\} \geq \varepsilon, \tag{2.3}$$

where $n(k)$ is the smallest index with this property, *i.e.*,

$$\max\bigl\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\bigr\} < \varepsilon. \tag{2.4}$$

From (2.4), we have

$$\begin{aligned} &\limsup_{k\longrightarrow\infty} \max\bigl\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), \\ &\quad G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\bigr\} \leq \varepsilon. \end{aligned} \tag{2.5}$$

From the rectangle inequality,

$$G(a_{m(k)}, a_{n(k)}, a_{n(k)}) \leq s\big[G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}) + G(a_{n(k)-1}, a_{n(k)}, a_{n(k)})\big]. \tag{2.6}$$

Similarly,

$$G(b_{m(k)}, b_{n(k)}, b_{n(k)}) \leq s\big[G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}) + G(b_{n(k)-1}, b_{n(k)}, b_{n(k)})\big] \tag{2.7}$$

and

$$G(c_{m(k)}, c_{n(k)}, c_{n(k)}) \leq s\big[G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1}) + G(c_{n(k)-1}, c_{n(k)}, c_{n(k)})\big]. \tag{2.8}$$

So,

$$\begin{aligned} &\max\big\{G(a_{m(k)}, a_{n(k)}, a_{n(k)}), G(b_{m(k)}, b_{n(k)}, b_{n(k)}), G(c_{m(k)}, c_{n(k)}, c_{n(k)})\big\} \\ &\quad\leq s\max\big\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\big\} \\ &\qquad+ s\max\big\{G(a_{n(k)-1}, a_{n(k)}, a_{n(k)}), G(b_{n(k)-1}, b_{n(k)}, b_{n(k)}), G(c_{n(k)-1}, c_{n(k)}, c_{n(k)})\big\}. \end{aligned} \tag{2.9}$$

Letting $k \to \infty$ as $\lim_{n\to\infty} \delta_n = 0$, by (2.3) and (2.4), we can conclude that

$$\begin{aligned} \frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \max\big\{&G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), \\ &G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\big\}. \end{aligned} \tag{2.10}$$

Since

$$G(a_{m(k)}, a_{n(k)}, a_{n(k)}) \leq sG(a_{m(k)}, a_{m(k)+1}, a_{m(k)+1}) + sG(a_{m(k)+1}, a_{n(k)}, a_{n(k)}) \tag{2.11}$$

and

$$G(b_{m(k)}, b_{n(k)}, b_{n(k)}) \leq sG(b_{m(k)}, b_{m(k)+1}, b_{m(k)+1}) + sG(b_{m(k)+1}, b_{n(k)}, b_{n(k)}), \tag{2.12}$$

and

$$G(c_{m(k)}, c_{n(k)}, c_{n(k)}) \leq sG(c_{m(k)}, c_{m(k)+1}, c_{m(k)+1}) + sG(c_{m(k)+1}, c_{n(k)}, c_{n(k)}), \tag{2.13}$$

we obtain that

$$\begin{aligned} &\max\big\{G(a_{m(k)}, a_{n(k)}, a_{n(k)}), G(b_{m(k)}, b_{n(k)}, b_{n(k)}), G(c_{m(k)}, c_{n(k)}, c_{n(k)})\big\} \\ &\quad\leq s\max\big\{G(a_{m(k)}, a_{m(k)+1}, a_{m(k)+1}), G(b_{m(k)}, b_{m(k)+1}, b_{m(k)+1}), G(c_{m(k)}, c_{m(k)+1}, c_{m(k)+1})\big\} \\ &\qquad+ s\max\big\{G(a_{m(k)+1}, a_{n(k)}, a_{n(k)}), G(b_{m(k)+1}, b_{n(k)}, b_{n(k)}), \\ &\qquad G(c_{m(k)+1}, c_{n(k)}, c_{n(k)})\big\}. \end{aligned} \tag{2.14}$$

If in the above inequality $k \to \infty$ as $\lim_{n\to\infty} \delta_n = 0$, from (2.3) we have

$$\begin{aligned} \frac{\varepsilon}{s} \le \limsup_{k\to\infty} \max\bigl\{&G(a_{m(k)+1}, a_{n(k)}, a_{n(k)}), G(b_{m(k)+1}, b_{n(k)}, b_{n(k)}), \\ &G(c_{m(k)+1}, c_{n(k)}, c_{n(k)})\bigr\}. \end{aligned} \tag{2.15}$$

As $n(k) > m(k)$, we have $gx_{m(k)} \preceq gx_{n(k)-1}$, $gy_{m(k)} \succeq gy_{n(k)-1}$ and $gz_{m(k)} \preceq gz_{n(k)-1}$. Putting $x = x_{m(k)}$, $y = y_{m(k)}$, $z = z_{m(k)}$, $u = x_{n(k)-1}$, $v = y_{n(k)-1}$, $w = z_{n(k)-1}$, $r = x_{n(k)-1}$, $s = y_{n(k)-1}$ and $t = z_{n(k)-1}$ in (2.1), we have

$$\begin{aligned} &\psi\bigl(s \cdot \max\bigl\{G(a_{m(k)+1}, a_{n(k)}, a_{n(k)}), G(b_{m(k)+1}, b_{n(k)}, b_{n(k)}), G(c_{m(k)+1}, c_{n(k)}, c_{n(k)})\bigr\}\bigr) \\ &\quad = \psi\bigl(s \cdot M_F(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1}, x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1})\bigr) \\ &\quad \le \psi\bigl(M_g(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1}, x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1})\bigr) \\ &\qquad - \varphi\bigl(M_g(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1}, x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1})\bigr) \\ &\quad = \psi\bigl(\max\bigl\{G(gx_{m(k)}, gx_{n(k)-1}, gx_{n(k)-1}), G(gy_{m(k)}, gy_{n(k)-1}, gy_{n(k)-1}), \\ &\qquad G(gz_{m(k)}, gz_{n(k)-1}, gz_{n(k)-1})\bigr\}\bigr) \\ &\qquad - \varphi\bigl(\max\bigl\{G(gx_{m(k)}, gx_{n(k)-1}, gx_{n(k)-1}), G(gy_{m(k)}, gy_{n(k)-1}, gy_{n(k)-1}), \\ &\qquad G(gz_{m(k)}, gz_{n(k)-1}, gz_{n(k)-1})\bigr\}\bigr) \\ &\quad = \psi\bigl(\max\bigl\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), \\ &\qquad G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\bigr\}\bigr) \\ &\qquad - \varphi\bigl(\max\bigl\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), \\ &\qquad G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\bigr\}\bigr). \end{aligned} \tag{2.16}$$

Letting $k \to \infty$ in (2.16),

$$\begin{aligned} \psi\left(s \cdot \frac{\varepsilon}{s}\right) &\le \psi\Bigl(s \cdot \limsup_{k\to\infty} \max\bigl\{G(a_{m(k)+1}, a_{n(k)}, a_{n(k)}), G(b_{m(k)+1}, b_{n(k)}, b_{n(k)}), \\ &\qquad G(c_{m(k)+1}, c_{n(k)}, c_{n(k)})\bigr\}\Bigr) \\ &\le \psi\Bigl(\limsup_{k\to\infty} \max\bigl\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), \\ &\qquad G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\bigr\}\Bigr) \\ &\qquad - \varphi\Bigl(\liminf_{k\to\infty} \max\bigl\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), \\ &\qquad G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\bigr\}\Bigr) \\ &\le \psi(\varepsilon) - \varphi\Bigl(\liminf_{k\to\infty} \max\bigl\{G(a_{m(k)}, a_{n(k)-1}, a_{n(k)-1}), G(b_{m(k)}, b_{n(k)-1}, b_{n(k)-1}), \\ &\qquad G(c_{m(k)}, c_{n(k)-1}, c_{n(k)-1})\bigr\}\Bigr). \end{aligned} \tag{2.17}$$

From (2.17), we have

$$\varphi\Big(\liminf_{k\to\infty}\max\big\{G(a_{m(k)},a_{n(k)-1},a_{n(k)-1}),G(b_{m(k)},b_{n(k)-1},b_{n(k)-1}),\\ G(c_{m(k)},c_{n(k)-1},c_{n(k)-1})\big\}\Big)\le 0.$$

Therefore,

$$\liminf_{k\to\infty}\max\big\{G(a_{m(k)},a_{n(k)-1},a_{n(k)-1}),G(b_{m(k)},b_{n(k)-1},b_{n(k)-1}),G(c_{m(k)},c_{n(k)-1},c_{n(k)-1})\big\}=0,$$

which is a contradiction to (2.10). Consequently, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are G_b-Cauchy.

Step II. We shall show that F and g have a tripled coincidence point.

First, let (a) hold, that is, F is G_b-continuous and (X,G) is G_b-complete.

Since X is G_b-complete and $\{a_n\}$ is G_b-Cauchy, there exists $a\in X$ such that

$$\lim_{n\to\infty}G(a_n,a_n,a)=\lim_{n\to\infty}G(gx_n,gx_n,a)=0. \tag{2.18}$$

Similarly, there exist $b,c\in X$ such that

$$\lim_{n\to\infty}G(b_n,b_n,b)=\lim_{n\to\infty}G(gy_n,gy_n,b)=0 \tag{2.19}$$

and

$$\lim_{n\to\infty}G(c_n,c_n,c)=\lim_{n\to\infty}G(gz_n,gz_n,c)=0. \tag{2.20}$$

Now, we prove that (a,b,c) is a tripled coincidence point of F and g.

Continuity of g and Lemma 1.25 yields that

$$\begin{aligned}0&=\frac{1}{s^3}G(ga,ga,ga)\le\lim\inf_{n\to\infty}G\big(g(gx_n),g(gx_n),ga\big)\\&\le\lim\sup_{n\to\infty}G\big(g(gx_n),g(gx_n),ga\big)\le s^3G(ga,ga,ga)=0.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}G\big(g(gx_n),g(gx_n),ga\big)=0 \tag{2.21}$$

and similarly,

$$\lim_{n\to\infty}G\big(g(gy_n),g(gy_n),gb\big)=0 \tag{2.22}$$

and

$$\lim_{n\to\infty}G\big(g(gz_n),g(gz_n),gc\big)=0. \tag{2.23}$$

Since $gx_{n+1}=F(x_n,y_n,z_n)$, $gy_{n+1}=F(y_n,x_n,y_n)$ and $gz_{n+1}=F(z_n,y_n,x_n)$, the commutativity of F and g yields that

$$g(gx_{n+1})=g\big(F(x_n,y_n,z_n)\big)=F(gx_n,gy_n,gz_n), \tag{2.24}$$

$$g(gy_{n+1}) = g\big(F(y_n, x_n, y_n)\big) = F(gy_n, gx_n, gy_n) \tag{2.25}$$

and

$$g(gz_{n+1}) = g\big(F(z_n, y_n, x_n)\big) = F(gz_n, gy_n, gx_n). \tag{2.26}$$

From the continuity of F and (2.24), (2.25) and (2.26) and Lemma 1.25, $\{g(gx_{n+1})\}$ is G_b-convergent to $F(a,b,c)$, $\{g(gy_{n+1})\}$ is G_b-convergent to $F(b,a,b)$ and $\{g(gz_{n+1})\}$ is G_b-convergent to $F(c,b,a)$. From (2.21), (2.22) and (2.23) and uniqueness of the limit, we have $F(a,b,c) = ga$, $F(b,a,b) = gb$ and $F(c,b,a) = gc$, that is, g and F have a tripled coincidence point.

In what follows, suppose that assumption (b) holds.

Following the proof of the previous step, there exist $u, v, w \in X$ such that

$$\lim_{n\to\infty} G(gx_n, gx_n, gu) = 0, \tag{2.27}$$

$$\lim_{n\to\infty} G(gy_n, gy_n, gv) = 0 \tag{2.28}$$

and

$$\lim_{n\to\infty} G(gz_n, gz_n, gw) = 0, \tag{2.29}$$

as $(g(X), G)$ is G_b-complete.

Now, we prove that $F(u,v,w) = gu$, $F(v,u,v) = gv$ and $F(w,v,u) = gw$. From regularity of X and using (2.1), we have

$$\begin{aligned}
&\psi\big(sM_F(x_n, y_n, z_n, u, v, w, u, v, w)\big) \\
&\quad\le \psi\big(\max\big\{G(gx_n, gu, gu), G(gy_n, gv, gv), G(gz_n, gw, gw)\big\}\big) \\
&\qquad - \varphi\big(\max\big\{G(gx_n, gu, gu), G(gy_n, gv, gv), G(gz_n, gw, gw)\big\}\big).
\end{aligned} \tag{2.30}$$

As $\{gx_n\}$ is G_b-convergent to gu, from Lemma 1.25, we have $\lim_{n\to\infty} G(gx_n, gu, gu) = 0$. Analogously, $\lim_{n\to\infty} G(gy_n, gv, gv) = \lim_{n\to\infty} G(gz_n, gw, gw) = 0$.

As ψ and φ are continuous, from (2.30) we have

$$\lim_{n\to\infty} M_F(x_n, y_n, z_n, u, v, w, u, v, w) = 0,$$

or, equivalently,

$$\lim_{n\to\infty} G\big(gx_{n+1}, F(u,v,w), F(u,v,w)\big) = 0. \tag{2.31}$$

Similarly,

$$\lim_{n\to\infty} G\big(gy_{n+1}, F(v,u,v), F(v,u,v)\big) = \lim_{n\to\infty} G\big(gz_{n+1}, F(w,v,u), F(w,v,u)\big) = 0. \tag{2.32}$$

On the other hand,

$$\begin{aligned}
&G(gu, F(u,v,w), F(u,v,w) \\
&\quad\le sG(gu, gx_{n+1}, gx_{n+1}) + sG(gx_{n+1}, F(u,v,w), F(u,v,w).
\end{aligned} \tag{2.33}$$

Taking the limit when $n \to \infty$ and using (2.27) and (2.31), we get

$$\begin{aligned} G\big(gu, F(u,v,w), F(u,v,w)\big) &\le s \lim_{n\to\infty} G(gu, gx_{n+1}, gx_{n+1}) \\ &\quad + s \lim_{n\to\infty} G(gx_{n+1}, F(u,v,w), F(u,v,w)) = 0, \end{aligned} \tag{2.34}$$

that is, $gu = F(u,v,w)$.

Analogously, we can show that $gv = F(v,u,v)$ and $gw = F(w,v,u)$.

Thus, we have proved that g and F have a tripled coincidence point. This completes the proof of the theorem. □

Let

$$M(x,y,z,u,v,w,r,s,t) = \max\big\{G(x,u,r), G(y,v,s), G(z,w,t)\big\}.$$

Taking $g = I_X$ (the identity mapping on X) in Theorem 2.1, we obtain the following tripled fixed point result.

Corollary 2.2 *Let $(X,\preceq,G)$ be a G_b-complete partially ordered G_b-metric space, and let $F : X^3 \to X$ be a mapping with the mixed monotone property. Assume that*

$$\begin{aligned} &\psi\big(sM_F(x,y,z,u,v,w,r,s,t)\big) \\ &\quad \le \psi\big(M(x,y,z,u,v,w,r,s,t)\big) - \varphi\big(M(x,y,z,u,v,w,r,s,t)\big) \end{aligned} \tag{2.35}$$

for every $x,y,z,u,v,w,r,s,t \in X$ with $x \preceq u \preceq r$, $y \succeq v \succeq s$ and $z \preceq w \preceq t$, or $r \preceq u \preceq x$, $s \succeq v \succeq y$ and $t \preceq w \preceq z$, where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are altering distance functions.

Also suppose that

(a) *either F is G_b-continuous, or*

(b) *(X,G) is regular.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0,y_0,z_0)$, $y_0 \succeq F(y_0,x_0,y_0)$ and $z_0 \preceq F(z_0,y_0,x_0)$, then F has a tripled fixed point in X.

Taking $\psi(t) = t$ and $\varphi(t) = \frac{t^2}{1+t}$ for all $t \in [0,\infty)$ in Corollary 2.2, we obtain the following tripled fixed point result.

Corollary 2.3 *Let $(X,\preceq,G)$ be a G_b-complete partially ordered G_b-metric space and $F : X^3 \to X$ with the mixed monotone property. Assume that*

$$sM_F(x,y,z,u,v,w,r,s,t) \le \frac{M(x,y,z,u,v,w,r,s,t)}{1 + M(x,y,z,u,v,w,r,s,t)} \tag{2.36}$$

for every $x,y,z,u,v,w,r,s,t \in X$ with $x \preceq u \preceq r$, $y \succeq v \succeq s$ and $z \preceq w \preceq t$, or $r \preceq u \preceq x$, $s \succeq v \succeq y$ and $t \preceq w \preceq z$.

Also suppose that

(a) *either F is G_b-continuous, or*

(b) *(X,G) is regular.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0,y_0,z_0)$, $y_0 \succeq F(y_0,x_0,y_0)$ and $z_0 \preceq F(z_0,y_0,x_0)$, then F has a tripled fixed point in X.

Remark 2.4 Theorem 1.8 is a special case of Theorem 2.1.

Remark 2.5 Theorem 2.1 part (a) holds if we replace the commutativity assumption of F and g by compatibility assumption (also see Remark 2.2 of [30]).

The following corollary can be deduced from our previously obtained results.

Corollary 2.6 *Let $(X,\preceq)$ be a partially ordered set and (X,G) be a G_b-complete G_b-metric space. Let $F:X^3\to X$ be a mapping with the mixed monotone property such that*

$$\psi\bigl(sM_F(x,y,z,u,v,w,r,s,t)\bigr)\le\psi\left(\frac{G(x,u,r)+G(y,v,s)+G(z,w,t)}{3}\right)-\varphi\bigl(\max\bigl\{G(x,u,r),G(y,v,s),G(z,w,t)\bigr\}\bigr) \tag{2.37}$$

for every $x,y,z,u,v,w,r,s,t\in X$ with $x\preceq u\preceq r$, $y\succeq v\succeq s$ and $z\preceq w\preceq t$, or $r\preceq u\preceq x$, $s\succeq v\succeq y$ and $t\preceq w\preceq z$.

Also suppose that

(a) *either F is G_b-continuous, or*

(b) *(X,G) is regular.*

If there exist $x_0,y_0,z_0\in X$ such that $x_0\preceq F(x_0,y_0,z_0)$, $y_0\succeq F(y_0,x_0,y_0)$ and $z_0\preceq F(z_0,y_0,x_0)$, then F has a tripled fixed point in X.

Proof If F satisfies (2.37), then F satisfies (2.35). So, the result follows from Theorem 2.1. □

In Theorem 2.1, if we take $\psi(t)=t$ and $\varphi(t)=(1-k)t$ for all $t\in[0,\infty)$, where $k\in[0,1)$, we obtain the following result.

Corollary 2.7 *Let $(X,\preceq)$ be a partially ordered set and (X,G) be a G_b-complete G_b-metric space. Let $F:X^3\to X$ be a mapping having the mixed monotone property and*

$$M_F(x,y,z,u,v,w,r,s,t)\le\frac{k}{s}M(x,y,z,u,v,w,r,s,t) \tag{2.38}$$

for every $x,y,z,u,v,w,r,s,t\in X$ with $x\preceq u\preceq r$, $y\succeq v\succeq s$ and $z\preceq w\preceq t$, or $r\preceq u\preceq x$, $s\succeq v\succeq y$ and $t\preceq w\preceq z$.

Also suppose that

(a) *either F is G_b-continuous, or*

(b) *(X,G) is regular.*

If there exist $x_0,y_0,z_0\in X$ such that $x_0\preceq F(x_0,y_0,z_0)$, $y_0\succeq F(y_0,x_0,y_0)$ and $z_0\preceq F(z_0,y_0,x_0)$, then F has a tripled fixed point in X.

Corollary 2.8 *Let $(X,\preceq)$ be a partially ordered set and (X,G) be a G_b-complete G_b-metric space. Let $F:X^3\to X$ be a mapping with the mixed monotone property such that*

$$M_F(x,y,z,u,v,w,r,s,t)\le\frac{k}{3s}\bigl[G(x,u,r)+G(y,v,s)+G(z,w,t)\bigr] \tag{2.39}$$

for every $x,y,z,u,v,w,r,s,t\in X$ with $x\preceq u\preceq r$, $y\succeq v\succeq s$ and $z\preceq w\preceq t$, or $r\preceq u\preceq x$, $s\succeq v\succeq y$ and $t\preceq w\preceq z$.

Also suppose that

(a) *either F is G_b-continuous, or*

(b) (X, G) *is regular.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$, then F has a tripled fixed point in X.

Proof If F satisfies (2.39), then F satisfies (2.38). □

Note that if $(X, \preceq)$ is a partially ordered set, then we can endow X^3 with the following partial order relation:

$$(x,y,z) \preceq (u,v,w) \quad \Longleftrightarrow \quad x \preceq u, \qquad y \succeq v, \qquad z \preceq w$$

for all $(x,y,z), (u,v,w) \in X^3$ (see [26]).

In the following theorem, we give a sufficient condition for the uniqueness of the common tripled fixed point (also see, *e.g.*, [4, 46, 50] and [51]).

Theorem 2.9 *In addition to the hypotheses of Theorem* 2.1, *suppose that for every* (x,y,z) *and* $(x^*, y^*, z^*) \in X \times X \times X$, *there exists* $(u,v,w) \in X^3$ *such that* $(F(u,v,w), F(v,u,v), F(w,v,u))$ *is comparable with* $(F(x,y,z), F(y,x,y), F(z,y,x))$ *and* $(F(x^*,y^*,z^*), F(y^*,x^*,y^*), F(z^*,y^*,x^*))$. *Then F and g have a unique common tripled fixed point.*

Proof From Theorem 2.1 the set of tripled coincidence points of F and g is nonempty. We shall show that if (x,y,z) and (x^*,y^*,z^*) are tripled coincidence points, that is,

$$g(x) = F(x,y,z), \qquad g(y) = F(y,x,y), \qquad g(z) = F(z,y,x)$$

and

$$g(x^*) = F(x^*,y^*,z^*), \qquad g(y^*) = F(y^*,x^*,y^*), \qquad g(z^*) = F(z^*,y^*,x^*),$$

then $gx = gx^*$ and $gy = gy^*$ and $gz = gz^*$.

Choose an element $(u,v,w) \in X^3$ such that $(F(u,v,w), F(v,u,v), F(w,v,u))$ is comparable with

$$\big(F(x,y,z), F(y,x,y), F(z,y,x)\big)$$

and

$$\big(F(x^*,y^*,z^*), F(y^*,x^*,y^*), F(z^*,y^*,x^*)\big).$$

Let $u_0 = u$, $v_0 = v$ and $w_0 = w$ and choose u_1, v_1 and $w_1 \in X$ so that $gu_1 = F(u_0, v_0, w_0)$, $gv_1 = F(v_0, u_0, v_0)$ and $gw_1 = F(w_0, v_0, u_0)$. Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{gu_n\}$, $\{gv_n\}$ and $\{gw_n\}$ such that $gu_{n+1} = F(u_n, v_n, w_n)$, $gv_{n+1} = F(v_n, u_n, v_n)$ and $gw_{n+1} = F(w_n, v_n, u_n)$. Since $(gx, gy, gz) = (F(x,y,z), F(y,x,y), F(w,y,x))$ and $(F(u,v,w), F(v,u,v), F(w,v,u)) = (gu_1, gv_1, gw_1)$ are comparable, we may assume that $(gx, gy, gz) \preceq (gu_1, gv_1, gw_1)$. Then $gx \preceq gu_1$, $gy \succeq gv_1$ and $gz \preceq gw_1$. Using the mathematical induction, it is easy to prove that $gx \preceq gu_n$, $gy \succeq gv_n$ and $gz \preceq gw_n$ for all $n \geq 0$.

Applying (2.1), as $gx \preceq gu_n$, $gy \succeq gv_n$ and $gz \preceq gw_n$, one obtains that

$$\begin{aligned}
&\psi\left(s\max\left\{G(gx,gu_{n+1},gu_{n+1}),G(gy,gv_{n+1},gv_{n+1}),G(gz,gw_{n+1},gw_{n+1})\right\}\right)\\
&\quad=\psi\left(sM_F(x,y,z,u_n,v_n,w_n,u_n,v_n,w_n)\right)\\
&\quad\le\psi\left(M_g(x,y,z,u_n,v_n,w_n,u_n,v_n,w_n)\right)-\varphi\left(M_g(x,y,z,u_n,v_n,w_n,u_n,v_n,w_n)\right)\\
&\quad=\psi\left(\max\left\{G(gx,gu_n,gu_n),G(gy,gv_n,gv_n),G(gz,gw_n,gw_n)\right\}\right)\\
&\qquad-\varphi\left(\max\left\{G(gx,gu_n,gu_n),G(gy,gv_n,gv_n),G(gz,gw_n,gw_n)\right\}\right).
\end{aligned}\tag{2.40}$$

From the properties of ψ, we deduce that

$$\left\{\max\left\{G(gx,gu_n,gu_n),G(gy,gv_n,gv_n),G(gz,gw_n,gw_n)\right\}\right\}$$

is nonincreasing.

Hence, if we proceed as in Theorem 2.1, we can show that

$$\lim_{n\to\infty}\max\left\{G(gx,gu_n,gu_n),G(gy,gv_n,gv_n),G(gz,gw_n,gw_n)\right\}=0,$$

that is, $\{gu_n\}$, $\{gv_n\}$ and $\{gw_n\}$ are G_b-convergent to gx, gy and gz, respectively.

Similarly, we can show that

$$\lim_{n\to\infty}\max\left\{G\left(gx^*,gu_n,gu_n\right),G\left(gy^*,gv_n,gv_n\right),G\left(gz^*,gw_n,gw_n\right)\right\}=0,$$

that is, $\{gu_n\}$, $\{gv_n\}$ and $\{gw_n\}$ are G_b-convergent to gx^*, gy^* and gz^*, respectively. Finally, since the limit is unique, $gx = gx^*$, $gy = gy^*$ and $gz = gz^*$.

Since $gx = F(x,y,z)$, $gy = F(y,x,y)$ and $gz = F(z,y,x)$, by commutativity of F and g, we have $g(gx) = g(F(x,y,z)) = F(gx,gy,gz)$, $g(gy) = g(F(y,x,y)) = F(gy,gx,gy)$ and $g(gz) = g(F(z,y,x)) = F(gz,gy,gx)$. Let $gx = a$, $gy = b$ and $g(z) = c$. Then $ga = F(a,b,c)$, $gb = F(b,a,b)$ and $gc = F(c,b,a)$. Thus, (a,b,c) is another tripled coincidence point of F and g. Then $a = gx = ga$, $b = gy = gb$ and $c = gz = gc$. Therefore, (a,b,c) is a tripled common fixed point of F and g.

To prove the uniqueness, assume that (p,q,r) is another tripled common fixed point of F and g. Then $p = gp = F(p,q,r)$, $q = gq = F(q,p,q)$ and $r = gr = F(r,p,q)$. Since (p,q,r) is a tripled coincidence point of F and g, we have $gp = gx$, $gq = gy$ and $gr = gz$. Thus, $p = gp = ga = a$, $q = gq = gb = b$ and $r = gr = gc = c$. Hence, the tripled common fixed point is unique. □

3 Examples

The following examples support our results.

Example 3.1 Let $X = (-\infty,\infty)$ be endowed with the usual ordering and the G_b-complete G_b-metric

$$G(x,y,z)=\left(|x-y|+|y-z|+|z-x|\right)^2,$$

where $s = 2$.

Define $F : X^3 \to X$ as

$$F(x,y,z) = \frac{x - 2y + 4z}{96}$$

for all $x, y, z \in X$ and $g : X \to X$ with $g(x) = \frac{x}{2}$ for all $x \in X$.

Let $\varphi : [0,\infty) \to [0,\infty)$ be defined by $\varphi(t) = \ln(t + 1)$, and let $\psi : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = \ln(\frac{4t+4}{t+4})$.

Now, from the fact that for $\alpha, \beta, \gamma \geq 0$, $(\alpha + \beta + \gamma)^p \leq 2^{2p-2}\alpha^p + 2^{2p-2}\beta^p + 2^{p-1}\gamma^p$, we have

$$\begin{aligned}
&\psi\bigl(sG\bigl(F(x,y,z), F(u,v,w), F(r,s,t)\bigr)\bigr)\\
&\quad = \ln\left(\begin{array}{c} 2(\frac{1}{96}[|(x-2y+4z)-(u-2v+4w)|] + \frac{1}{96}[|(u-2v+4w)-(r-2s+4t)|] \\ + \frac{1}{96}[|(r-2s+4t)-(x-2y+4z)|])^2 + 1 \end{array}\right)\\
&\quad \leq \ln\left(\begin{array}{c} 2(\frac{1}{48}|\frac{x}{2}-\frac{u}{2}| + \frac{1}{24}|\frac{y}{2}-\frac{v}{2}| + \frac{1}{12}|\frac{z}{2}-\frac{w}{2}| + \frac{1}{48}|\frac{u}{2}-\frac{r}{2}| + \frac{1}{24}|\frac{v}{2}-\frac{s}{2}| \\ + \frac{1}{12}|\frac{w}{2}-\frac{t}{2}| + \frac{1}{48}|\frac{r}{2}-\frac{x}{2}| + \frac{1}{24}|\frac{s}{2}-\frac{y}{2}| + \frac{1}{12}|\frac{t}{2}-\frac{z}{2}|)^2 + 1 \end{array}\right)\\
&\quad = \ln\left(\begin{array}{c} 2(\frac{1}{48}[|\frac{x}{2}-\frac{u}{2}| + |\frac{u}{2}-\frac{r}{2}| + |\frac{r}{2}-\frac{x}{2}|] + \frac{1}{24}[|\frac{y}{2}-\frac{v}{2}| + |\frac{v}{2}-\frac{s}{2}| + |\frac{s}{2}-\frac{y}{2}|] \\ + \frac{1}{12}[|\frac{z}{2}-\frac{w}{2}| + |\frac{w}{2}-\frac{t}{2}| + |\frac{t}{2}-\frac{z}{2}|])^2 + 1 \end{array}\right)\\
&\quad \leq \ln\left(\begin{array}{c} \frac{8}{48^2}([|\frac{x}{2}-\frac{u}{2}| + |\frac{u}{2}-\frac{r}{2}| + |\frac{r}{2}-\frac{x}{2}|]^2 + \frac{8}{24^2}[|\frac{y}{2}-\frac{v}{2}| + |\frac{v}{2}-\frac{s}{2}| + |\frac{s}{2}-\frac{y}{2}|]^2 \\ + \frac{4}{12^2}[|\frac{z}{2}-\frac{w}{2}| + |\frac{w}{2}-\frac{t}{2}| + |\frac{t}{2}-\frac{z}{2}|]^2) + 1 \end{array}\right)\\
&\quad \leq \ln\left(\begin{array}{c} \frac{1}{12}([|\frac{x}{2}-\frac{u}{2}| + |\frac{u}{2}-\frac{r}{2}| + |\frac{r}{2}-\frac{x}{2}|]^2 + \frac{1}{12}[|\frac{y}{2}-\frac{v}{2}| + |\frac{v}{2}-\frac{s}{2}| + |\frac{s}{2}-\frac{y}{2}|]^2 \\ + \frac{1}{12}[|\frac{z}{2}-\frac{w}{2}| + |\frac{w}{2}-\frac{t}{2}| + |\frac{t}{2}-\frac{z}{2}|]^2) + 1 \end{array}\right)\\
&\quad \leq \ln\left(\begin{array}{c} \frac{1}{4}\max\{[|\frac{x}{2}-\frac{u}{2}| + |\frac{u}{2}-\frac{r}{2}| + |\frac{r}{2}-\frac{x}{2}|]^2, [|\frac{y}{2}-\frac{v}{2}| + |\frac{v}{2}-\frac{s}{2}| + |\frac{s}{2}-\frac{y}{2}|]^2, \\ [|\frac{z}{2}-\frac{w}{2}| + |\frac{w}{2}-\frac{t}{2}| + |\frac{t}{2}-\frac{z}{2}|]^2\} + 1 \end{array}\right)\\
&\quad \leq \ln\biggl(\frac{1}{4}\max\bigl\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\bigr\} + 1\biggr)\\
&\quad = \ln\bigl(\max\bigl\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\bigr\} + 1\bigr)\\
&\qquad - \ln\biggl(\frac{4\max\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\} + 4}{\max\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\} + 4}\biggr)\\
&\quad = \psi\bigl(\max\bigl\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\bigr\}\bigr)\\
&\qquad - \varphi\bigl(\max\bigl\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\bigr\}\bigr).
\end{aligned}$$

Analogously, we can show that

$$\begin{aligned}
&\psi\bigl(G\bigl(F(y,x,y), F(v,u,v), F(s,r,s)\bigr)\bigr)\\
&\quad \leq \psi\bigl(\max\bigl\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\bigr\}\bigr)\\
&\qquad - \varphi\bigl(\max\bigl\{G(gx,gu,gr), G(gy,gv,gs), G(gz,gw,gt)\bigr\}\bigr)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\bigl(G\bigl(F(z,y,x),F(w,v,u),F(t,s,r)\bigr)\bigr)\\
&\quad\le \psi\bigl(\max\bigl\{G(gx,gu,gr),G(gy,gv,gs),G(gz,gw,gt)\bigr\}\bigr)\\
&\qquad-\varphi\bigl(\max\bigl\{G(gx,gu,gr),G(gy,gv,gs),G(gz,gw,gt)\bigr\}\bigr).
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi\bigl(sM_F(x,y,z,u,v,w,r,s,t)\bigr) &\le \psi\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\\
&\quad-\varphi\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr).
\end{aligned}$$

Hence, all of the conditions of Theorem 2.1 are satisfied. Moreover, $(0,0,0)$ is the unique common tripled fixed point of F and g.

The following example has been constructed according to Example 2.12 of [2].

Example 3.2 Let $X=\{(x,0,x)\}\cup\{(0,x,0)\}\subset R^3$, where $x\in[0,\infty]$ with the order $\preceq$ defined as

$$(x_1,y_1,z_1)\preceq(x_2,y_2,z_2)\quad\Longleftrightarrow\quad x_1\le x_2,\qquad y_1\le y_2,\qquad z_1\le z_2.$$

Let d be given as

$$d(x,y)=\max\bigl\{|x_1-x_2|^2,|y_1-y_2|^2,|z_1-z_2|^2\bigr\}$$

and

$$G(x,y,z)=\max\bigl\{d(x,y),d(y,z),d(z,x)\bigr\},$$

where $x=(x_1,y_1,z_1)$ and $y=(x_2,y_2,z_2)$. (X,G) is, clearly, a G_b-complete G_b-metric space.

Let $g:X\to X$ and $F:X^3\to X$ be defined as follows:

$$F(x,y,z)=x$$

and

$$g\bigl((x,0,x)\bigr)=(0,x,0)\quad\text{and}\quad g\bigl((0,x,0)\bigr)=(x,0,x).$$

Let $\psi,\varphi:[0,\infty)\to[0,\infty)$ be as in the above example.

According to the order on X and the definition of g, we see that for any element $x\in X$, $g(x)$ is comparable only with itself.

By a careful computation, it is easy to see that all of the conditions of Theorem 2.1 are satisfied. Finally, Theorem 2.1 guarantees the existence of a unique common tripled fixed point for F and g, *i.e.*, the point $((0,0,0),(0,0,0),(0,0,0))$.

4 Applications

In this section, we obtain some tripled coincidence point theorems for a mapping satisfying a contractive condition of integral type in a complete ordered G_b-metric space.

We denote by Λ the set of all functions $\mu : [0,+\infty) \to [0,+\infty)$ verifying the following conditions:

(I) μ is a positive Lebesgue integrable mapping on each compact subset of $[0,+\infty)$.
(II) For all $\varepsilon > 0$, $\int_0^\varepsilon \mu(t)\,dt > 0$.

Corollary 4.1 *Replace the contractive condition* (2.1) *of Theorem* 2.1 *by the following condition*:

There exists $\mu \in \Lambda$ *such that*

$$\begin{aligned}&\int_0^{\psi(sM_F(x,y,z,u,v,w,r,s,t))} \mu(t)\,dt\\&\quad\le \int_0^{\psi(M_g(x,y,z,u,v,w,r,s,t))} \mu(t)\,dt - \int_0^{\varphi(M_g(x,y,z,u,v,w,r,s,t))} \mu(t)\,dt.\end{aligned}\tag{4.1}$$

If the other conditions of Theorem 2.1 *are satisfied, then F and g have a tripled coincidence point.*

Proof Consider the function $\Gamma(x) = \int_0^x \mu(t)\,dt$. Then (4.1) becomes

$$\begin{aligned}&\Gamma\bigl(\psi\bigl(sM_F(x,y,z,u,v,w,r,s,t)\bigr)\bigr)\\&\quad\le \Gamma\bigl(\psi\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\bigr) - \Gamma\bigl(\varphi\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\bigr).\end{aligned}$$

Taking $\psi_1 = \Gamma o \psi$ and $\varphi_1 = \Gamma o \varphi$ and applying Theorem 2.1, we obtain the proof (it is easy to verify that ψ_1 and φ_1 are altering distance functions). □

Corollary 4.2 *Substitute the contractive condition* (2.1) *of Theorem* 2.1 *by the following condition*:

There exists $\mu \in \Lambda$ *such that*

$$\begin{aligned}&\psi\biggl(\int_0^{sM_F(x,y,z,u,v,w,r,s,t)} \mu(t)\,dt\biggr)\\&\quad\le \psi\biggl(\int_0^{M_g(x,y,z,u,v,w,r,s,t)} \mu(t)\,dt\biggr) - \varphi\biggl(\int_0^{M_g(x,y,z,u,v,w,r,s,t)} \mu(t)\,dt\biggr).\end{aligned}\tag{4.2}$$

If the other conditions of Theorem 2.1 *are satisfied, then F and g have a tripled coincidence point.*

Proof Again, as in Corollary 4.1, define the function $\Gamma(x) = \int_0^x \mu(t)\,dt$. Then (4.2) changes to

$$\begin{aligned}\psi\bigl(\Gamma\bigl(sM_F(x,y,z,u,v,w,r,s,t)\bigr)\bigr) &\le \psi\bigl(\Gamma\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\bigr)\\&\quad - \varphi\bigl(\Gamma\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\bigr).\end{aligned}$$

Now, if we define $\psi_1 = \psi o\Gamma$ and $\varphi_1 = \varphi o\Gamma$ and apply Theorem 2.1, then the proof is completed. □

Corollary 4.3 *Replace the contractive condition* (2.1) *of Theorem* 2.1 *by the following condition*:

There exists $\mu \in \Lambda$ *such that*

$$\psi_1\left(\int_0^{\psi_2(sM_F(x,y,z,u,v,w,r,s,t))}\mu(t)\,dt\right) \le \psi_1\left(\int_0^{\psi_2(M_g(x,y,z,u,v,w,r,s,t))}\mu(t)\,dt\right) - \varphi_1\left(\int_0^{\varphi_2(M_g(x,y,z,u,v,w,r,s,t))}\mu(t)\,dt\right) \tag{4.3}$$

for altering distance functions ψ_1, ψ_2, φ_1 *and* φ_2. *If the other conditions of Theorem* 2.1 *are satisfied, then* F *and* g *have a tripled coincidence point.*

Similar to [52], let $N \in \mathbb{N}$ be fixed. Let $\{\mu_i\}_{1\le i\le N}$ be a family of N functions which belong to Λ. For all $t \ge 0$, we define

$$\begin{aligned}
I_1(t) &= \int_0^t \mu_1(s)\,ds,\\
I_2(t) &= \int_0^{I_1 t} \mu_2(s)\,ds = \int_0^{\int_0^t \mu_1(s)\,ds} \mu_2(s)\,ds,\\
I_3(t) &= \int_0^{I_2 t} \mu_3(s)\,ds = \int_0^{\int_0^{\int_0^t \mu_1(s)\,ds} \mu_2(s)\,ds} \mu_3(s)\,ds,\\
&\cdots,\\
I_N(t) &= \int_0^{I_{(N-1)}t} \mu_N(s)\,ds.
\end{aligned}$$

We have the following result.

Corollary 4.4 *Replace inequality* (2.1) *of Theorem* 2.1 *by the following condition*:

$$\begin{aligned}
\psi\bigl(I_N\bigl(sM_F(x,y,z,u,v,w,r,s,t)\bigr)\bigr) &\le \psi\bigl(I_N\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\bigr)\\
&\quad - \varphi\bigl(I_N\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\bigr).
\end{aligned} \tag{4.4}$$

If the other conditions of Theorem 2.1 *are satisfied, then* F *and* g *have a tripled coincidence point.*

Proof Consider $\hat{\Psi} = \psi o I_N$ and $\hat{\Phi} = \varphi o I_N$. Then the above inequality becomes

$$\begin{aligned}
\hat{\Psi}\bigl(sM_F(x,y,z,u,v,w,r,s,t)\bigr) &\le \hat{\Psi}\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr)\\
&\quad - \hat{\Phi}\bigl(M_g(x,y,z,u,v,w,r,s,t)\bigr).
\end{aligned}$$

Applying Theorem 2.1, we obtain the desired result (it is easy to verify that $\hat{\Psi}$ and $\hat{\Phi}$ are altering distance functions). □

Another consequence of the main theorem is the following result.

Corollary 4.5 *Substitute contractive condition* (2.1) *of Theorem* 2.1 *by the following condition*:

There exist $\mu_1, \mu_2 \in \Lambda$ *such that*

$$\begin{aligned} &\int_0^{sM_F(x,y,z,u,v,w,r,s,t)} \mu_1(t)\,dt \\ &\quad \leq \int_0^{M_g(x,y,z,u,v,w,r,s,t)} \mu_1(t)\,dt - \int_0^{M_g(x,y,z,u,v,w,r,s,t)} \mu_2(t)\,dt. \end{aligned}$$

If the other conditions of Theorem 2.1 *are satisfied, then* F *and* g *have a tripled coincidence point*.

Proof It is clear that the function $s \to \int_0^s \mu_i(t)\,dt$ for $i = 1, 2$ is an altering distance function. □

Motivated by [46], we study the existence of solutions for nonlinear integral equations using the results proved in the previous section.

Consider the integral equations in the following system.

$$\begin{aligned} x(t) &= \omega(t) + \int_0^T S(t,r)\big[f\big(r, x(r)\big) + k\big(r, y(r)\big) + h\big(r, z(r)\big)\big]\,dr, \\ y(t) &= \omega(t) + \int_0^T S(t,r)\big[f\big(r, y(r)\big) + k\big(r, x(r)\big) + h\big(r, y(r)\big)\big]\,dr, \\ z(t) &= \omega(t) + \int_0^T S(t,r)\big[f\big(\lambda, z(r)\big) + k\big(r, y(r)\big) + h\big(r, x(r)\big)\big]\,dr. \end{aligned} \tag{4.5}$$

We will consider system (4.5) under the following assumptions:

(i) $f, k, h : [0, T] \times R \to R$ are continuous,

(ii) $\omega : [0, T] \to R$ is continuous,

(iii) $S : [0, T] \times R \to [0, \infty)$ is continuous,

(iv) there exists $q > 0$ such that for all $x, y \in R$,

$$\begin{aligned} &0 \leq f(r, y) - f(r, x) \leq q(y - x), \\ &0 \leq k(r, x) - k(r, y) \leq q(y - x) \end{aligned}$$

and

$$0 \leq h(r, y) - h(r, x) \leq q(y - x).$$

(v) We suppose that

$$2^{3p-3} 3q^p \sup_{t \in [0,T]} \left(\int_0^T \big|S(t,r)\big|\,dr \right)^p < 1.$$

(vi) There exist continuous functions $\alpha, \beta, \gamma : [0, T] \to R$ such that

$$\alpha(t) \leq \omega(t) + \int_0^T S(t,r)\big[f\big(r,\alpha(r)\big) + k\big(r,\beta(r)\big) + h\big(r,\gamma(r)\big)\big]\,dr,$$

$$\beta(t) \geq \omega(t) + \int_0^T S(t,r)\big[f\big(r,\beta(r)\big) + k\big(r,\alpha(r)\big) + h\big(r,\beta(r)\big)\big]\,dr$$

and

$$\gamma(t) \leq \omega(t) + \int_0^T S(t,r)\big[f\big(r,\gamma(r)\big) + k\big(r,\beta(r)\big) + h\big(r,\alpha(r)\big)\big]\,dr.$$

We consider the space $X = C([0,T],R)$ of continuous functions defined on $[0,T]$ endowed with the G_b-metric given by

$$G(\theta,\varphi,\psi) = \Big(\max_{t\in[0,T]} \big|\theta(t) - \varphi(t)\big|^p, \max_{t\in[0,T]} \big|\varphi(t) - \psi(t)\big|^p, \max_{t\in[0,T]} \big|\psi(t) - \theta(t)\big|^p\Big)$$

for all $\theta, \varphi, \psi \in X$, where $s = 2^{p-1}$ and $p \geq 1$ (see Example 1.12).

We endow X with the partial ordered $\preceq$ given by

$$x \preceq y \quad \Longleftrightarrow \quad x(t) \leq y(t), \quad \text{for all } t \in [0,T].$$

On the other hand, (X,d) is regular [53].

Our result is the following.

Theorem 4.6 *Under assumptions* (i)-(vi), *system* (4.5) *has a solution in* X^3 *where* $X = (C[0,T],\mathbb{R})$.

Proof As in [46], we consider the operators $F : X^3 \to X$ and $g : X \to X$ defined by

$$F(x_1,x_2,x_3)(t) = \omega(t) + \int_0^T S(t,r)\big[f\big(r,x_1(r)\big) + k\big(r,x_2(r)\big) + h\big(r,x_3(r)\big)\big]\,dr$$

and

$$g(x) = x$$

for all $t \in [0,T]$ and $x_1, x_2, x_3, x \in X$.

F has the mixed monotone property (see Theorem 25 of [46]).

Let $x, y, z, u, v, w \in X$ be such that $x \geq u$, $y \leq v$ and $z \geq w$. Since F has the mixed monotone property, we have

$$F(u,v,w) \leq F(x,y,z).$$

On the other hand,

$$G\big(F(x,y,z),F(u,v,w),F(a,b,c)\big) = \max\left\{\begin{array}{l} \max_{t\in[0,T]} |F(x,y,z)(t) - F(u,v,w)(t)|^p, \\ \max_{t\in[0,T]} |F(u,v,w)(t) - F(a,b,c)(t)|^p, \\ \max_{t\in[0,T]} |F(a,b,c)(t) - F(x,y,z)(t)|^p \end{array}\right\}.$$

Now, for all $t \in [0,T]$ from (iv) and the fact that for $\alpha, \beta, \gamma \geq 0$, $(\alpha + \beta + \gamma)^p \leq 2^{2p-2}\alpha^p + 2^{2p-2}\beta^p + 2^{p-1}\gamma^p$, we have

$$\begin{aligned}
&\bigl|F(x,y,z)(t) - F(u,v,w)(t)\bigr|^p \\
&\quad = \left| \begin{array}{c} \int_0^T S(t,r)[f(r,x(r)) - f(r,u(r))]\,dr \\ + \int_0^T S(t,r)[k(r,y(r)) - k(r,v(r))]\,dr \\ + \int_0^T S(t,r)[h(r,z(r)) - h(r,w(r))]\,dr \end{array} \right|^p \\
&\quad \leq \left(\begin{array}{c} |\int_0^T S(t,r)[f(r,x(r)) - f(r,u(r))]\,dr| \\ + |\int_0^T S(t,r)[k(r,y(r)) - k(r,v(r))]\,dr| \\ + |\int_0^T S(t,r)[h(r,z(r)) - h(r,w(r))]\,dr| \end{array} \right)^p \\
&\quad \leq \left(\begin{array}{c} 2^{2p-2}|\int_0^T S(t,r)[f(r,x(r)) - f(r,u(r))]\,dr|^p \\ + 2^{2p-2}|\int_0^T S(t,r)[k(r,y(r)) - k(r,v(r))]\,dr|^p \\ + 2^{p-1}|\int_0^T S(t,r)[h(r,z(r)) - h(r,w(r))]\,dr|^p \end{array} \right) \\
&\quad \leq 2^{2p-2} \left[\begin{array}{c} (\int_0^T |S(t,r)[f(r,x(r)) - f(r,u(r))]|\,dr)^p \\ + (\int_0^T |S(t,r)[k(r,y(r)) - k(r,v(r))]|\,dr)^p \\ + (\int_0^T |S(t,r)[h(r,z(r)) - h(r,w(r))]|\,dr)^p \end{array} \right] \\
&\quad \leq 2^{2p-2}q^p \Bigl[\Bigl(\max_{r\in[0,T]} \bigl|x(r) - u(r)\bigr| \Bigr)^p + \Bigl(\max_{r\in[0,T]} \bigl|y(r) - v(r)\bigr| \Bigr)^p \\
&\qquad + \Bigl(\max_{r\in[0,T]} \bigl|z(r) - w(r)\bigr| \Bigr)^p \Bigr] \biggl(\int_0^T \bigl|S(t,r)\bigr|\,dr \biggr)^p \\
&\quad = 2^{2p-2}q^p \left[\begin{array}{c} \max_{r\in[0,T]} |x(r) - u(r)|^p \\ + \max_{r\in[0,T]} |y(r) - v(r)|^p \\ + \max_{r\in[0,T]} |z(r) - w(r)|^p \end{array} \right] \biggl(\int_0^T \bigl|S(t,r)\bigr|\,dr \biggr)^p.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\max_{t\in[0,T]} \bigl|F(x,y,z)(t) - F(u,v,w)(t)\bigr|^p \\
&\quad \leq 2^{2p-2}3q^p \sup_{t\in[0,T]} \biggl(\int_0^T \bigl|S(t,r)\bigr|\,dr \biggr)^p \\
&\qquad \times \max\Bigl\{ \max_{r\in[0,T]} \bigl|x(r) - u(r)\bigr|^p, \max_{r\in[0,T]} \bigl|y(r) - v(r)\bigr|^p, \max_{r\in[0,T]} \bigl|z(r) - w(r)\bigr|^p \Bigr\}.
\end{aligned} \tag{4.6}$$

Repeating this idea and using the definition of the G_b-metric G, we obtain

$$\begin{aligned}
&\max_{t\in[0,T]} \bigl|F(u,v,w)(t) - F(a,b,c)(t)\bigr|^p \\
&\quad \leq 2^{2p-2}3q^p \sup_{t\in[0,T]} \biggl(\int_0^T \bigl|S(t,r)\bigr|\,dr \biggr)^p \\
&\qquad \times \max\Bigl\{ \max_{r\in[0,T]} \bigl|u(r) - a(r)\bigr|^p, \max_{r\in[0,T]} \bigl|v(r) - b(r)\bigr|^p, \max_{r\in[0,T]} \bigl|w(r) - c(r)\bigr|^p \Bigr\}
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\max_{t\in[0,T]}\left|F(a,b,c)(t)-F(x,y,z)(t)\right|^p\\
&\quad\le 2^{2p-2}3q^p\sup_{t\in[0,T]}\left(\int_0^T\left|S(t,r)\right|dr\right)^p\\
&\qquad\times\max\Bigl\{\max_{r\in[0,T]}\left|a(r)-x(r)\right|^p,\max_{r\in[0,T]}\left|b(r)-y(r)\right|^p,\max_{r\in[0,T]}\left|c(r)-z(r)\right|^p\Bigr\}.
\end{aligned}\tag{4.8}$$

So, from (4.6), (4.7) and (4.8), we have

$$\begin{aligned}
&G\bigl(F(x,y,z),F(u,v,w),F(a,b,c)\bigr)\\
&\quad\le 2^{2p-2}3q^p\sup_{t\in[0,T]}\left(\int_0^T\left|S(t,r)\right|dr\right)^p\\
&\qquad\times\max\left\{\begin{array}{c}
\max\{\max_{r\in[0,T]}|x(r)-u(r)|^p,\max_{r\in[0,T]}|y(r)-v(r)|^p,\\
\max_{r\in[0,T]}|z(r)-w(r)|^p\},\\
\max\{\max_{r\in[0,T]}|u(r)-a(r)|^p,\max_{r\in[0,T]}|v(r)-b(r)|^p,\\
\max_{r\in[0,T]}|w(r)-c(r)|^p\},\\
\max\{\max_{r\in[0,T]}|a(r)-x(r)|^p,\max_{r\in[0,T]}|b(r)-y(r)|^p,\\
\max_{r\in[0,T]}|c(r)-z(r)|^p\}
\end{array}\right\}.
\end{aligned}\tag{4.9}$$

Similarly, we can obtain

$$\begin{aligned}
&G\bigl(F(y,x,y),F(v,u,v),F(b,a,b)\bigr)\\
&\quad\le 2^{2p-2}3q^p\sup_{t\in[0,T]}\left(\int_0^T\left|S(t,r)\right|dr\right)^p\\
&\qquad\times\max\left\{\begin{array}{c}
\max\{\max_{r\in[0,T]}|y(r)-v(r)|^p,\max_{r\in[0,T]}|x(r)-u(r)|^p,\\
\max_{r\in[0,T]}|y(r)-v(r)|^p\},\\
\max\{\max_{r\in[0,T]}|v(r)-b(r)|^p,\max_{r\in[0,T]}|u(r)-a(r)|^p,\\
\max_{r\in[0,T]}|v(r)-b(r)|^p\},\\
\max\{\max_{r\in[0,T]}|b(r)-y(r)|^p,\max_{r\in[0,T]}|a(r)-x(r)|^p,\\
\max_{r\in[0,T]}|b(r)-y(r)|^p\}
\end{array}\right\}
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
&G\bigl(F(z,y,x),F(w,v,u),F(c,b,a)\bigr)\\
&\quad\le 2^{2p-2}3q^p\sup_{t\in[0,T]}\left(\int_0^T\left|S(t,r)\right|dr\right)^p\\
&\qquad\times\max\left\{\begin{array}{c}
\max\{\max_{r\in[0,T]}|z(r)-w(r)|^p,\max_{r\in[0,T]}|y(r)-v(r)|^p,\\
\max_{r\in[0,T]}|x(r)-u(r)|^p\},\\
\max\{\max_{r\in[0,T]}|w(r)-c(r)|^p,\max_{r\in[0,T]}|v(r)-b(r)|^p,\\
\max_{r\in[0,T]}|u(r)-a(r)|^p\},\\
\max\{\max_{r\in[0,T]}|c(r)-z(r)|^p,\max_{r\in[0,T]}|b(r)-y(r)|^p,\\
\max_{r\in[0,T]}|a(r)-x(r)|^p\}
\end{array}\right\}.
\end{aligned}\tag{4.11}$$

Now, from (4.9), (4.10) and (4.11), we have

$$\begin{aligned}
&\max\left\{\begin{matrix} G(F(x,y,z),F(u,v,w),F(a,b,c)), \\ G(F(y,x,y),F(v,u,v),F(b,a,b)), \\ G(F(z,y,x),F(w,v,u),F(c,b,a)) \end{matrix}\right\} \\
&\quad\le 2^{2p-2}3q^p \sup_{t\in[0,T]}\left(\int_0^T \left|S(t,r)\right| dr\right)^p \\
&\quad\quad\times \max\left\{\begin{matrix} \max\{\max_{r\in[0,T]}|x(r)-u(r)|^p, \max_{r\in[0,T]}|u(r)-a(r)|^p, \\ \max_{r\in[0,T]}|x(r)-a(r)|^p\}, \\ \max\{\max_{r\in[0,T]}|y(r)-v(r)|^p, \max_{r\in[0,T]}|v(r)-b(r)|^p, \\ \max_{r\in[0,T]}|y(r)-b(r)|^p\}, \\ \max\{\max_{r\in[0,T]}|z(r)-w(r)|^p, \max_{r\in[0,T]}|w(r)-c(r)|^p, \\ \max_{r\in[0,T]}|z(r)-c(r)|^p\} \end{matrix}\right\} \\
&\quad= 2^{2p-2}3q^p \sup_{t\in[0,T]}\left(\int_0^T \left|S(t,r)\right| dr\right)^p \max\bigl\{G(x,u,a),G(y,v,b),G(z,w,c)\bigr\} \\
&\quad= \frac{2^{3p-3}3q^p \sup_{t\in[0,T]}(\int_0^T |S(t,r)|\,dr)^p}{2^{p-1}} \max\bigl\{G(x,u,a),G(y,v,b),G(z,w,c)\bigr\}.
\end{aligned}$$

But from (v), we have

$$2^{3p-3}3q^p \sup_{t\in[0,T]}\left(\int_0^T \left|S(t,r)\right| dr\right)^p < 1.$$

This proves that the operator F satisfies the contractive condition appearing in Corollary 2.7.

Let α, β, γ be the functions appearing in assumption (vi), then by (vi), we get

$$\alpha \le F(\alpha,\beta,\gamma), \qquad \beta \ge F(\beta,\alpha,\beta), \qquad \gamma \le F(\gamma,\beta,\alpha).$$

Applying Corollary 2.7, we deduce the existence of $x_1, x_2, x_3 \in X$ such that $x_1 = F(x_1, x_2, x_3)$, $x_2 = F(x_2, x_1, x_2)$ and $x_3 = F(x_3, x_2, x_1)$. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

1. Gnana Bhaskar, T, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. TMA **65**, 1379-1393 (2006)
2. Abbas, M, Ali Khan, M, Radenović, S: Common coupled fixed point theorems in cone metric spaces for *w*-compatible mappings. Appl. Math. Comput. **217**, 195-202 (2010)

3. Aydi, H, Postolache, M, Shatanawi, W: Coupled fixed point results for (ψ,φ)-weakly contractive mappings in ordered G-metric spaces. Comput. Math. Appl. **63**, 298-309 (2012)
4. Berinde, V: Coupled fixed point theorems for contractive mixed monotone mappings in partially ordered metric spaces. Nonlinear Anal. **75**, 3218-3228 (2012)
5. Choudhury, BS, Maity, P: Coupled fixed point results in generalized metric spaces. Math. Comput. Model. **54**, 73-79 (2011)
6. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73**, 2524-2531 (2010)
7. Ćirić, L, Damjanović, B, Jleli, M, Samet, B: Coupled fixed point theorems for generalized Mizoguchi-Takahashi contraction and applications to ordinary differential equations. Fixed Point Theory Appl. **2012**, Article ID 51 (2012)
8. Ding, HS, Li, L, Radenović, S: Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces. Fixed Point Theory Appl. **2012**, Article ID 96 (2012)
9. Hussain, N, Dorić, D, Kadelburg, Z, Radenović, S: Suzuki-type fixed point results in metric type spaces. Fixed Point Theory Appl. (2012). doi:10.1186/1687-1812-2012-126
10. Khan, MS, Swaleh, M, Sessa, S: Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. **30**, 1-9 (1984)
11. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**(12), 4341-4349 (2009)
12. Luong, NV, Thuan, NX: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74**, 983-992 (2011)
13. Luong, NV, Thuan, NX: Coupled fixed point theorems in partially ordered G-metric spaces. Math. Comput. Model. **55**, 1601-1609 (2012)
14. Razani, A, Parvaneh, V: Coupled coincidence point results for (ψ,α,β)-weak contractions in partially ordered metric spaces. J. Appl. Math. **2012**, Article ID 496103 (2012). doi:10.1155/2012/496103
15. Shatanawi, W: Coupled fixed point theorems in generalized metric spaces. Hacet. J. Math. Stat. **40**, 441-447 (2011)
16. Shatanawi, W, Abbas, M, Nazir, T: Common coupled coincidence and coupled fixed point results in two generalized metric spaces. Fixed Point Theory Appl. **2011**, Article ID 80 (2011)
17. Shatanawi, W, Samet, B, Abbas, M: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. Math. Comput. Model. **55**(3-4), 680-687 (2012)
18. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces. Fixed Point Theory Appl. **2011**, Article ID 81 (2011)
19. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled fixed point theorems for weak contraction mapping under F-invariant set. Abstr. Appl. Anal. **2012**, Article ID 324874 (2012)
20. Sintunavarat, W, Kumam, P: Coupled coincidence and coupled common fixed point theorems in partially ordered metric spaces. Thai J. Math. **10**(3), 551-563 (2012)
21. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled fixed-point theorems for contraction mapping induced by cone ball-metric in partially ordered spaces. Fixed Point Theory Appl. **2012**, Article ID 128 (2012)
22. Sintunavarat, W, Petrusel, A, Kumam, P: Common coupled fixed point theorems for w^*-compatible mappings without mixed monotone property. Rend. Circ. Mat. Palermo **61**, 361-383 (2012)
23. Sintunavarat, W, Kumam, P, Cho, YJ: Coupled fixed point theorems for nonlinear contractions without mixed monotone property. Fixed Point Theory Appl. **2012**, Article ID 170 (2012)
24. Karapınar, E, Kumam, P, Sintunavarat, W: Coupled fixed point theorems in cone metric spaces with a c-distance and applications. Fixed Point Theory Appl. **2012**, Article ID 194 (2012)
25. Agarwal, RP, Sintunavarat, W, Kumam, P: Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property. Fixed Point Theory Appl. **2013**, Article ID 22 (2013)
26. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74**, 4889-4897 (2011)
27. Borcut, M: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**, 7339-7346 (2012)
28. Borcut, M, Berinde, V: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**, 5929-5936 (2012)
29. Choudhury, BS, Karapınar, E, Kundu, A: Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces. Int. J. Math. Math. Sci. **2012**, Article ID 329298 (2012). doi:10.1155/2012/329298
30. Radenović, S, Pantelić, S, Salimi, P, Vujaković, J: A note on some tripled coincidence point results in G-metric spaces. Int. J. Math. Sci. Eng. Appl. **6**(6), 23-38 (2012)
31. Aydi, H, Abbas, M, Sintunavarat, W, Kumam, P: Tripled fixed point of W-compatible mappings in abstract metric spaces. Fixed Point Theory Appl. **2012**, Article ID 134 (2012)
32. Abbas, M, Ali, B, Sintunavarat, W, Kumam, P: Tripled fixed point and tripled coincidence point theorems in intuitionistic fuzzy normed spaces. Fixed Point Theory Appl. **2012**, Article ID 187 (2012)
33. Mustafa, Z, Sims, B: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7**(2), 289-297 (2006)
34. Mustafa, Z: Common fixed points of weakly compatible mappings in G-metric spaces. Appl. Math. Sci. **6**(92), 4589-4600 (2012)
35. Mustafa, Z: Some new common fixed point theorems under strict contractive conditions in G-metric spaces. J. Appl. Math. **2012**, Article ID 248937 (2012). doi:10.1155/2012/248937
36. Mustafa, Z: Mixed g-monotone property and quadruple fixed point theorems in partially ordered G-metric spaces using $(\phi-\psi)$ contractions. Fixed Point Theory Appl. **2012**, Article ID 199 (2012)
37. Mustafa, Z, Aydi, H, Karapınar, E: Mixed g-monotone property and quadruple fixed point theorems in partially ordered metric spaces. Fixed Point Theory Appl. **2012**, Article ID 71 (2012)
38. Mustafa, Z, Aydi, H, Karapınar, E: On common fixed points in G-metric spaces using (E.A) property. Comput. Math. Appl. **64**, 1944-1956 (2012)
39. Mustafa, Z, Awawdeh, F, Shatanawi, W: Fixed point theorem for expansive mappings in G-metric spaces. Int. J. Contemp. Math. Sci. **5**, 49-52 (2010)

40. Mustafa, Z, Khandagjy, M, Shatanawi, W: Fixed point results on complete G-metric spaces. Studia Sci. Math. Hung. **48**(3), 304-319 (2011)
41. Mustafa, Z, Obiedat, H, Awawdeh, F: Some of fixed point theorem for mapping on complete G-metric spaces. Fixed Point Theory Appl. **2008**, Article ID 189870 (2008)
42. Mustafa, Z, Shatanawi, W, Bataineh, M: Existence of fixed point result in G-metric spaces. Int. J. Math. Math. Sci. **2009**, Article ID 283028 (2009)
43. Mustafa, Z, Sims, B: Fixed point theorems for contractive mappings in complete G-metric space. Fixed Point Theory Appl. **2009**, Article ID 917175 (2009)
44. Abbas, M, Sintunavarat, W, Kumam, P: Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces. Fixed Point Theory Appl. **2012**, Article ID 31 (2012)
45. Chandok, S, Sintunavarat, W, Kumam, P: Some coupled common fixed points for a pair of mappings in partially ordered G-metric spaces. Math. Sci. **7**, 24 (2013)
46. Aydi, H, Karapınar, E, Shatanawi, W: Tripled coincidence point results for generalized contractions in ordered generalized metric spaces. Fixed Point Theory Appl. **2012**, Article ID 101 (2012)
47. Aghajani, A, Abbas, M, Roshan, JR: Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces Filomat (2013, in press)
48. Aghajani, A, Abbas, M, Roshan, JR: Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Math. Slovaca (2012, in press)
49. Mustafa, Z, Rezaei Roshan, J, Parvaneh, V: Coupled coincidence point results for (ψ,φ)-weakly contractive mappings in partially ordered G_b-metric spaces. Fixed Point Theory Appl. **2013**, Article ID 206 (2013). doi:10.1186/1687-1812-2013-206
50. Aydi, H, Karapınar, E, Shatanawi, W: Coupled fixed point results for (ψ,φ)-weakly contractive condition in ordered partial metric spaces. Comput. Math. Appl. **62**, 4449-4460 (2011)
51. Cho, YJ, Rhoades, BE, Saadati, R, Samet, B, Shatanawi, W: Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type. Fixed Point Theory Appl. **2012**, Article ID 8 (2012)
52. Nashine, HK, Samet, B: Fixed point results for mappings satisfying (ψ,φ)-weakly contractive condition in partially ordered metric spaces. Nonlinear Anal. **74**, 2201-2209 (2011)
53. Nieto, JJ, Rodriguez-López, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Order **22**(3), 223-239 (2005)

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