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On the Continuity of the Real Roots of an Algebraic Equation

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ON THE CONTINUITY OF THE REAL ROOTS OF AN ALGEBRAIC EQUATION

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1. Introduction. It is well known that the root of an algebraic equation is a continuous multiple-valued function of its coefficients [**5**, p. 3]. However, it is not necessarily true that a root can be given by a continuous single-valued function. A complete solution of this problem has long been known in the case where the coefficients are themselves polynomials in a complex variable [**3**, chap. V]. For most purposes the concept of the Riemann surface enables one to bypass the problem. However, in the study of the ideal structure of rings of continuous functions, the general problem must be met directly.

This paper is confined to an investigation of the continuity of the real roots of an algebraic equation; the results obtained are used to establish a theorem stated, but not correctly proved, by Hewitt [**2**, Theorem 42] on rings of real-valued continuous functions.

2. Multiple-valued functions. DEFINITION. A multiple-valued function $\mathcal{J}$ from a space X to a space Y will be called a continuous n-valued function on X to Y and will be symbolized by $\mathcal{J}: X \to^n Y$ provided

(i) to each $x \in X$, $\mathcal{J}$ assigns m_x values $y_1, \cdots, y_{m_x}$, in Y, with associated multiplicities k_i such that $\sum_{i=1}^{m_x} k_i = n$.

(ii) to each neighborhood $N(y_i)$ in Y there corresponds a neighborhood $U(x)$ in X such that for z in $U(x)$ there are k_i values of $\mathcal{J}(z)$ in $N(y_i)$, counting multiplicities. (Note—k_i depends on x.)

All spaces considered will be Hausdorff. Unless otherwise specified, in any mention of the number of values of $\mathcal{J}(x)$ it is supposed that multiplicities are counted.

LEMMA 1. *For any $\mathcal{J}: X \to^n R$, where R is the real line, the least value $f(x)$ of $\mathcal{J}(x)$ is a continuous function.*

PROOF. For any $x_0 \in X$ and any $\epsilon > 0$, there is a neighborhood U of x_0 such that for $z \in U$, all n values of $\mathcal{J}(z)$ are greater than $f(x_0) - \epsilon$. There is another neighborhood V of x_0 in which at least one value of $\mathcal{J}(z)$ is less than $f(x_0) + \epsilon$. Hence if $z \in U \cap V$, then $f(x_0) - \epsilon < f(z) < f(x_0) + \epsilon$; so f is continuous at x_0.

LEMMA 2. *Any $\mathcal{J}: X \to^n R$ can be decomposed into n continuous single-valued functions.*

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The proof is by finite induction.

LEMMA 3. *If $\mathcal{J}$: $X \to^n Y$ has always exactly m values in an open or closed subspace W of Y, then the restriction of the values of $\mathcal{J}(x)$ to W defines an $\mathcal{J}': X \to^m W$.*

PROOF. For each $x_0 \in X$, let N, N^* be disjoint open sets of Y containing respectively the m values of $\mathcal{J}'(x_0)$ and the remaining $n-m$ values of $\mathcal{J}(x_0)$. If W is open, $N \cap W$ is open in Y, so a discontinuity of $\mathcal{J}'$ would be a discontinuity of $\mathcal{J}$.

If W is closed, choose a neighborhood U of x such that if $z \in U$, then $n-m$ values of $\mathcal{J}(z)$ are in $N^* \cap (Y-W)$. Hence there is a neighborhood V of x such that if $z \in U \cap V$, the only values of $\mathcal{J}(z)$ in N are also in W. It easily follows that $\mathcal{J}': X \to^m W$.

3. **Real roots of polynomials.** Let $P(\alpha, w) = P(a_0, \cdots, a_{n-1}; w) = w^n + \sum_{j=0}^{n-1} a_j w^j$ be a polynomial with complex coefficients. As was noted above, the root $r(\alpha)$ of $P(\alpha, w) = 0$ is a continuous n-valued function of α; that is, $r: K^n \to^n K$. If $\phi(\alpha)$, $\xi(\alpha)$ are the real and imaginary parts of $r(\alpha)$, clearly $\phi: K^n \to^n R$ and $\xi: K^n \to^n R$. By Lemma 2, either (but not necessarily both together) can be decomposed into n continuous single-valued functions, ϕ_j or ξ_j. Now if the coefficients are given by a continuous function $\alpha: X \to K^n$, the ϕ_j or ξ_j are continuous on the space X.

THEOREM 1. *If $P(\alpha(x), w) \equiv P(x, w) = 0$ has a real root for each $x \in X$, then* (i) *there is an open set U on which a real root is given by a continuous function r_U;* (ii) *if the number of real roots is constant over X, we may take $U = X$.*

PROOF. We begin with a lemma.

LEMMA 4. *Each set $B_j = [x \in X \mid P(x, w) = 0$ has at least j real roots$]$ is the set of zeros of a continuous real function, for $j = 1, 2, \cdots, n$.*

Having decomposed ξ into continuous functions $\xi_1, \cdots, \xi_n$, we may write $B_j = [x \in X \mid \prod_{(i_1, \ldots, i_j)} \sum_{k=1}^{j} \xi_{i_k}^2(x) = 0]$, where the product is extended over all choices of j different indices.

(i) Noting that $B_1 = X$, let j_0 be the greatest j for which B_j has a nonvoid interior V. Then $U = V - B_{j_0+1}$ (letting B_{n+1} be empty) is a nonvoid open set. By Lemma 3, the real root is a continuous j_0-valued function on U, and by Lemma 1 the least real root is a continuous function on U, which we may take as r_U.

(ii) The above construction yields $U = V = X$ in the special case. The following example of a polynomial whose real root is necessarily discontinuous on the Cantor perfect set suggests that the result above cannot be substantially improved.

EXAMPLE. Let the real parts of the roots of a cubic be given by $\phi_k = k$, $k = 1, 2, 3$. Define the imaginary parts ξ_k as follows. In the real interval $[0, 1]$, let $\delta(x)$ be the distance of x from the ternary Cantor set C. This set is obtained by excluding successively intervals of lengths $1/3^n$. Let I_m be the union of the removed intervals of length $1/3^m$. Define $\xi_k = 0$ identically on C; on I_m let $\xi_k = 0$, $\delta(x)$, $-\delta(x)$, according as $m - k \equiv 0, 1, 2 \pmod 3$. The polynomial $\prod_{k=1}^{3}(w - \phi_k - i\xi_k)$ has continuous coefficients, but obviously in any neighborhood of any point of C there are excluded intervals of lengths $1/3^n$ and $1/3^{n+1}$. Thus $r(\alpha)$, however chosen, is discontinuous at every point of C.

4. **Rings of real-valued continuous functions.** In [2], Hewitt considered the ring $C(X, R)$ of all real-valued continuous functions on a completely regular space X. He stated [2, Theorem 42] that for any maximal ideal M of $C(X, R)$, $C_M = C(X, R)/M$ is a real closed field. While he established that C_M is ordered and that every positive element has a square root, his proof that every polynomial of odd degree has a root in C_M depends on the assumption that the least real root of a real polynomial of odd degree is a continuous function of its coefficients. In his review [1] of the paper, Dieudonné observed the error.[1] Nevertheless the theorem is true for normal spaces, as will be shown.

THEOREM 2. *If X is normal, C_M is a real closed field.*

PROOF. By the above, we need only show that every polynomial $P(x, w) = w^{2n+1} + \sum_{k=0}^{2n} a_k(x) w^k$, $a_k \in C(X, R)$, has a root in C_M. If $f \in C(X, R)$, let $Z(f) = [x \in X \mid f(x) = 0]$, $Z(M) = [Z(f) \mid f \in M]$. Consider the sets B_j defined as in Lemma 4. Since $B_1 = X$, there is a greatest j_0 such that B_{j_0} meets all elements of $Z(M)$; and by [2, Theorem 36] there is an $f \in M$ such that $Z(f) \subset B_{j_0} - B_{j_0+1}$. By Theorem 1, (ii), there is a continuous root function $r_{Z(f)}$ on $Z(f)$. By Tietze's extension theorem [4, p. 28], $r_{Z(f)}$ has an extension r continuous on X; so $P(x, r) \equiv 0 \pmod M$.

The authors do not know if Theorem 2 holds for non-normal spaces.

Added in proof. In a paper currently being prepared by L. Gillman and M. Henriksen for submission to Trans. Amer. Math. Soc., examples are given of completely regular non-normal spaces in which

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[1] Professor W. F. Eberlein has communicated to the senior author an example of a real polynomial whose real root cannot be chosen continuously.

every $Z(f)$ is open and closed (see also Bull. Amer. Math. Soc. Abstract 59-4-446). It is easily seen that the conclusion of Theorem 2 holds for such spaces.

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