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Research Article

Estimation on Reliability Models of Bearing Failure Data

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The failure data of bearing products is random and discrete and shows evident uncertainty. Is it accurate and reliable to use Weibull distribution to represent the failure model of product? The Weibull distribution, log-normal distribution, and an improved maximum entropy probability distribution were compared and analyzed to find an optimum and precise reliability analysis model. By utilizing computer simulation technology and k-s hypothesis testing, the feasibility of three models was verified, and the reliability of different models obtained via practical bearing failure data was compared and analyzed. The research indicates that the reliability model of two-parameter Weibull distribution does not apply to all situations, and sometimes, two-parameter log-normal distribution model is more precise and feasible; compared to three-parameter log-normal distribution model, the three-parameter Weibull distribution manifests better accuracy but still does not apply to all cases, while the novel proposed model of improved maximum entropy probability distribution fits not only all kinds of known distributions but also poor information issues with unknown probability distribution, prior information, or trends, so it is an ideal reliability analysis model with least error at present.

1. Introduction

In machinery products and engineering projects, bearings are the joints and wearing parts in the whole transmission system. Their operational reliability is the basis to establish optimization and improvement strategies and implement failure factor analysis, which directly relates to the operation security of product during service time. In bearing reliability estimation, the selection of failure distribution model is of great importance, because it directly relates to the precision of reliability prediction and has a huge influence on the usability of bearing. If the predicted reliability value is too high and the product performance exceeds a fatigue limit of normal operation, particularly for aerospace, high-speed rail, nuclear reactor, precision meter, and such systems, it will result in major vicious accident or even affect the national security [1, 2]. If the predicted reliability value is too low, the product function cannot be fully exploited, making the product lose its environmental adaption and leading to huge waste of conditional resources. Therefore, in order to guarantee the safe and stable operation of product system, it is essential to implement effective monitoring and diagnosing and precise reliability model estimation for bearings [3–7].

Reliability analysis is aimed at searching for the failure distribution information which can exactly reflect that the failure mechanism of product components accords with the analysis results of failure data. After fitting the fault or failure data into certain distribution form, the reliability estimation and prediction will be carried out, in which the distribution function of product failure time is the basis to study reliability. Since the failure state of bearings may be affected by different operation conditions, such as structural composition, material, load, lubrication, and numerous uncertainty factors, the failure life of actual situation is random, accompanied by multiple failure modes among which each mode can be mutually affected, acted, and dynamically varied [8].

There are some difficulties how to quickly and effectively utilize failure data for precise reliability analysis and model selection of production information. Recently, there is a lack of exact plan to describe their gradual change process during operation, and no perfect theoretical system is formed yet [9]. Traditional reliability estimation theory is established on the basis of a large number of failure data. However, in many cases, the probability distribution of problems met in engineering and experiment are not normal ones. For example, the aircraft bearing, due to its high cost, very few failure data, and extremely high requirements for precision and reliability, has a harsh demand for test equipment. This makes it difficult to implement large sample life test or obtain failure data within limited testing time. So, when the product's life probability distribution is unknown, and we only have small sample data for reference, it is impossible for us to use existing reliability theory to accurately describe its failure evolving law.

At present, the researches of bearing failure data are mostly about Weibull distribution, log-normal distribution, gamma distribution, and binomial distribution reliability analysis methods. In particular, the Weibull distribution and log-normal distribution are widely used in reliability theoretical analysis. Though this has achieved certain results, they show large error and low precision in the process of product reliability estimation [10–15]. The accuracy of reliability models increasingly got the attention of scholars and experts. Rodriguez-Picon et al. [16] considered a gamma process to marginally model the degradation of a performance characteristic through two degradation test phases performed sequentially and obtained a robust model to get reliability estimates considering the effect of two serial degradation tests. Reuben et al. [17, 18] proposed a reliability evaluation method by using Weibull equation and made reliability estimation on product failure data of gearbox bearing and ceramic material, respectively, where results showed very small discrepancy between its fitting curve and the points of failure data. Through the reliability modeling disposing of failure process of large-scale and complicated machinery equipment, Pulcini [19] declared that its failure strength is not so monotonous, and on this basis, he proposed the reliability analysis model of nonhomogeneous Poisson process. To solve the real-time online reliability problems, Hong and Meeker [20] proposed an intelligent reliability estimation method based on dynamic state information changes, which brought much convenience to timely judge the dynamic running state of workpiece. Khaleghei and Makis [21] proposed a new competing risk model to calculate the conditional mean residual life and conditional reliability function of a system subject to two dependent failure modes, namely, degradation failure and catastrophic failure. In order to ensure that the classifier can correctly inspect the system failure information, Hwang and Lee [22] presented a new approach to overcome class imbalance problem and human factor influence by using classification technique, thus speedily and effectively implementing system reliability estimation. Zhang et al. [23] applied ANSYS/PDS module to make simulated analysis on the reliability of agricultural machinery chassis drive axle housing and probed into the influence of random variables such as geometric dimension, load, and material strength on drive axle housing. By exploring discrete random variable and analyzing the expectation interval and information capacity of certain entropy, Aviyente et al. [24, 25] successfully solved the time frequency distribution problem and interval forecast problem of entropy. Xia [26] proposed a grey bootstrap method based on poor information theory, which conducted a reliability analysis of zero-failure data when the probability distribution information is known or unknown in life test, thus providing a strong theoretical reference to the reliability of poor information of zero-failure data.

Based on this, lots of topics and literature sources are mentioned associated with bearing capacity, distribution types, reliability, lifetime, and so on, because there is a very close relationship between them. Firstly, bearing capacity, lubrication condition, rotational speed, and other working conditions are important determinants affecting the bearing lifetime, and the set of the same batch bearing lifetime makes up a number of failure data under the above working conditions. Secondly, the distribution types of a number of failure data can be obtained according to statistical theory, and then its probability density function can be acquired easily. As we all know, the probability density function is the hub of data analysis and solution, and then, according to the probability density function and the given integral interval, the failure probability of bearings can be obtained during their service. Finally, using the unit one to subtract the failure probability, the reliability of bearing failure data is acquired. Therefore, these topics on bearing capacity, distribution types, reliability, and lifetime have a very close coherent interlocking, and all of them have an evidently direct or indirect relationship with the calculation of the reliability of bearing failure data.

This article used the failure data obtained from simulated test and bearing life failure test and made comparative analysis via log-normal distribution, Weibull distribution, and improved maximum entropy distribution, so as to select the optimum and precise reliability analysis model. First, the reliability empirical value calculated by Johnson [27] method was taken as standard. In specific analysis, two-parameter log-normal distribution was compared with two-parameter Weibull distribution, and three-parameter log-normal distribution was compared with three-parameter Weibull distribution, while in parameter estimation process of three-parameter log-normal distribution, the integral transformation moment method, linear moment method, and probability weighted moment method were used for comparative analysis, respectively. The research indicates that Weibull distribution is not applicable to all bearing failure conditions, and sometimes, the log-normal distribution has a smaller standard deviation and lower relative life error in reliability analysis. Then, the reliability estimation method for improved maximum entropy probability distribution was put forward to make reliability analysis on failure data, and this novel method has a high fitting degree and can be applied to all failure cases; when comparatively analyzing with Weibull distribution model and log-normal distribution model, both of the standard deviation and relative life error between its reliability truth-value vector and empirical value vector are minimum.

2. Mathematical Model

2.1. Classical Reliability Empirical Value. Suppose $\mathbf{X}_0$ is a failure data series group of research object, and each failure data is unequal and nonredundant, which is denoted as

$$\mathbf{X}_0 = \{x_{0i}\}; \quad x_{01} < x_{02} < \cdots < x_{0i} < \cdots < x_{0n};\ i = 1, 2, \ldots, n, \tag{1}$$

where $\mathbf{X}_0$ is the vector composed of such failure data group, x_{0i} is the ith failure data of this data series, i is the serial number of ith failure data, and n is the number of failure data.

In case that the probability distribution or distribution parameter of failure data is unknown, the reliability of life failure data of research object can be nonparametric estimated using Johnson's median rank empirical value formula. The reliability empirical formula [28, 29] of such method can be expressed by vector as

$$\mathbf{R}0 = \{r(x_{0i})\}; \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\mathbf{R}0$ refers to reliability empirical value vector.

The formula to calculate reliability median rank empirical value is

$$r(x_{0i}) = 1 - \frac{i - 0.3}{n + 0.4}; \quad i = 1, 2, \ldots, n, \tag{3}$$

where i is the ith failure data and n is the number of failure data.

2.2. Two-Parameter Log-Normal Distribution and Weibull Distribution. Both distributions are common reliability models in engineering applications, especially the Weibull distribution which is widely used in analyzing bearing failure data and has achieved good research results.

The probability density function of two-parameter log-normal distribution is

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma t} \exp\left\{-\frac{[\ln t - \mu]^2}{2\sigma^2}\right\}. \tag{4}$$

The reliability function is

$$f(t) = 1 - \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma t} \exp\left\{-\frac{[\ln t - \mu]^2}{2\sigma^2}\right\} dt, \tag{5}$$

where t is the random variable of life, μ is the proportional parameter, σ is the shape parameter, $t > 0$, $\mu > 0$, and $\sigma > 0$.

The probability density function of two-parameter Weibull distribution is

$$f(t) = \frac{\sigma}{\mu}\left(\frac{t}{\mu}\right)^{\sigma-1} \exp\left(-\left(\frac{t}{\mu}\right)^{\sigma}\right). \tag{6}$$

The reliability function is

$$f(t) = \exp\left(-\left(\frac{t}{\mu}\right)^{\sigma}\right), \tag{7}$$

where t is the random variable of life, μ is the proportional parameter, σ is the shape parameter, $t > 0$, $\mu > 0$, and $\sigma > 0$.

2.2.1. Parameter Estimation. Maximum likelihood method [30, 31] is widely used in the parameter estimation of all kinds of reliability models, which is one of the frequently used parameter estimation methods. For two-parameter log-normal distribution and two-parameter Weibull distribution, the maximum likelihood method is used for parameter estimation of these two models, respectively.

When the maximum likelihood method is used to estimate two-parameter log-normal distribution, the likelihood equation set is obtained as below:

$$\begin{aligned} \frac{\partial \ln L(\mu, \sigma^2)}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^{n} (\ln t_i - \mu) = 0 \\ \frac{\partial \ln L(\mu, \sigma^2)}{\partial \sigma^2} &= \frac{1}{2\sigma^4} \sum_{i=1}^{n} (\ln t_i - \mu)^2 = 0, \end{aligned} \tag{8}$$

where $L(\mu, \sigma^2)$ is the likelihood function of log-normal distribution.

When the maximum likelihood method is used to estimate two-parameter Weibull distribution, the likelihood equation set is obtained as below:

$$\begin{aligned} \frac{\partial \ln L(\sigma, \mu)}{\partial \sigma} &= \frac{2n}{\mu} + (\ln \mu)\mu^{-\sigma} \sum_{i-1}^{n} t_i^{\sigma} - \frac{1}{\mu^{\sigma}} \sum_{i=1}^{n} \sigma t_i^{\sigma-1} \\ \frac{\partial \ln L(\sigma, \mu)}{\partial \mu} &= -\frac{n\sigma}{\mu^2} - \frac{n(\sigma - 1)}{\mu^2} + \frac{\sigma}{\mu^{\sigma-1}} \sum_{i=1}^{n} t_i^{\sigma}, \end{aligned} \tag{9}$$

where $L(\sigma, \mu)$ is the likelihood function of Weibull distribution. Using the iterative method to solve the equation set, we can acquire the estimated values of two parameters in two-parameter Weibull distribution.

2.3. Three-Parameter Log-Normal Distribution and Weibull Distribution. The probability density function of three-parameter log-normal distribution is

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma t} \exp\left\{-\frac{[\ln(t - \tau) - \mu]^2}{2\sigma^2}\right\}. \tag{10}$$

The reliability function is

$$f(t) = 1 - \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma t} \exp\left\{-\frac{[\ln(t - \tau) - \mu]^2}{2\sigma^2}\right\} dt, \tag{11}$$

where t is the random variable of life, (μ, σ) is the parameter of log-normal distribution: μ is the proportional parameter and σ is the shape parameter, and τ is the location parameter. $t > 0$, $\mu > 0$, and $\sigma > 0$.

The probability density function of three-parameter Weibull distribution is

$$f(t;\mu,\sigma,\tau)=\frac{\sigma}{\mu}\left(\frac{t-\tau}{\mu}\right)^{\sigma-1}\exp\left(-\left(\frac{t-\tau}{\mu}\right)^{\sigma}\right)$$
$$t\geq\tau>0;\ \sigma>0;\ \mu>0. \quad (12)$$

The reliability function is

$$R(t;\mu,\sigma,\tau)=1-F(t)=\exp\left(-\left(\frac{t-\tau}{\mu}\right)^{\sigma}\right), \quad (13)$$

where t is the random variable of life, μ is the proportional parameter, σ is the shape parameter, and τ is the location parameter. $t>0$, $\mu>0$, and $\sigma>0$.

2.3.1. The Parameter Estimation of Three-Parameter Log-Normal Distribution. In the process of parameter estimation of three-parameter log-normal distribution, the integral transformation moment method [32], linear moment method [33], and probability weighted moment method [34] were used for comparative analysis, respectively. These parameter estimation methods are mature and widely used in many fields, and the details are as follows.

(1) Integral Transformation Moment Method. In the following formulas, τ is the location parameter, which can be determined by mean value Y, coefficient of variation C_v, and coefficient of skew C_s, that is,

$$\begin{aligned}\tau&=Y\left(1-\frac{C_v}{\eta}\right)\\ \sigma&=\sqrt{\ln\left(1-\eta^2\right)}\\ \mu&=\frac{1}{2}\ln\left[\frac{1+\eta^2}{(Y-\tau)^2}\right]\\ \eta&=\left(\frac{C_s-\sqrt{C_v^{\,2}+4}}{2}\right)^{1/3}-\left(\frac{-C_s-\sqrt{C_s^{\,2}+4}}{2}\right)^{1/3},\end{aligned} \quad (14)$$

where

$$\begin{aligned}Y&=\frac{1}{n}\sum_{i=1}^{n}t_i\\ C_v&=\frac{1}{Y}\sqrt{\frac{\sum_{i=1}^{n}(t_i-1/Y)}{n-1}}\\ C_s&=3\Phi+\Phi^3,\end{aligned} \quad (15)$$

with

$$\Phi=\sqrt{\exp\left(\sigma^2\right)-1}, \quad (16)$$

where Φ stands for standard normal distribution.

(2) L-Moment Method

$$\begin{aligned}\sigma&=\tau_3\frac{E_0+E_1{\tau_3}^2+E_2{\tau_3}^4+E_3{\tau_3}^6}{1+F_1{\tau_3}^2+F_2{\tau_3}^4+F_3{\tau_3}^6}\\ \mu&=\ln\left[\frac{-\lambda_2e^{-\sigma^2/2}}{1-2\Phi\left(\sigma/\sqrt{2}\right)}\right]\\ \tau&=\lambda_1-e^{\mu+\sigma^2/2},\end{aligned} \quad (17)$$

where E_0, E_1, E_2, E_3, F_1, F_2, and F_3 are constants. Φ refers to standard normal distribution, and linear moments λ_1, λ_2, and τ_3 are determined by life data of given sample.

(3) Probability Weighted Moment Method

$$\tau=u\left(1-\frac{C_v}{k}\right) \quad (18)$$

$$\sigma=\sqrt{\ln\left(1+k^2\right)} \quad (19)$$

$$u=\ln(u-\tau)-\frac{1}{2}\ln\left(1+k^2\right) \quad (20)$$

$$k=\left(\frac{\sqrt{C_s^{\,2}+4}+C_s}{2}\right)^{1/2}-\left(\frac{\sqrt{C_s^{\,2}+4}-C_s}{2}\right)^{1/3} \quad (21)$$

$$C_v=H\left(\frac{M_1}{M_0}-\frac{1}{2}\right) \quad (22)$$

$$R=\frac{M_2-M_0/3}{M_1-M_0/2}, \quad (23)$$

where H and R are function of C_s, both of which cannot be expressed by explicit formulation; C_v and C_s are coefficient of variation and coefficient of skew of variable t, respectively. M_0, M_1, and M_2 are the probability weighted moments [35] of zero-order, first-order, and two-order, respectively.

$$H=3.545+34.7v+220v^2+178v^3+12160v^4 \quad (24)$$

with

$$\begin{aligned}v&=\frac{(R-1)^2}{(4/3-R)^{0.2}},\quad\left(1\leq R<\frac{4}{3}\right),\\ C_s&=12.83w+3.8w^2+40.5w^3+203w^4+855w^5\\ w&=\frac{(R-1)}{(4/3-R)^{0.3}},\quad\left(1\leq R<\frac{4}{3}\right)\end{aligned} \quad (25)$$

where v and w are transition variables to solve function value for H and C_s.

2.3.2. The Parameter Estimation of Three-Parameter Weibull Distribution. The k-order exceeding probability weighted moment [36] equation of Weibull distribution is

$$M_{1,0,k} = \frac{\gamma}{1+k} + \frac{\eta\Gamma(1+1/\sigma)}{(1+k)^{(1+1/\sigma)}}, \tag{26}$$

where Γ is gamma function; for convenience, k is valued as $k = 0, 1, 3$, and we can obtain $M_{1,0,0}$, $M_{1,0,1}$, and $M_{1,0,3}$, and the three parameters of three-parameter Weibull distribution are

$$\begin{aligned} \sigma &= \frac{\ln 2}{\ln\left(\left(M_{1,0,0} - 2M_{1,0,1}\right)/2\left(M_{1,0,1} - 2M_{1,0,3}\right)\right)}, \\ \tau &= \frac{4\left(M_{1,0,3}M_{1,0,0} - M_{1,0,1}^2\right)}{4M_{1,0,3} + M_{1,0,0} - 4M_{1,0,1}} \\ \mu &= \frac{M_{1,0,0} - \tau}{\Gamma[1/\sigma]}. \end{aligned} \tag{27}$$

The exceeding probability weighted moments of observed sample are

$$\begin{aligned} M_{1,0,0} &= \frac{1}{n}\sum_{i=1}^{n} t_i \\ M_{1,0,1} &= \frac{1}{n}\sum_{i=1}^{n} t_i\left(1 - \frac{i-0.35}{n}\right) \\ M_{1,0,3} &= \frac{1}{n}\sum_{i=1}^{n} t_i\left(1 - \frac{i-0.35}{n}\right)^3. \end{aligned} \tag{28}$$

2.4. Improved Maximum Entropy Reliability Model. Improved maximum entropy method can make an optimal estimation with minimum subjective bias on unknown probability distribution. Firstly, according to reliability empirical formula, the reliability empirical vector **R**0 can be obtained for failure data. Secondly, using the empirical value of vector **R**0 to adversely deduce a frequency vector for discrete failure, a statistical histogram is acquired, which is convenient to be used to calculate Lagrangian multipliers, and it is different for the traditional maximum entropy to use amount of sample data to solve the Lagrangian multipliers. Then, based on an internal mapped method, probability density function $f(x)$ for the improved maximum entropy can be obtained. Finally, reliability function for estimate true value is then acquired by integrating the function of $f(x)$.

2.4.1. Discrete Failure Frequency Vector. According to statistic theory, from the reliability empirical vector **R**0 in (2), we can get the discrete cumulative failure probability vector $\mathbf{F}_1$:

$$\mathbf{F}_1 = \{f_i\} = 1 - R = \{1 - r(x_{0i})\}; \quad i = 1, 2, \ldots, n, \tag{29}$$

where $r(x_{0i})$ is the reliability of the ith failure data of initial data $\mathbf{X}_0$ in (1) and n is the number of initial data.

Suppose the corresponding discrete failure probability of each failure life data is p_{0i}. For the first data, that is, when $i = 1$, let its failure probability be $p_{01} = f_1$. So, from the second data, that is, when $i = 2, 3, \ldots, n$, the corresponding failure probability of each failure life can be obtained by cumulatively subtracting the elements in vector $\mathbf{F}_1$ successively as follows: $p_{0i} = f_i - f_{i-1}$, $i = 2, 3, \ldots, n$.

So the discrete failure frequency vector of its failure life data is

$$\mathbf{P}_{0i} = \{p_{0i}\} = \begin{cases} \varphi_1 = f_1, & i = 1 \\ \varphi_i = \{f_i - f_{i-1}\}, & i = 2, 3, \ldots, n. \end{cases} \tag{30}$$

Let (30) correspond to statistic histogram, in which the abscissa is discrete failure life data x_{0i}, and ordinate is the frequency $p_q = p_{0(q-1)}$, $q = 2, \ldots, n+1$ that corresponds to class mid-value $x_q = x_{0(q-1)}$ of each group. Normally, the histogram can be expanded to $n+2$ group, that is, $q = 1, 2, \ldots, n+2$, and let $p_1 = p_{n+2} = 0$, $x_1 = x_{01} - (x_{02} - x_{01})$, $x_{n+2} = x_{0n} + (x_{02} - x_{01})$. Here, the processing of histogram is beneficial for utilizing Newton's method to solve Lagrangian multipliers in maximum entropy probability density function in the following.

2.4.2. Maximum Entropy Probability Distribution Density Function. Suppose the probability distribution density function with maximum entropy is

$$f(t) = \exp\left(\sum_{k=0}^{m} c_k t^k\right), \tag{31}$$

where t is the random variable of life; m is origin moment order, generally let $m = 3$~8, and commonly $m = 5$; c_k is the kth Lagrangian multiplier, $k = 0, 1, \ldots, m$, totally $m+1$.

The first Lagrangian multiplier c_0 is

$$c_0 = -\ln\left(\int_S \exp\left(\sum_{k=1}^{m} c_k t^k\right) dt\right). \tag{32}$$

Other m Lagrangian multipliers shall satisfy

$$g_k = g(c_k) = 1 - \frac{\int_S t^k \exp\left(\sum_{j=1}^{m} c_j t^j\right) dt}{m_k \int_S \exp\left(c_k t^k\right) dt} = 0 \qquad k = 1, 2, \ldots, m. \tag{33}$$

Newton iteration method can be used to solve Lagrangian multiplier vector **c**.

2.4.3. Improved Maximum Entropy Probability Distribution Numerical Solution. There are some difficulties to obtain the solution procedure of probability distribution by improved maximum entropy method. To achieve a quick numerical solution with good convergence, this article adopted the internal mapped Newton iteration method. First, the failure data series were mapped onto dimensionless interval $[-e, e]$, $e = 2.718282$. Then, the mapping data were sorted from small to large into Q-2 groups, and histogram was drawn, to obtain mid-value t_q and frequency p_q of each group. Later, the histogram is extended into Q groups; namely, $p_1 = p_Q = 0$, $q = 1, 2, \ldots, Q$.

The value of k order origin moment m_k changes to

$$m_k = \sum_{q=1}^{n+2} t_q^k p_q \quad k = 0, 1, \ldots, m;\ m_0 = 1. \tag{34}$$

The integration variable t turns into mapped variable x, and integrating range S is mapped onto $[-e, e]$. The improved maximum entropy probability distribution density function changes to

$$f(x) = \exp\left[c_0 + \sum_{k=1}^{m} c_k (ax + b)^k\right], \tag{35}$$

where a and b are interval mapping parameters.

Integrate the improved maximum entropy distribution density function $f(x)$ in interval $S = [x_{01}, x_{0n}]$. The obtained cumulative failure probability function F is

$$F = \int_S f(x)\, \mathrm{d}x. \tag{36}$$

Therefore, the improved maximum entropy reliability estimation truth function can be expressed as

$$R(x) = 1 - F. \tag{37}$$

3. Computer Simulation Verification

3.1. Reliability Median Ranks Empirical Model Verification. Through comparing the reliability discrete vector obtained from empirical equation (3) with the reliability of simulated failure data of known Weibull distribution and log-normal distribution, we verified the feasibility of median ranks empirical model of failure data.

Example 1 (two-parameter Weibull distribution simulation example). Suppose that two-parameter Weibull (TWPW) distribution parameter $(\mu, \sigma) = (80, 3)$. Let the value range of reliability simulated vector $\mathbf{R}_1$ be 0.95~0.05 with interval of −0.05. According to the inverse function of two-parameter Weibull reliability, we got 19 simulation data $\mathbf{T}_1$, which is failure data. Then according to reliability median ranks empirical model in (3), empirical point can be figured out to compose empirical vector $\mathbf{R}0$, and the results are shown in Figure 1.

$$\begin{gathered}\mathbf{R}_1 = [0.95\ \ 0.90\ \ 0.85\ \ 0.80\ \ 0.75\ \ 0.70\ \ 0.65\ \ 0.60\ \ 0.55\ \ 0.50\ \ 0.45\ \ 0.40\ \ 0.35\ \ 0.30\ \ 0.25\ \ 0.20\ \ 0.15\ \ 0.10\ \ 0.05] \\ \mathbf{T}_1 = [29.72\ \ 37.78\ \ 43.66\ \ 48.52\ \ 52.81\ \ 56.73\ \ 60.42\ \ 63.95\ \ 67.39\ \ 70.80\ \ 74.22\ \ 77.70\ \ 81.31\ \ 85.11\ \ 89.20\ \ 93.75\ \ 99.03\ \ 105.64\ \ 115.33] \\ \mathbf{R}0 = [0.96\ \ 0.91\ \ 0.86\ \ 0.81\ \ 0.76\ \ 0.71\ \ 0.65\ \ 0.60\ \ 0.55\ \ 0.50\ \ 0.45\ \ 0.40\ \ 0.35\ \ 0.29\ \ 0.24\ \ 0.19\ \ 0.14\ \ 0.09\ \ 0.04].\end{gathered} \tag{38}$$

Example 2 (three-parameter Weibull distribution simulation example). Suppose that three-parameter Weibull (THPW) distribution parameter $(\mu, \sigma, \tau) = (80, 3, 10)$. Let the value range of reliability simulated vector $\mathbf{R}_1$ be 0.95~0.05 with interval of −0.05. From inverse function of three-parameter Weibull distribution reliability, we got 19 simulation data $\mathbf{T}_2$, which is failure data. According to reliability median ranks empirical model in (3), empirical point can be figured out to compose empirical vector $\mathbf{R}0$, and the results are shown in Figure 2.

$$\mathbf{T}_2 = [39.72\ \ 47.78\ \ 53.66\ \ 58.52\ \ 62.81\ \ 66.73\ \ 70.42\ \ 73.95\ \ 77.39\ \ 80.80\ \ 84.22\ \ 87.70\ \ 91.31\ \ 95.11\ \ 99.20\ \ 103.75\ \ 109.03\ \ 115.64\ \ 125.33]. \tag{39}$$

Example 3 (two-parameter log-normal distribution simulation example). Suppose that two-parameter log-normal (TWPLN) distribution parameter $(\mu, \sigma) = (4, 0.3)$. Let the value range of reliability simulated vector $\mathbf{R}_1$ be 0.95~0.05 with interval of −0.05. From inverse function of two-parameter log-normal distribution reliability, we got 19 simulation data $\mathbf{T}_3$, which is failure data. According to reliability median ranks empirical model in (3), empirical point can be figured out to compose empirical vector $\mathbf{R}0$, and the results are shown in Figure 3.

$$\mathbf{T}_3 = [33.33\ \ 37.17\ \ 40.01\ \ 42.41\ \ 44.60\ \ 46.61\ \ 48.64\ \ 50.60\ \ 52.58\ \ 54.60\ \ 56.69\ \ 58.91\ \ 61.29\ \ 63.90\ \ 66.48\ \ 70.28\ \ 74.5\ \ 80.19\ \ 89.43]. \tag{40}$$

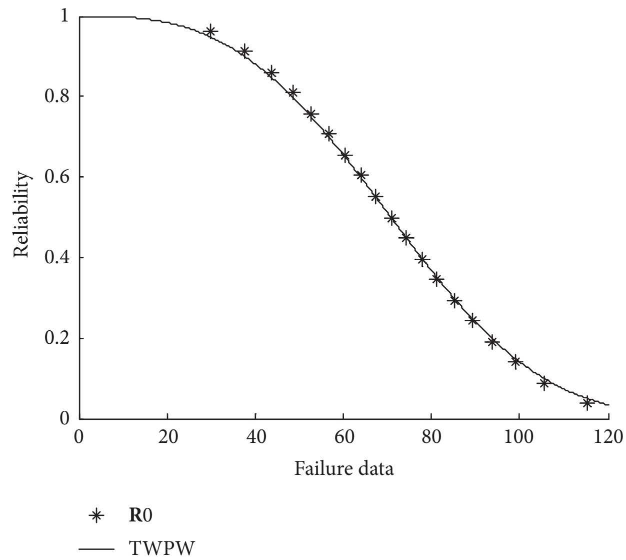

FIGURE 1: Verification of empirical model by two-parameter Weibull model.

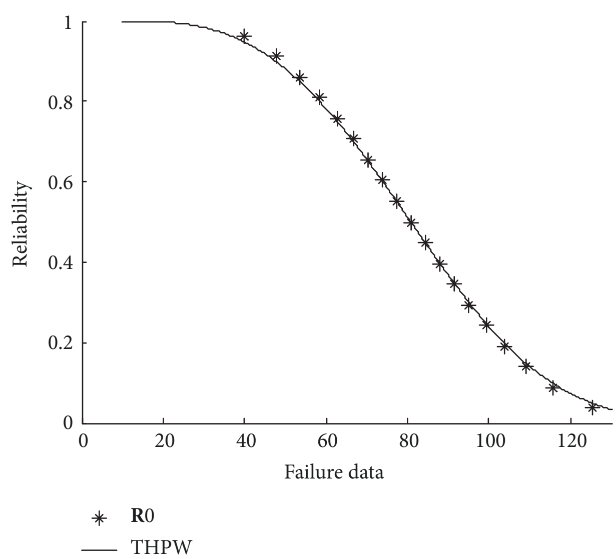

FIGURE 2: Verification of empirical model by three-parameter Weibull model.

Example 4 (three-parameter log-normal distribution simulation example). Suppose that three-parameter log-normal (THPLN) distribution parameter $(\mu, \sigma, \tau) = (4, 0.3, 10)$. Let the value range of reliability simulated vector $\mathbf{R}_1$ be 0.95~0.05 with interval of −0.05. From inverse function of three-parameter log-normal distribution reliability, we got 19 simulation data $\mathbf{T}_4$, which is failure data. According to reliability median ranks empirical model in (3), empirical point can be figured out to compose empirical vector **R**0, and the results are shown in Figure 4.

$$\mathbf{T}_4 = [34.33 \quad 38.17 \quad 41.01 \quad 43.41 \quad 45.60 \quad 47.61 \quad 49.64 \quad 51.60 \quad 53.58 \quad 55.60 \quad 57.69 \quad 59.91 \quad 62.29 \quad 64.90 \quad 67.48 \quad 71.28 \quad 75.5 \quad 81.19 \quad 90.43]. \tag{41}$$

In the above four simulation examples, it can be seen that the median ranks empirical method can excellently describe two-parameter Weibull distribution, three-parameter Weibull distribution, two-parameter log-normal distribution, and three-parameter log-normal distribution. The reliability empirical values almost completely fall on the known

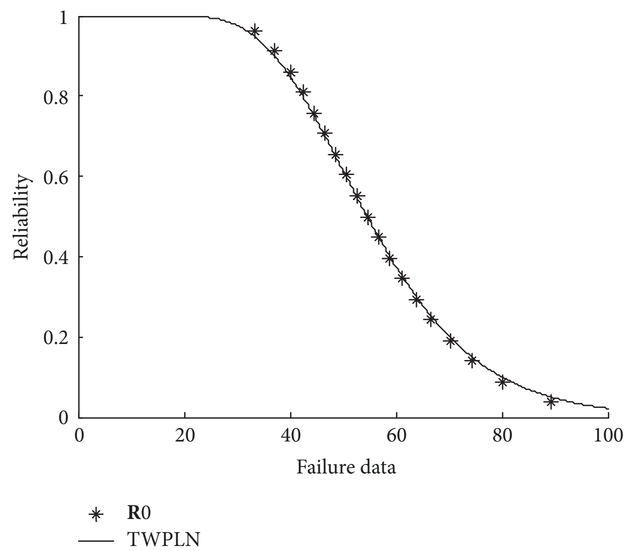

Figure 3: Verification of empirical model by two-parameter log-normal model.

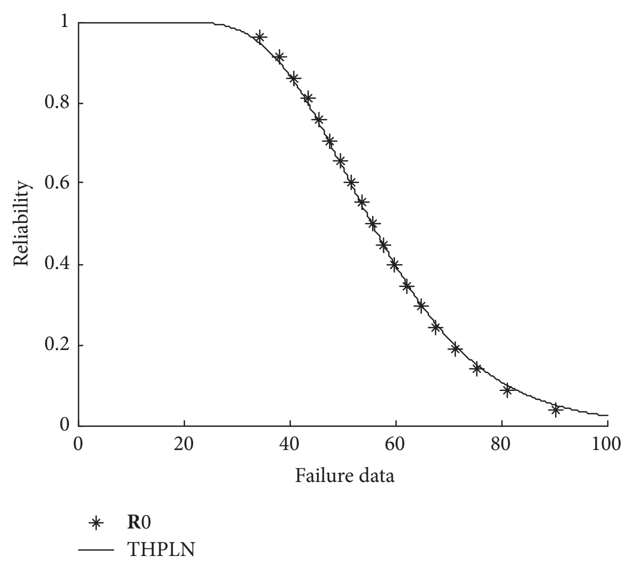

Figure 4: Verification of empirical model by three-parameter log-normal model.

distribution function curve, which accurately describes the distribution law of failure data. In four simulation examples, the standard deviations between empirical vector **R**0 and known distribution $\mathbf{R}_1$ are all 0.0079, which indicates there is tiny standard deviation between the known reliability of failure data and that obtained by the empirical method. The research shows that median ranks empirical model can well estimate the reliability of product failure data with known probability distribution, without estimating the parameters, which avoids possible errors in parameter estimation and greatly improves the accuracy of estimation. So, in the following reliability model estimation, empirical vector will be taken as criterion to comparatively analyze the fitting degree of other models and empirical value, so as to determine whether reliability model is good or bad.

Though this empirical formula does not need parameter estimation, is easy to use, and can precisely compute reliability, the estimation results of reliability truth are discrete, fluctuant, and uncertain, making it hard to conduct continuous estimation. Moreover, when failure data repeatedly appear, this formula cannot estimate the failure probability precisely. In order to get more accurate and continuous reliability estimation model, it is necessary to make further study on fitting curve of failure data (researched later).

3.2. Two-Parameter Model and Improved Maximum Entropy Model Verification. If using two-parameter log-normal distribution and two-parameter Weibull distribution as reliability model, it is required to verify the accuracy of

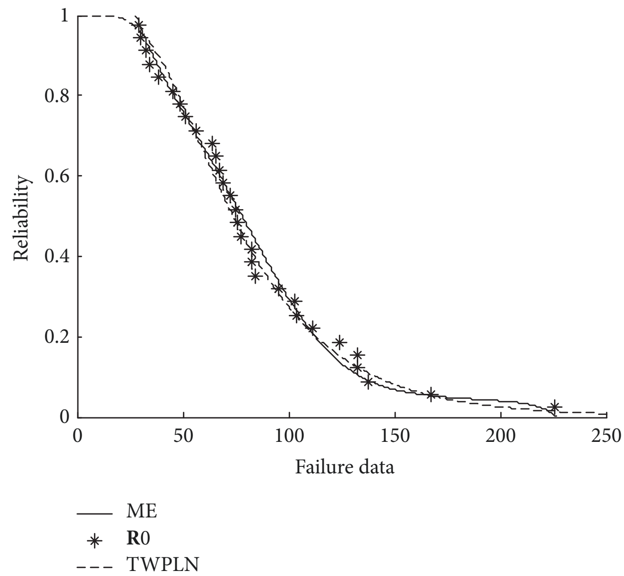

Figure 5: Reliability model simulation of two-parameter log-normal distribution and improved maximum entropy distribution.

parameter estimation on two-parameter Weibull distribution and two-parameter log-normal distribution by maximum likelihood method. The improved maximum entropy method does not take function distribution into account, but using simulated data of two-parameter Weibull distribution and two-parameter log-normal distribution to verify the practicability of improved maximum entropy method.

First, generate a group of two-parameter log-normal distributed random numbers by computer, with parameter setting as $(\mu, \sigma) = (4, 0.4)$, $n = 30$, and n is the number of simulated data.

$$t1 = [29.09 \;\; 30.22 \;\; 32.78 \;\; 34.66 \;\; 38.35 \;\; 45.32 \;\; 48.84 \;\; 51.61 \;\; 56.07 \;\; 63.90 \;\; 65.64 \;\; 67.08 \;\; 69.18 \;\; 72.84 \;\; 74.99 \;\; 75.45 \;\; 77.45 \;\; 82.25 \;\; 82.55 \;\; 84.01 \;\; 95.65 \;\; 103.22 \;\; 103.85 \;\; 111.42 \;\; 124.51 \;\; 132.89 \;\; 133.01 \;\; 137.55 \;\; 167.45 \;\; 225.53]. \quad (42)$$

Use the reliability empirical equation (3) for empirical value (**R**0) to implement point estimation on these 30 random numbers.

Use maximum likelihood method's equation (8) for two-parameter log-normal (TWPLN) to implement parameter estimation on these 30 random numbers:

$$\mu = 4.3024, \quad \sigma = 0.5088. \quad (43)$$

Use internal mapped method's equation (35) for improved maximum entropy (ME) to implement probability density estimation on these 30 random numbers.

Substitute the estimated parameters into (5) and (37) to obtain relevant reliability function, and substitute $t1$ into (3) to obtain empirical point **R**0. The results are shown in Figure 5.

From Figure 5, it is observed that reliability curve of two-parameter log-normal distribution conforms to the distribution of empirical points. Let significance level $\alpha = 0.05$. We can get the k-s test values of log-normal distribution fitting model and improved maximum entropy model which are 0.07 and 0.09, respectively, and the critical value is $Dc = 0.2417$, which shows that the test values are all less than critical values. Furthermore, this proves that using maximum likelihood method to make parameter estimation on two-parameter log-normal distribution can get better effect, and improved maximum entropy probability distribution model can be well applied in two-parameter log-normal distribution.

Generate a group of random numbers on two-parameter Weibull distribution by computer, with parameter setting as $(\mu, \sigma) = (60, 2.5)$, $n = 30$.

$$t2 = [10.2 \;\; 15.76 \;\; 17.91 \;\; 22.11 \;\; 28.86 \;\; 32.11 \;\; 36.24 \;\; 37.83 \;\; 39.54 \;\; 40.82 \;\; 43.79 \;\; 46.44 \;\; 47.32 \;\; 47.4 \;\; 48.79 \;\; 50.85 \;\; 55.76 \;\; 59.56 \;\; 59.64 \;\; 61.19 \;\; 61.42 \;\; 64.26 \;\; 67.89 \;\; 69.02 \;\; 71.94 \;\; 76.63 \;\; 76.76 \;\; 77.27 \;\; 85.23 \;\; 114.4]. \quad (44)$$

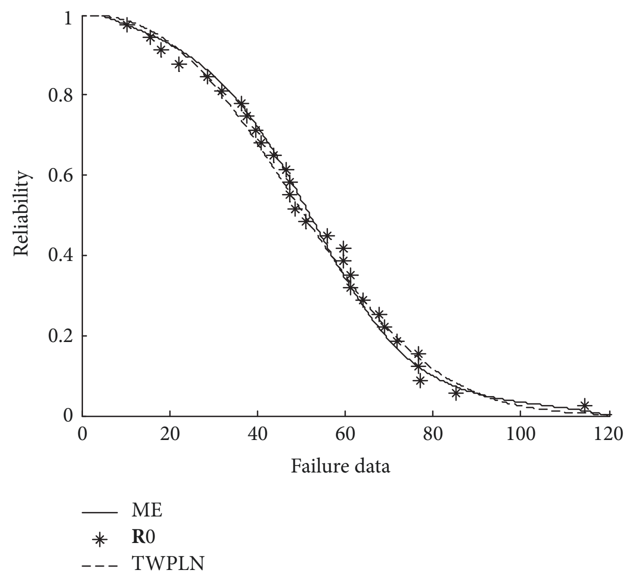

FIGURE 6: Reliability model simulation of two-parameter Weibull distribution and improved maximum entropy distribution.

Use the reliability empirical equation (3) for empirical value (**R**0) to implement point estimation on these 30 random numbers.

Use maximum likelihood method's equation (9) for two-parameter Weibull (TWPW) to implement parameter estimation on these 30 random numbers:

$$\mu = 58.8290,$$
$$\sigma = 2.4744. \tag{45}$$

Use internal mapped method's equation (35) for improved maximum entropy (ME) to implement probability density estimation on these 30 random numbers.

Substitute the estimated parameters into (7) and (37) to obtain relevant reliability function, and substitute $t2$ into (3) to obtain empirical point **R**0. The results are shown in Figure 6.

From Figure 6, it is observed that reliability curve of two-parameter Weibull distribution conforms to the distribution of empirical points. Use k-s test method, and let significance level $\alpha = 0.05$. With estimated parameters of maximum likelihood method, we can get the k-s test values of two-parameter Weibull distribution fitting model and improved maximum entropy fitting model which are 0.06 and 0.07, respectively, and the critical value is $Dc = 0.2417$, which shows that the test values are all less than the critical values. Furthermore, this proves that using maximum likelihood method to make parameter estimation on two-parameter Weibull distribution is feasible, and improved maximum entropy probability distribution model can be well applied in two-parameter Weibull distribution.

To sum up, from the results of parameter estimation on random numbers of two-parameter log-normal distribution and two-parameter Weibull distribution, it is known that taking maximum likelihood method as parameter estimation method is feasible for these two models with highly accurate estimation results; improved maximum entropy reliability function basically completely coincides with empirical value vector, which proves that this model is suitable for the above two distributions and has small error and high precision.

3.3. Three-Parameter Model and Improved Maximum Entropy Model Verification

3.3.1. Three-Parameter Log-Normal Distribution and Improved Maximum Entropy Distribution. If using integral transformation moment method, linear moment method, and probability weighted moment method as the parameter estimation method of three-parameter log-normal distribution, it is a must to verify the feasibility of these three-parameter estimation methods to three-parameter log-normal distribution.

Let distribution parameter $(\mu, \sigma, \tau) = (4, 0.4, 10)$ generate a group of three-parameter log-normal distributed random numbers by computer ($n = 30$):

$$t3 = [24.05\ \ 24.36\ \ 24.69\ \ 27.19\ \ 28.97\ \ 30.58\ \ 31.38\ \ 32.34\ \ 34.85\ \ 36.61\ \ 36.80\ \ 37.42\ \ 37.74\ \ 44.16\ \ 45.62\ \ 46.79\ \ 54.68\ \ 55.41\ \ 57.81\ \ 61.06\ \ 62.04\ \ 63.48\ \ 63.87\ \ 64.42\ \ 67.65\ \ 69.73\ \ 70.56\ \ 72.37\ \ 83.01\ \ 116.64]. \tag{46}$$

Empirical value (**R**0) uses the reliability empirical equation (3) for point estimation on this random numbers group.

Three-parameter log-normal uses integral transformation moment (ITM) method's equation (14) for parameter estimation on this random numbers group:

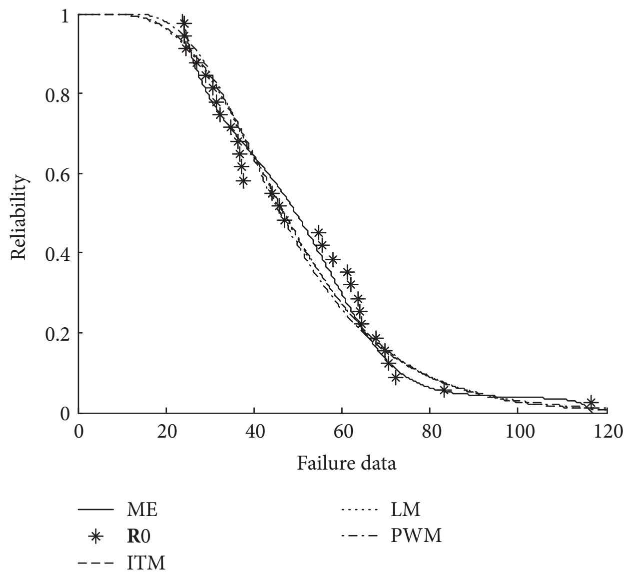

Figure 7: Reliability model simulation of three-parameter log-normal distribution and improved maximum entropy model.

$$\tau = -12.6478, \quad \mu = 4.0870, \quad \sigma = 0.3283. \tag{47}$$

Three-parameter log-normal uses linear moment (LM) method's equation (17) for parameter estimation on this random numbers group:

$$\tau = -13.4789, \quad \mu = 4.4101, \quad \sigma = 0.3287. \tag{48}$$

Three-parameter log-normal uses probability weighted moment (PWM) method's equations (18), (19), and (20) for parameter estimation on this random numbers group:

$$\tau = 1.5063, \quad \mu = 3.7966, \quad \sigma = 0.4222. \tag{49}$$

Improved maximum entropy (ME) uses internal mapped method's equation (35) for probability density estimation on this random numbers group.

Substitute the estimated parameters into (11) and (37) to obtain relevant reliability function, and substitute $t3$ into (3) to obtain empirical point $\mathbf{R}0$, as shown in Figure 7.

It is observed in Figure 7 that the reliability curve basically conforms to the distribution of empirical points, and three-parameter estimation methods and improved maximum entropy probability distribution demonstrate perfect fitting degree. Use k-s test method and let significance level $\alpha = 0.05$ to make hypothesis testing on the results, as shown in Table 1.

In reliability function image, it can be found that three estimation methods have good fitting degree. And it is known from k-s test that the k-s test values of three-parameter estimation methods are less than critical values; thus, three methods are all suitable for three-parameter log-normal distribution. So, when we conduct parameter estimation of three-parameter log-normal distribution on test failure data, the above three estimation methods can be used for estimation. Meanwhile, k-s test value of improved maximum entropy is also less than critical value, indicating that improved maximum entropy probability distribution model can be perfectly applied in three-parameter log-normal distribution with perfect estimation effect.

3.3.2. Three-Parameter Weibull Distribution and Improved Maximum Entropy Distribution. In order to verify the feasibility of k-order exceeding probability weighted moment based parameter estimation method to three-parameter Weibull distribution, let Weibull parameter $(\mu, \sigma, \tau) = (60, 2.5, 10)$, and simulate three-parameter Weibull distribution and life data by computer system ($n = 25$) as below:

$$t4 = [22.60, 29.71, 34.91, 38.15, 41.42, 44.38, 47.12, 49.72, 52.23, 54.66, 57.06, 59.44, 61.82, 64.23, 66.68, 69.21, 71.85, 74.62, 77.59, 80.83, 84.45, 88.63, 93.76, 100.75, 113.54]. \tag{50}$$

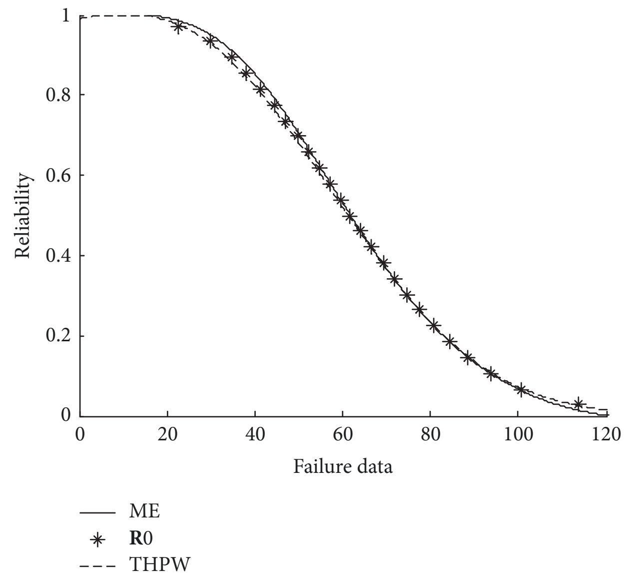

Figure 8: Reliability simulation of three-parameter Weibull distribution and improved maximum entropy model.

Table 1: Comparison of three-parameter log-normal distribution and improved maximum entropy distribution.

Estimation method	k-s critical value	k-s test value	Results
ITM	0.2417	0.1126	Valid
LM	0.2417	0.1063	Valid
PWM	0.2417	0.1131	Valid
ME	0.2417	0.0916	Valid

Empirical value (**R**0) uses the reliability empirical equation (3) for point estimation on this group of random numbers.

Three-parameter Weibull (THPW) distribution uses k-order exceeding probability weighted moment method's equation (27) for parameter estimation on this group of random numbers:

$$\tau = 10.8908,$$
$$\mu = 59.0072, \quad (51)$$
$$\sigma = 2.3293.$$

Improved maximum entropy (ME) uses internal mapped method's equation (35) for probability density estimation on this group of random numbers.

Substitute the estimated parameters into (13) and (37) to obtain relevant reliability function, and substitute $t4$ into (3) to obtain empirical point **R**0, as shown in Figure 8.

From Figure 8, it is observed that reliability curves of three-parameter Weibull distribution and improved maximum entropy probability distribution basically conform to the distribution of empirical points. Use k-s test method, and let significance level $\alpha = 0.05$. We can get the k-s test value of three-parameter Weibull distribution model which is 0.01, the values of improved maximum entropy fitting model are 0.02, and the critical value is $Dc = 0.2641$. As the test values are all less than critical values, it is feasible to use k-order exceeding probability weighted moment method in parameter estimation on three-parameter Weibull distribution, and improved maximum entropy probability distribution model can be well applied in three-parameter Weibull distribution.

To sum up, it can be known from the simulation results of random numbers obeying three-parameter Weibull distribution, three-parameter log-normal distribution, and improved maximum entropy probability distribution that the reliability estimation methods of the above three models are all feasible with highly accurate estimation results. Improved maximum entropy reliability function basically completely coincides with empirical value vector and is suitable for the above two distribution models with small error and high precision. Meanwhile, it also proves that the distribution of sample data can be ignored in improved maximum entropy application process, and its major feature is that it applies to the poor information issues with unknown probability distribution, trends, or prior information. This is because the improved maximum entropy model does not need parameter estimation or consider any distribution, but objectively processing experimental data. That is to say, there is no ideal model artificially presupposed before data processing, which overcomes the influence produced by subjective factor and parameter estimation error, so that the certainty rule in data change can be directly reflected.

At the same time, median rank empirical model can effectively assess the reliability of product failure data with high precision, so, in the following experimental research section, this empirical value should be taken as a criterion to decide whether the reliability model is good or bad; in two-parameter and three-parameter reliability model verification, multiple parameter estimation methods are feasible and effective with high precision and small error; therefore, they are safe and reliable in the following practical case applications; the novel improved maximum entropy model is suitable for all the above conditions, which has laid a foundation to search for quasi-ideal model in the following sections.

4. Experimental Research

4.1. Experimental Facility and Conditions. This bearing life reliability research adopted NTN bearing life experimental facility and material samples. The experimental facility is Φ12 point-contact life testing machine, and the material samples are cylindrical roller of Φ12 mm × 22 mm obtained by processing different batches of material under the same heat treatment conditions. The experiment was conducted at motor speed of 3900 r/min, indoor temperature of 26°C, and humidity of 53%. By applied load from compression spring, the stress imposed on roller and steel ball is 2.55 KN, and the experimental applied maximum contact stress is 5.88 GPa. In the timing from the start of experiment, when surface peeling occurs on contact region of cylindrical roller and steel ball, it will make cylindrical roller vibrate more. And if the vibration value reaches certain amplitude, the inductor switch will automatically jump up, the motor will stop to run, and the experiment ends. At this time, read the test run time of testing machine, which is the failure life of this roller material.

4.2. Test Data. There are 3 groups of failure data test in total. The failure time recorded by experimental facility is initial data with unit in minute. For easy calculation, failure data is conversed into data with unit in hour and sorted from small to large into a group of vector.

In the first batch of test, the failure data is expressed by $T1$ with $n = 26$:

$$T1 = [0.38\ \ 1.685\ \ 1.687\ \ 1.71\ \ 1.8\ \ 1.86\ \ 1.89\ \ 2.06\ \ 2.14\ \ 2.2\ \ 2.42\ \ 2.46\ \ 3.88\ \ 4.89\ \ 6.2\ \ 7.73\ \ 12.46\ \ 12.5\ \ 12.88\ \ 13.33\ \ 31.97\ \ 38.57\ \ 47.5\ \ 50.2\ \ 51.77\ \ 58.71]. \tag{52}$$

In the second batch of test, the failure data is expressed by $T2$ with $n = 30$:

$$T2 = [0.61\ \ 0.69\ \ 1.66\ \ 1.81\ \ 1.91\ \ 1.93\ \ 2.34\ \ 2.36\ \ 2.38\ \ 3.07\ \ 3.075\ \ 3.08\ \ 3.63\ \ 11.80\ \ 12.67\ \ 14.18\ \ 14.29\ \ 16.27\ \ 17.84\ \ 18.83\ \ 26.10\ \ 28.00\ \ 29.79\ \ 47.52\ \ 47.86\ \ 52.91\ \ 53.15\ \ 53.57\ \ 80.20\ \ 90.11]. \tag{53}$$

In the third batch of test, the failure data is expressed by $T3$ with $n = 23$:

$$T3 = [1.46\ \ 1.685\ \ 1.687\ \ 1.88\ \ 2.06\ \ 2.13\ \ 2.25\ \ 2.257\ \ 2.39\ \ 2.48\ \ 2.58\ \ 4.32\ \ 4.97\ \ 8.55\ \ 11.34\ \ 12.78\ \ 15.75\ \ 22.66\ \ 32.49\ \ 69.72\ \ 71.54\ \ 86.36\ \ 86.91]. \tag{54}$$

4.3. Experimental Investigation of Two-Parameter Model and Improved Maximum Entropy Method.

Example 1. Substitute failure data $T1$ into the reliability empirical equation (3) for empirical value (**R**0) estimation.

Substitute failure data $T1$ into two-parameter log-normal (TWPLN) distribution model's equation (8) for parameter estimation:

$$\mu = 1.7859, \quad \sigma = 1.3713. \tag{55}$$

Substitute failure data $T1$ into two-parameter Weibull (TWPW) distribution model's equation (9) for parameter estimation:

$$\mu = 12.0224, \quad \sigma = 0.7589. \tag{56}$$

Substitute failure data $T1$ into improved maximum entropy (ME) model's equation (35) for probability density estimation.

Their reliability curves are shown in Figure 9.

Example 2. Substitute failure data $T2$ into the reliability empirical formula for empirical value estimation.

Substitute failure data $T2$ into two-parameter log-normal distribution for parameter estimation:

$$\mu = 2.2075, \quad \sigma = 1.4681. \tag{57}$$

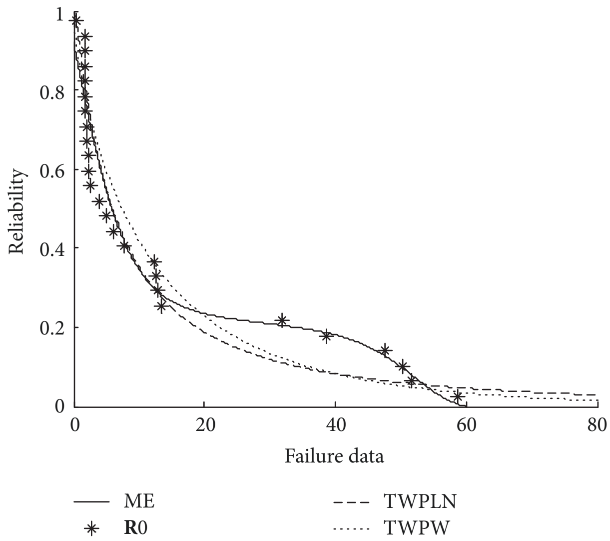

Figure 9: Reliability function image of failure data $T1$.

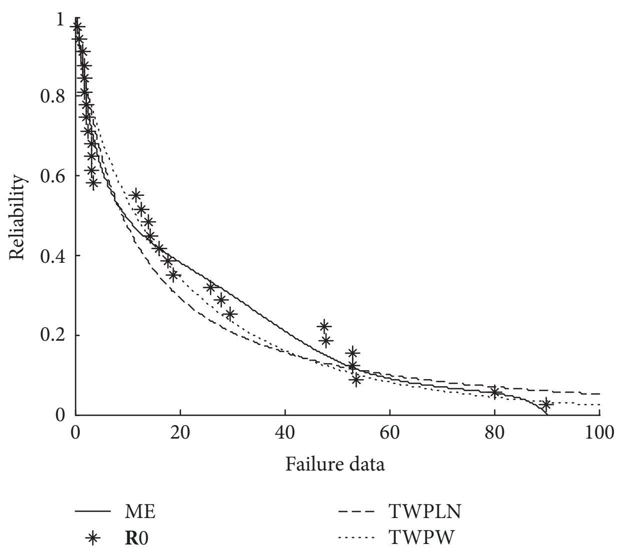

Figure 10: Reliability function image of failure data $T2$.

Substitute failure data $T2$ into two-parameter Weibull distribution for parameter estimation:

$$\mu = 18.6897, \quad \sigma = 0.7861. \tag{58}$$

Substitute failure data $T2$ into improved maximum entropy model for probability density estimation.

Their reliability curves are shown in Figure 10.

Example 3. Substitute failure data $T3$ into the reliability empirical formula for empirical value estimation.

Substitute failure data $T3$ into two-parameter log-normal distribution for parameter estimation:

$$\mu = 1.9509, \quad \sigma = 1.4091. \tag{59}$$

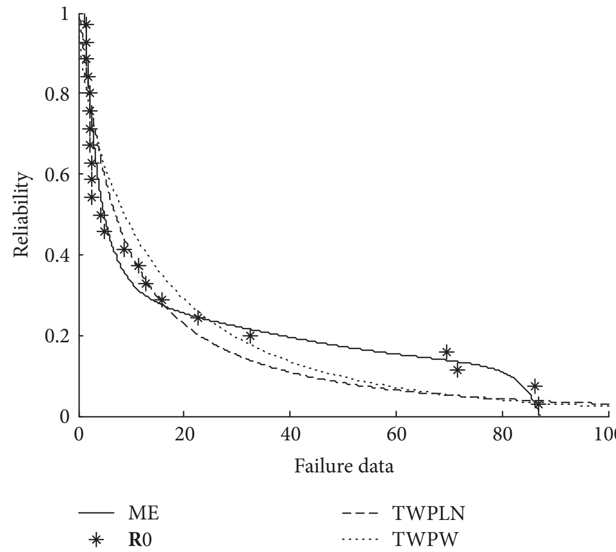

Figure 11: Reliability function image of failure data $T3$.

Substitute failure data $T3$ into two-parameter Weibull distribution for parameter estimation:

$$\mu = 14.7678, \quad \sigma = 0.6938. \tag{60}$$

Substitute failure data $T3$ into improved maximum entropy model for probability density estimation.

Their reliability curves are shown in Figure 11.

From Figures 9 to 11, it can be discovered that three reliability curves basically conform to distribution of empirical points and show good fitting degree. Use k-s test method and let significance level $\alpha = 0.05$, to make hypothesis testing of parameter estimation results; the results are shown in Table 2. The research indicates that k-s test values of three failure data distribution models in each example are all less than the critical values, so three curves can describe the distribution rule of failure data.

Calculate the standard deviation of reliability function value and empirical points and substitute failure data into empirical equation (3), two-parameter log-normal distribution reliability equation (5), two-parameter Weibull distribution reliability equation (7), and improved maximum entropy probability distribution reliability equation (37). Then we can obtain reliability empirical value vector $\mathbf{R}0$ and reliability estimated truth vectors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ and substitute them into standard deviation equation:

$$\sigma_j = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{R}0 - \mathbf{R}_j\right)^2}, \tag{61}$$

where n is the number of failure data of each group and $j = 1, 2, 3$.

According to (61), we can get the standard deviation when each failure data group takes two-parameter log-normal (TWPLN) distribution, two-parameter Weibull (TWPW)

Table 2: Comparative results of three reliability models of failure data.

Distribution model	k-s test value	Critical value	Standard deviation	$P1$	$P2$
$T1$ TWPLN	0.2024	0.2591	0.0802	1.0295	5.9662
$T1$ TWPW	0.2024	0.2591	0.0899	0.6197	7.4173
$T1$ ME	0.1631	0.2591	0.0737	0.2483	5.8670
$T2$ TWPLN	0.1547	0.2417	0.0757	1.3854	9.0933
$T2$ TWPW	0.1696	0.2417	0.0685	1.0675	11.7251
$T2$ ME	0.1140	0.2417	0.0498	1.2390	9.7120
$T3$ TWPLN	0.2400	0.2749	0.0950	1.1562	7.0353
$T3$ TWPW	0.2205	0.2749	0.1008	0.5763	8.7075
$T3$ ME	0.1872	0.2749	0.0701	1.6342	4.9040

Table 3: Relative errors of each group data under different distributions.

Relative error	$T1$ TWPLN	$T1$ TWPW	$T2$ TWPLN	$T2$ TWPW	$T3$ TWPLN	$T3$ TWPW
f_1	314.62%	149.58%	11.81%	13.84%	29.25%	64.74%
f_2	1.69%	26.42%	6.37%	20.73%	43.46%	77.56%

distribution, and improved maximum entropy (ME) probability distribution as reliability models. At the same time, figure out the life value of three distributions when their life failure probability is 10% and 50%, respectively. That is to say, when reliability function $R(t) = 0.9$, the value of life t is $P1$, and when $R(t) = 0.5$, t value is $P2$. Comparing the life values under these two failure probabilities, and integrating standard deviation, we can determine which reliability model has small error and high precision; the results are shown in Table 2. This is the main problem with this article to be revealed; the difference of the standard deviation for each model may be small, but the difference of life value t under life reliability of 90% and 50% may be extremely large. Thus, an accurate reliability model of bearing failure data is vital and necessary.

It can be known from Table 2 that the k-s test values of the three reliability models are all less than critical values, and all models conform to these failure data groups and thus can be taken as their reliability model. There is very small difference between the calculated standard deviations of each data group by three models, all at 0.01 orders of magnitude (except 0.1008), which indicates that three models have high precision in reliability estimation. However, in comparison with $P1$ or $P2$ value of each example, it is not difficult to discover that there is large difference between $P1$ and $P2$ values for different distribution models. That is also the greatest distinction in numerical solution by three models. In other words, under equivalent reliability, different estimation model will cause different bearing failure life.

In addition, the life value t under life reliability of 90% and 50% is an important indicator to study reliability. We took the distribution of small standard deviation, that is, the improved maximum entropy probability distribution, as datum to calculate the relative error of other two distributions under failure probability of 10% and 50%. When reliability is 90%, the relative error of $T1$ log-normal distribution is

$$f\,(T1)_1 = \frac{1.0295 - 0.2483}{0.2483} = 3.1458 = 314.62\%. \quad (62)$$

When reliability is 50%, the relative error of $T1$ log-normal distribution is

$$f\,(T1)_2 = \frac{5.9662 - 5.8670}{5.8670} = 0.0169 = 1.69\%. \quad (63)$$

Similarly, the relative errors of each data group under different distributions can be obtained as shown in Table 3.

From Table 3, though there is very small difference between the standard deviations of three models, under reliability of 90% and 50%, their life values can be far different and the maximum relative error reaches 314.62%. Hence, the selection of distribution model will directly affect the precision of predicted reliability, which should not be blindly chosen in practical applications. That is to say, in reliability function image, provided a small change in ordinate value, it may bring very large change in abscissa value. To reduce life estimation error, this article judged the fitting degree of reliability model by whether standard deviation is large or small.

In the table, the standard deviations of reliability estimation truth-value vector and reliability empirical value vector for $T1$ and $T3$ two-parameter log-normal distribution groups are less than those of two-parameter Weibull distribution. But $T2$ groups are the opposite. This illustrates that on premise of that both distributions satisfy this failure data group, sometimes, the two-parameter log-normal distribution being taken as a reliability model may have higher fitting and smaller error than two-parameter Weibull distribution. So, in daily reliability analysis on bearing life, people should not merely use two-parameter Weibull distribution as reliability model for analysis, in order to prevent large error in bearing reliability predication and the occurrence of vicious accident. Regarding the above three examples, all the improved maximum entropy curves participate in the fitting, and fitting curves almost coincide with empirical values. What is more important is that the standard deviations of this model are all the smallest in reliability estimation on $T1$, $T2$, and $T3$ failure data groups. In summary, compared to two-parameter

Table 4: Comparison results of failure data $T1$ by three models.

Estimating method	k-s test value	k-s critical value	Result	Standard deviation
ITM	0.2127	0.2591	Valid	0.1116
LM	0.3715	0.2591	Invalid	0.1486
PWM	0.4615	0.2591	Invalid	0.2183
THPW	0.1847	0.2591	Valid	0.0773
ME	0.1631	0.2591	Valid	0.0737

log-normal distribution and Weibull distribution, the novel improved maximum entropy reliability estimation model is more accurate to reflect the general change rule of failure data and it possesses the smallest error and the highest precision in bearing reliability prediction.

4.4. Experimental Research of Three-Parameter Model and Improved Maximum Entropy Method.

Example 1. Empirical value (**R**0) uses the reliability empirical equation (3) for point estimation on failure data $T1$.

There-parameter log-normal uses integral transformation moment (ITM) method's equation (14) for parameter estimation on failure data $T1$:

$$\tau = -8.0525,\quad \mu = 2.8467,\quad \sigma = 0.7287. \tag{64}$$

There-parameter log-normal uses linear moment (LM) method's equation (17) for parameter estimation on failure data $T1$:

$$\tau = -8.4671,\quad \mu = 2.5606,\quad \sigma = 1.0676. \tag{65}$$

There-parameter log-normal uses probability weighted moment (PWM) method's equation (18) for parameter estimation on failure data $T1$:

$$\tau = 3.3308,\quad \mu = 1.5137,\quad \sigma = 1.3358. \tag{66}$$

Three-parameter Weibull (THPW) uses k-order exceeding probability weighted moment method's equation (27) for parameter estimation on failure data $T1$:

$$\tau = 0.3655,\quad \mu = 10.5119,\quad \sigma = 0.6631. \tag{67}$$

Improved maximum entropy (ME) uses internal mapped method's equation (35) for probability density estimation on failure data $T1$.

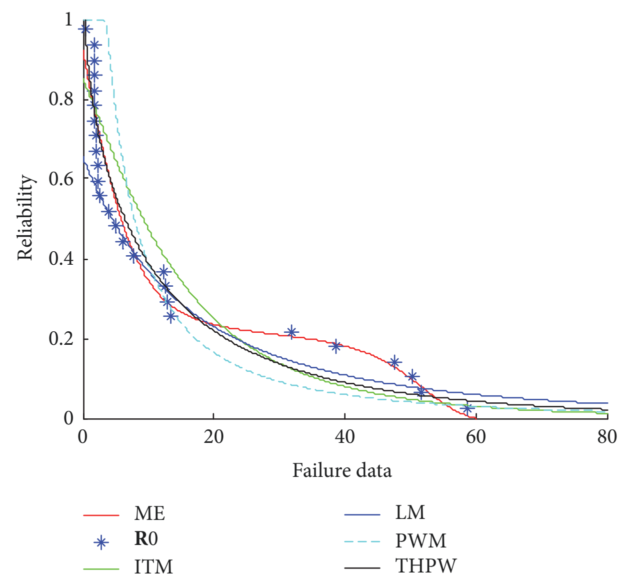

Figure 12: Reliability function image of failure data $T1$.

Their reliability images are shown in Figure 12 and model comparative results are shown in Table 4.

Example 2. Empirical value uses the reliability empirical formula for point estimation on failure data $T2$.

There-parameter log-normal uses integral transformation moment method for parameter estimation on failure data $T2$:

$$\tau = -14.4719,\quad \mu = 3.3865,\quad \sigma = 0.6245. \tag{68}$$

There-parameter log-normal uses linear moment method for parameter estimation on failure data $T2$:

$$\tau = -18.7008,\quad \mu = 3.3481,\quad \sigma = 0.8303. \tag{69}$$

There-parameter log-normal uses probability weighted moment method for parameter estimation on failure data $T2$:

$$\tau = 1.1823,\quad \mu = 2.4503,\quad \sigma = 1.0573. \tag{70}$$

Table 5: Comparison results of failure data $T2$ by three models.

Estimating method	k-s test value	k-s critical value	Result	Standard deviation
ITM	0.2172	0.2417	Valid	0.0813
LM	0.3204	0.2417	Invalid	0.1160
PWM	0.3627	0.2417	Invalid	0.1541
THPW	0.1795	0.2417	Valid	0.0651
ME	0.1140	0.2417	Valid	0.0498

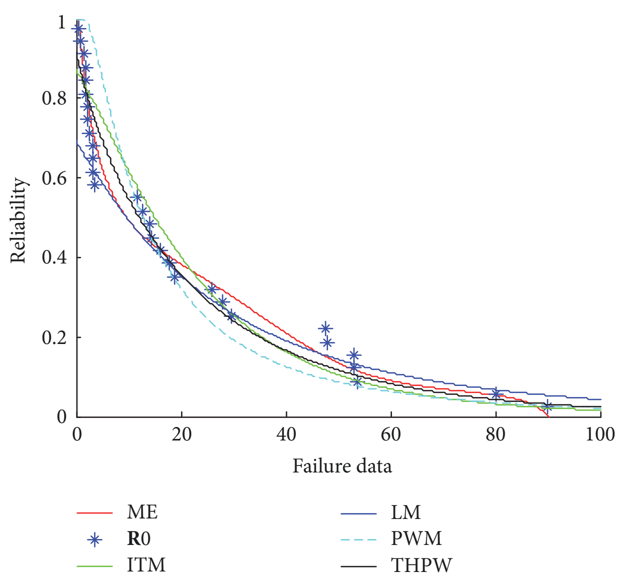

Figure 13: Reliability function image of failure data $T2$.

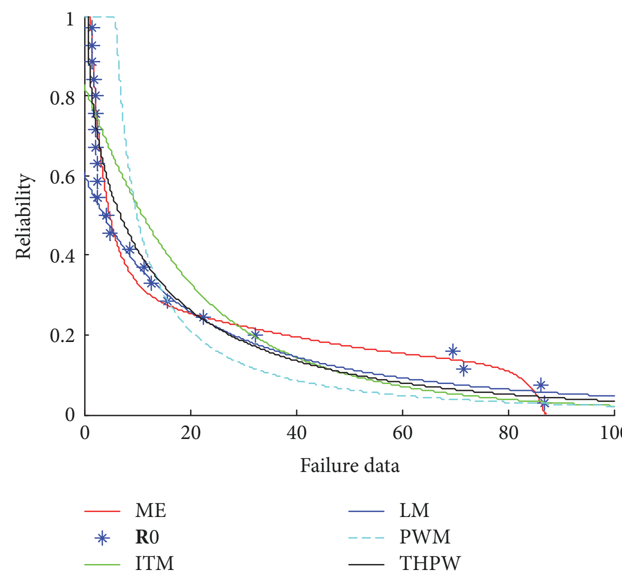

Figure 14: Reliability function image of failure data $T3$.

Three-parameter Weibull uses k-order exceeding probability weighted moment method for parameter estimation on failure data $T2$:

$$\tau = -1.0515,$$
$$\mu = 20.4337, \quad (71)$$
$$\sigma = 0.8334.$$

Improved maximum entropy uses internal mapped method for probability density estimation on failure data $T2$.

Their reliability images are shown in Figure 13 and model comparative results are shown in Table 5.

Example 3. Empirical value uses the reliability empirical formula for point estimation on failure data $T3$.

There-parameter log-normal uses integral transformation moment method for parameter estimation on failure data $T3$:

$$\tau = -9.4007,$$
$$\mu = 3.0202, \quad (72)$$
$$\sigma = 0.8322.$$

There-parameter log-normal uses linear moment method for parameter estimation on failure data $T3$:

$$\tau = -9.2346,$$
$$\mu = 2.5617,$$
$$\sigma = 1.2641. \quad (73)$$

There-parameter log-normal uses probability weighted moment method for parameter estimation on failure data $T3$:

$$\tau = 5.8718,$$
$$\mu = 1.4258, \quad (74)$$
$$\sigma = 1.5440.$$

Three-parameter Weibull uses k-order exceeding probability weighted moment method for parameter estimation on failure data $T3$:

$$\tau = 0.6897,$$
$$\mu = 11.7192, \quad (75)$$
$$\sigma = 0.5706.$$

Improved maximum entropy uses internal mapped method for probability density estimation on failure data $T3$.

Their reliability images are shown in Figure 14 and model comparative results are shown in Table 6.

Table 6: Comparison results of failure data $T3$ by three models.

Estimating method	k-s test value	k-s critical value	Result	Standard deviation
ITM	0.2304	0.2749	Valid	0.1207
LM	0.4397	0.2749	Invalid	0.1727
PWM	0.5652	0.2749	Invalid	0.2533
THPW	0.1907	0.2749	Valid	0.0831
ME	0.1872	0.2749	Valid	0.0702

Three failure data groups were estimated by three-parameter log-normal distribution, three-parameter Weibull distribution, and improved maximum entropy probability distribution. Their reliability images are shown in Figures 12–14, in which, three-parameter log-normal distribution used integral transformation moment, linear moment, and probability weighted moment methods for parameter estimation, three-parameter Weibull distribution used k-order exceeding probability weighted moment methods for parameter estimation, and the novel improved maximum entropy directly made probability fitting operation without considering failure data distribution condition. Then, the k-s test method is used with significance level $\alpha = 0.05$ to implement hypothesis testing on each method and results; finally, it figured out standard deviations according to (61), so as to easily analyze and compare the precision of reliability models; the results are shown in Tables 4–6.

In Tables 4–6, it can be known from hypothesis test results that, in three methods on parameter estimation of three-parameter log-normal distribution, only k-s test values of integral transformation moment method are less than critical values: namely, only integral transformation moment method is suitable for three-parameter log-normal distribution model. But such "Suitable" only satisfies hypothesis test of data verification; in practical applying, the location parameter τ of three-parameter log-normal distribution means the minimum failure data; that is, τ should be larger than zero. While the location parameter τ obtained by using integral transformation moment method is all less than zero in three groups of failure life tests, which goes against the practical significance of this formula in life test application, so integral transformation moment method does not apply to these failure data groups. That is to say, none of the above three-parameter estimations can be taken as an estimation approach for three-parameter log-normal distribution in failure data reliability model application. As a result, at present stage, it is considered that three-parameter log-normal distribution cannot be perfectly used in reliability estimation of bearing life, and its parameter estimation approach needs further exploration.

The k-s test values of three-parameter Weibull distribution are less than the critical values with good fitting; namely, k-order exceeding probability weighted moment method performs well when it is used for parameter estimation of three-parameter Weibull distribution. But for $T2$ failure data group, the location parameter of three-parameter Weibull distribution is $\tau = -1.0515 < 0$, which goes against real meaning of life. This also indicates that three-parameter Weibull distribution performs better on bearing life failure than three-parameter log-normal distribution does, but it does not be the same with all situations. However, in engineering practices, it is widely believed that three-parameter Weibull distribution can better describe the distribution rule of bearing performance failure data, because it has more parameters and comprehensive information mining so that we can consider the general characteristics of research object from multiple aspects. But this research shows that three-parameter Weibull distribution has a large misunderstanding during its wide application in engineering practices, and it may not accurately recognize product's performance reliability in all cases.

Improved maximum entropy reliability curves participate in fitting in three examples, and the fitting curves almost coincide with empirical values. Its standard deviation is the minimum in three groups of life test compared to previous two models, declaring its highest fitting degree. It is observed in Figures 12–14 that the novel improved maximum entropy reliability curves trend to be deck chair shape, directly showing the change trend of initial failure data. The obtained curve does not require parameter estimation, thus preventing calculation error that may be brought by traditional mathematical hypothesis definition and condition limitation. In summary, relative to three-parameter log-normal distribution and Weibull distribution, the improved maximum entropy probability distribution can make reliability estimation with the highest precision and minimum error and directly reflect the general change rule of failure data. No matter there is truly distribution model (quasi-ideal distribution model) to calculate bearing failure data reliability or not, at least, the novel improved maximum entropy probability distribution is a precise simulation of quasi-ideal failure data distribution model.

4.5. Improved Maximum Entropy Reliability Test. Through calculation in test examples, it can be obtained that the reliability estimation method of improved maximum entropy probability distribution is suitable for all situations, and it is observed that improved maximum entropy reliability model estimation approach has a perfect effect on reliability estimation on bearing failure data. In reliability model estimation on three failure data groups, the two-parameter log-normal distribution of $T1$ and $T3$ is superior to two-parameter Weibull distribution, and the two-parameter log-normal distribution of $T2$ is inferior to two-parameter Weibull distribution; the three-parameter Weibull distribution of $T1$ and $T3$ is superior to three-parameter log-normal distribution, and both distributions of $T2$ group are not satisfactory. So we compared the two-parameter log-normal (TWPLN) distribution of $T1$ and $T3$, two-parameter Weibull (TWPW) distribution of $T2$,

Table 7: Compare the standard deviations of three groups of experimental feasible reliability models.

Group number	2-parameter std. deviation	3-parameter std. deviation	ME std. deviation
$T1$	0.0802 (TWPLN)	0.0773 (THPW)	0.0737
$T2$	0.0685 (TWPW)	N/A	0.0498
$T3$	0.0950 (TWPLN)	0.0831 (THPW)	0.0702

Table 8: Compare the relative life errors of three groups of experimental feasible models.

Group number	2-parameter f_1	2-parameter f_2	3-parameter f_1	3-parameter f_2
$T1$	314.62%	1.69%	189.37%	9.32%
$T2$	13.84%	20.73%	N/A	N/A
$T3$	29.25%	43.46%	43.91%	39.78%

and three-parameter Weibull (THPW) distribution of $T1$ and $T3$ with improved maximum entropy (ME) method, respectively, and the results are shown in Tables 7 and 8.

The results display that, in comparison with three reliability models, the standard deviation between estimated truth-value of improved maximum entropy reliability model and empirical value vector is the smallest. Taking the improved maximum entropy model as datum, other two models possess a larger relative life error under failure probability of 10% and 50%, with maximum relative error of 314.58%, which means the log-normal distribution and Weibull distribution possess low precision and large error in reliability prediction of bearing failure data. Once again, this proves that the novel improved maximum entropy probability distribution being taken as reliability model estimation method possesses the best effect and lowest error in estimating bearing failure life data.

In order to verify that the novel proposed model of maximum entropy distribution can be applied to poor information problem with small sample and unknown probability distribution, another $T4$ group test was carried out, with data number $n = 5$.

Initial data series are

$$T4 = \begin{bmatrix} 4.36 & 14.68 & 43.08 & 64.26 & 71.54 \end{bmatrix}. \qquad (76)$$

With the help of improved maximum entropy method, the reliability estimation results of this bearing failure data group are shown in Figure 15. The reliability empirical value vector of this example is $\mathbf{R}0 = \begin{bmatrix} 0.8704 & 0.6852 & 0.5000 & 0.3148 & 0.1296 \end{bmatrix}$, and the reliability estimation truth-value vector obtained by improved maximum entropy method is $\mathbf{R}3 = \begin{bmatrix} 0.9425 & 0.7509 & 0.5486 & 0.2800 & 0.0895 \end{bmatrix}$.

Depending on reliability empirical value vector $\mathbf{R}0$ and reliability estimation truth-value vector $\mathbf{R}3$ of improved maximum entropy method, it can be figured out that the standard deviation between $\mathbf{R}3$ and $\mathbf{R}0$ is very small at 0.0542, and the maximum difference value between both reliabilities is only 0.0721. The research results show that it is effective and feasible to use the reliability estimation truth-value function obtained by improved maximum entropy method to assess the reliability of small sample failure data with unknown probability distribution. In the condition of having

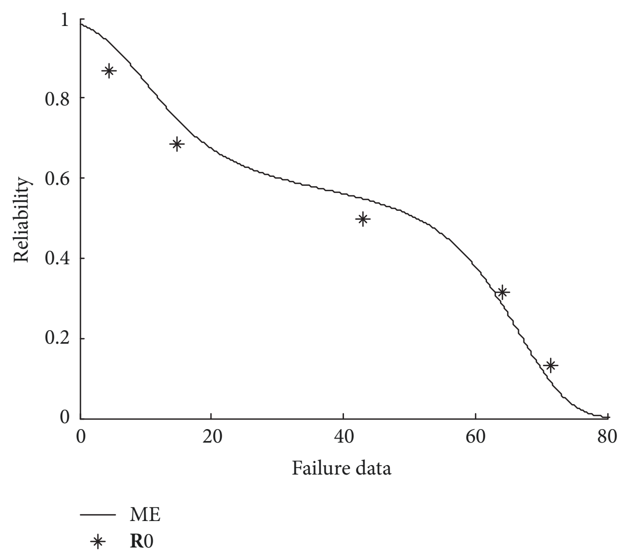

Figure 15: Improved maximum entropy reliability function image of small sample data $T4$.

failure data but without probability distribution or any prior information, the improved maximum entropy method can perfectly estimate the reliability function. And, at the same given life, the difference value between the reliability obtained by improved maximum entropy method and that obtained by empirical value is very small. This method also remedies the defect of existing method to only solve reliability estimation with known probability distribution.

To sum up, in simulation test, three reliability models display good fitting. In other words, in theoretical state, log-normal distribution, Weibull distribution, and improved maximum entropy model can be applied in reliability analysis on product performance failure problems. But in experiment part, we made comparative analysis of three reliability estimation methods according to actual bearing failure data, with the results showing that both standard deviation and relative error are the smallest between reliability empirical value vector $\mathbf{R}0$ and reliability estimation truth-value vector $\mathbf{R}3$ of improved maximum entropy method. Besides, this method can solve the poor information issue with small sample and unknown distribution that cannot be assessed by classical statistics. Therefore, based on all failure data series in this article, maximum reliability estimation method is the optimal reliability estimation approach due to its best fitting and highest precision.

5. Conclusions

Though the reliability estimation of bearing performance failure data can be realized by the above models, their prediction precision is far different. In practical application, we cannot rush to a conclusion by reliability obtained from single model.

The standard deviations of two-parameter log-normal distribution for $T1$ and $T3$ failure data groups are 0.0802 and 0.0950, respectively, whose performance is superior to two-parameter Weibull distribution. Although the three-parameter Weibull distribution taken as reliability model in example test of $T1$ and $T3$ groups performs better than three-parameter log-normal distribution, it cannot be applied to all analysis cases. So, in daily reliability analysis of bearing life, we should not only use Weibull distribution as reliability model for analysis.

For a novel improved maximum entropy model compared to Weibull distribution and log-normal distribution, the relative life error and standard deviation of its truth-value vector and empirical value vector are the smallest, in which, the maximum relative error of log-normal distribution reaches 314.58%, and Weibull distribution reaches 189.37%.

Whether there are really other distribution models (quasi-ideal distribution model) to calculate bearing failure performance reliability or not, at least, the improved maximum entropy probability distribution is a precise simulation to such quasi-ideal distribution model.

The novel proposed model of maximum entropy probability distribution does not take parameter distribution into account and allows poor information issues with unknown probability distribution, unknown prior information, or trends. This provides important theoretical reference to many uncertain information and poor information issues in engineering and even aerospace field and remedies the deficiency of classic statistics.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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