

## RESEARCH

## Open Access

# Some generalizations of 2D Bernoulli polynomials

Da-Qian Lu<sup>1</sup> and Qiu-Ming Luo<sup>2\*</sup>

\*Correspondence:

luomath2007@163.com

<sup>2</sup>Department of Mathematics,  
Chongqing Normal University,  
Chongqing Higher Education Mega  
Center, Huxi Campus, Chongqing,  
401331, People's Republic of China  
Full list of author information is  
available at the end of the article**Abstract**

As a generalization of 2D Bernoulli polynomials, neo-Bernoulli polynomials are introduced from a point of view involving the use of nonexponential generating functions. Their relevant recurrence relations, the differential equations satisfied by them and some other properties are obtained. Especially, we obtain the relationships between them and neo-Hermite polynomials. We also study some other generalizations of 2D Bernoulli polynomials.

**MSC:** Primary 11B68; secondary 33C99; 34A35**Keywords:** 2D Bernoulli polynomials; neo-Bernoulli polynomials; generating function; recurrence relation; umbral calculus

## 1 Introduction, definitions and motivation

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP)  $H_n(x, y)$  [1] are defined by

$$H_n(x, y) = n! \sum_{r=0}^{[n/2]} \frac{x^{n-2r} y^r}{r!(n-2r)!} \quad (1.1)$$

with the generating function

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \quad (1.2)$$

The neo-Hermite polynomials  $\phi_n(x, y)$  [2] are defined by

$$\phi_n(x, y) = n! \sum_{s=0}^{[n/2]} \frac{f_{n-s} x^{n-2s} y^s}{(n-2s)! s!} \quad (1.3)$$

with the generating function

$$f(xt + yt^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \phi_n(x, y), \quad (1.4)$$

where  $f(x)$  is a continuous and infinitely differentiable function and it can be expanded in series as follows:

$$f(x) = \sum_{s=0}^{\infty} \frac{f_s}{s!} x^s.$$

And it has an operational definition

$$\phi_n(x, y) = \exp \left[ (\hat{f}^{-1}y) \frac{\partial^2}{\partial x^2} \right] f_n x^n. \tag{1.5}$$

As it is well-known, the HKdF polynomials are generated by (1.4) when  $f(x)$  reduces to an exponential function.

The classical Bernoulli polynomials  $B_n(x)$  are defined by [3]

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \quad (|z| < 2\pi), \tag{1.6}$$

and consequently, the classical Bernoulli numbers  $B_n := B_n(0)$  can be obtained by the generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \tag{1.7}$$

It is well known that

$$\begin{aligned} B_n(0) &= B_n(1) = B_n, \quad n \neq 1, \\ B_n(x) &= \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k, \\ B'_n(x) &= nB_{n-1}(x). \end{aligned} \tag{1.8}$$

The generalized Bernoulli polynomials  $B_n^{(\alpha)}(x)$  are defined by [4]

$$\left( \frac{z}{e^z - 1} \right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < 2\pi). \tag{1.9}$$

Clearly, the generalized Bernoulli numbers  $B_n^{(\alpha)}$  are given by

$$B_n^{(\alpha)} := B_n^{(\alpha)}(0)$$

and

$$B_n(x) := B_n^{(1)}(x) \quad (n \in \mathbb{N}_0).$$

In 2005, Luo defined the Apostol-Bernoulli numbers  $\mathcal{B}_n(\lambda)$  and polynomials  $\mathcal{B}_n(x; \lambda)$  as follows:

$$\frac{z}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda) \frac{z^n}{n!}, \tag{1.10}$$

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{z^n}{n!}, \tag{1.11}$$

$$(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log \lambda| \text{ when } \lambda \neq 1).$$

The generalized Apostol-Bernoulli numbers  $\mathcal{B}_n^{(\alpha)}(\lambda)$  and polynomials  $\mathcal{B}_n^{(\alpha)}(x; \lambda)$  are defined by [5]

$$\left(\frac{z}{\lambda e^z - 1}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(\lambda) \frac{z^n}{n!}, \tag{1.12}$$

$$\left(\frac{z}{\lambda e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{z^n}{n!}, \tag{1.13}$$

$$(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log \lambda| \text{ when } \lambda \neq 1).$$

The 2D Bernoulli polynomials  $B_n^{(2)}(x, y)$  are defined [6] by the generating function

$$\frac{t}{e^t - 1} e^{xt + yt^2} = \sum_{n=0}^{\infty} B_n^{(2)}(x, y) \frac{t^n}{n!}. \tag{1.14}$$

In this paper, we will give some generalizations of the 2D Bernoulli polynomials. And some properties of them will be given.

## 2 The Bernoulli polynomials from a general point of view

We consider a continuous and infinitely differentiable function  $f(x)$  and associate it with the following generating function:

$$\frac{t}{e^t - 1} f(xt + yt^2) = \sum_{n=0}^{\infty} b_n^{(2)}(x, y) \frac{t^n}{n!}. \tag{2.1}$$

As it is well known, the 2D Bernoulli polynomials  $B_n^{(2)}(x, y)$  are generated by (2.1) when  $f(x)$  reduces to an exponential function.

It is possible to find an explicit form of the polynomials  $b_n^{(2)}(x, y)$  in terms of the neo-Hermite polynomials  $\phi_n(x, y)$  defined by (1.4).

**Theorem 2.1** *The following representation formulas hold true:*

$$\begin{aligned} b_n^{(2)}(x, y) &= \sum_{h=0}^n \binom{n}{h} B_{n-h} \phi_h(x, y) \\ &= n! \sum_{h=0}^n \frac{B_{n-h}}{(n-h)!} \sum_{s=0}^{\lfloor h/2 \rfloor} \frac{f_{h-2s} x^{h-2s} y^s}{s!(h-2s)!}, \end{aligned} \tag{2.2}$$

where  $B_k$  denotes the Bernoulli numbers;

$$\phi_n(x, y) = \sum_{h=0}^n \binom{n}{h} \frac{1}{n-h+1} b_h^{(2)}(x, y). \tag{2.3}$$

*Proof* Equation (2.2) is obtained starting from the generating function (2.1) by using the Cauchy product of the series expansion (1.4) and (1.7), and then using the identity principle of power series.

Equation (2.3) is obtained in the same way, starting from the equation

$$f(xt + yt^2) = \frac{e^t - 1}{t} \sum_{n=0}^{\infty} b_n^{(2)}(x, y) \frac{t^n}{n!}. \quad \square$$

By assuming that  $f(x)$  can be expanded in series as follows:

$$f(x) = \sum_{s=0}^{\infty} \frac{f_s}{s!} x^s, \tag{2.4}$$

we introduce the operator  $\hat{f}$  defined in such a way that

$$(\hat{f})^s = f_s, \quad (\hat{f}^r) f_s = f_{s+r}. \tag{2.5}$$

Thus, we can write

$$f(x) = \exp[\hat{f}x] = \sum_{s=0}^{\infty} \frac{(\hat{f}x)^s}{s!} = \sum_{s=0}^{\infty} \frac{f_s}{s!} x^s. \tag{2.6}$$

Within the context of such a formalism, we have

$$\begin{aligned} f(x + y) &= \exp[\hat{f}(x + y)] \\ &= \exp[(\hat{f}x) + (\hat{f}y)] \end{aligned} \tag{2.7}$$

$$= \exp[(\hat{f}x)] \exp[(\hat{f}y)]. \tag{2.8}$$

Making use of (2.1), (2.7), we can obtain

$$\frac{\partial}{\partial x} b_n^{(2)}(x, y) = n \hat{f} b_{n-1}^{(2)}(x, y), \tag{2.9}$$

$$\frac{\partial}{\partial y} b_n^{(2)}(x, y) = n(n-1) \hat{f} b_{n-2}^{(2)}(x, y). \tag{2.10}$$

By using (1.6), (2.5), (2.8) and (2.9), we know that the polynomials  $b_n^{(2)}(x, y)$  satisfy the following partial differential equation:

$$\frac{\partial}{\partial y} b_n^{(2)}(x, y) = \hat{f}^{-1} \frac{\partial^2}{\partial x^2} b_n^{(2)}(x, y) \tag{2.11}$$

with

$$b_n^{(2)}(x, 0) = B_n[\hat{f}x].$$

Thus, the polynomials  $b_n^{(2)}(x, y)$  can be constructed according to the following operational rule:

$$b_n^{(2)}(x, y) = \exp \left[ \left( \hat{f}^{-1} y \frac{\partial^2}{\partial x^2} \right) \right] B_n[\hat{f}x]. \tag{2.12}$$

Then we can derive some relations of the polynomials  $b_n^{(2)}(x, y)$  by using the relations of the Bernoulli polynomials  $B_n(x)$  along with the operational rule (2.12).

**Remark 2.2** Upon the proof, the series (2.1) is the absolute convergence and uniformly convergence, thus we can differentiate the both sides of (2.1) with respect to the variable  $x$  or  $y$ .

We recall the following functional equations involving the Bernoulli polynomials  $B_n(x)$  [4]:

$$\begin{aligned} B_n(x) &= \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k, \\ B_n(x) &= x^n - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k(x), \\ B_n(x+1) - B_n(x) &= nx^{n-1}. \end{aligned}$$

Now, substituting  $x$  with  $\hat{f}x$  in the above equations and performing the operation  $\exp[\hat{f}^{-1}y\frac{\partial^2}{\partial x^2}]$  on the results, then by using (1.5), (2.5) and (2.11), we get the following identities involving  $b_n^{(2)}(x, y)$ .

**Theorem 2.3** *We have the following relationships between the neo-Hermite polynomials and the neo-Bernoulli polynomials:*

$$\begin{aligned} b_n^{(2)}(x, y) &= \sum_{k=0}^n \binom{n}{k} B_{n-k} \phi_k(x, y), \\ b_n^{(2)}(x, y) &= \phi_n(x, y) - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} b_k^{(2)}(x, y), \\ b_n^{(2)}(x+1, y) - b_n^{(2)}(x, y) &= n\phi_{n-1}(x, y). \end{aligned}$$

A recurrence relation for the polynomials  $b_n^{(2)}(x, y)$  is given by the following theorem.

**Theorem 2.4** *For any integral  $n \geq 1$ , the following linear homogeneous recurrence relation for the 2D Bernoulli polynomials  $b_n^{(2)}(x, y)$  holds true:*

$$\begin{aligned} b_0^{(2)}(x, y) &= 1, \\ b_n^{(2)}(x, y) &= -\frac{1}{n} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} b_k^{(2)}(x, y) + \left( \hat{f}x - \frac{1}{2} \right) b_{n-1}^{(2)}(x, y) \\ &\quad + 2(n-1)\hat{f}y b_{n-2}^{(2)}(x, y), \end{aligned} \tag{2.13}$$

where  $B_k$  denotes the Bernoulli numbers.

*Proof* Differentiating both sides of Eq. (2.1) with respect to  $t$ , recalling the generating functions (1.7) of the Bernoulli numbers, and using (2.7) and some elementary algebra and the identity principle of power series, we get recursion (2.13) easily.  $\square$

A natural further extension is obtained by considering the following case:

$$\frac{t}{e^t - 1} f(xt + yt^m) = \sum_{n=0}^{\infty} b_n^{(m)}(x, y) \frac{t^n}{n!}, \tag{2.14}$$

where

$$\begin{aligned} b_n^{(m)}(x, y) &= \sum_{h=0}^n \binom{n}{h} B_{n-h} \phi_n^{(m)}(x, y) \\ &= n! \sum_{h=0}^n \frac{B_{n-h}}{(n-h)!} \sum_{s=0}^{[h/m]} \frac{f_{h-(m-1)s} x^{h-ms} y^s}{s!(h-ms)!}. \end{aligned} \tag{2.15}$$

We can also extend the polynomials  $b_n^{(2)}(x, y)$  to  $b_n^{(2;\alpha)}(x, y; \lambda)$  defined by

$$\left( \frac{t}{\lambda e^t - 1} \right)^\alpha f(xt + yt^2) = \sum_{n=0}^{\infty} b_n^{(2;\alpha)}(x, y; \lambda) \frac{t^n}{n!}, \tag{2.16}$$

where

$$\begin{aligned} b_n^{(2;\alpha)}(x, y) &= \sum_{h=0}^n \binom{n}{h} \mathcal{B}_{n-h}^{(\alpha)}(\lambda) \phi_h(x, y) \\ &= n! \sum_{h=0}^n \frac{\mathcal{B}_{n-h}^{(\alpha)}(\lambda)}{(n-h)!} \sum_{s=0}^{[h/2]} \frac{f_{h-2s} x^{h-2s} y^s}{s!(h-2s)!}. \end{aligned} \tag{2.17}$$

The properties of these polynomials can be obtained by using a similar method.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally in writing this paper, and read and approved the final manuscript.

**Author details**

<sup>1</sup>School of Mathematics Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, People's Republic of China.

<sup>2</sup>Department of Mathematics, Chongqing Normal University, Chongqing Higher Education Mega Center, Huxi Campus, Chongqing, 401331, People's Republic of China.

**Acknowledgements**

Dedicated to Professor Hari M Srivastava.

The present investigation was supported by the *National Natural Science Foundation of China* under Grant 11226281, *Fund of Science and Innovation of Yangzhou University, China* under Grant 2012CXJ005, *Research Project of Science and Technology of Chongqing Education Commission, China* under Grant KJ120625 and *Fund of Chongqing Normal University, China* under Grant 10XLR017 and 2011XLZ07.

Received: 10 December 2012 Accepted: 24 February 2013 Published: 19 March 2013

**References**

1. Appell, P, Kampé de Fériet, J: Fonctions hypergéométriques et hypersphériques: Polynômes d'Hermite. Gauthier-Villars, Paris (1926)

2. Dattoli, G, Srivastava, HM, Sacchetti, D: The Hermite polynomials and the Bessel functions from a general point of view. *Int. J. Math. Math. Sci.* **57**, 3633-3642 (2003)
3. Gradshteyh, IS, Ryzhik, IM: *Table of Series, and Products*. Academic Press, New York (1980)
4. Magnus, W, Oberhettinger, F, Soni, RP: *Formulas and Theorems for the Special Functions of Mathematical Physics*. 3rd. enlarged edn. Springer, New York (1966)
5. Luo, Q-M, Srivastava, HM: Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials. *J. Math. Anal. Appl.* **308**, 290-302 (2005)
6. Bretti, G, Ricci, PE: Multidimensional extensions of the Bernoulli and Appell polynomials. *Taiwan. J. Math.* **8**, 415-428 (2004)

doi:10.1186/1029-242X-2013-110

**Cite this article as:** Lu and Luo: Some generalizations of 2D Bernoulli polynomials. *Journal of Inequalities and Applications* 2013 **2013**:110.

**Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:**

- ▶ Convenient online submission
- ▶ Rigorous peer review
- ▶ Immediate publication on acceptance
- ▶ Open access: articles freely available online
- ▶ High visibility within the field
- ▶ Retaining the copyright to your article

---

Submit your next manuscript at ▶ [springeropen.com](http://springeropen.com)

---